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FIXED POINT THEOREMS FOR SELF MAPPINGS IN GENERALIZED METRIC SPACES

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Abstract. In this paper a fixed point theorem in D- metric space using contractive type mappings is discussed. This work is an extension of the result obtained in [2].

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1. Introduction

In 1906 Mauric Frechet introduced metric spaces in his work surquelaues points du calcul fonctionel. However the name is due to Felix Hausdorff. In 1984 B. C. Dhage first introduced generalized i.e. D-metric space[2].

Definition 1.1: Let S be a nonempty set. A point $x \in S$ is said to be a fixed of a function $F : S \to S$ if $F(x) = x$.

Definition 1.2[1]: Let R denote the real line and X denote a nonempty set. Let $D : X \times X \times X \to R$ be a function satisfying the following properties.

(1) $D(x, y, z) \geq 0, \forall x, y, z \in X$
(2) $D(x, y, z) = 0, \forall x, y, z \in X$ if and only if x = y = z(coincidence),
(3) $D(x, y, z) = D(p(x, y, z))$, where $p(x, y, z)$ is a permutation of x, y, z (symmetry),
(4) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z), \forall x, y, z, a \in X$. (Tetrahedral inequality).

Then the function D is called a D-metric (also called a generalised metric) on X. The set X together with a D-metric D is called a D-metric space denoted by (X, D). Generally, the usual ordinary metric is called the distance function. D-metric is called diameter function of the point of X [1]. In[2] a D-metric continuous function on X^3 in the topology of D-metric convergence.

Let (A, D) be a D-metric space and $g : A \to A$. The orbit of g at point $a \in A$ is the set $O(a) = \{a, ga, g^2a, ...\}$. An orbit of a is said to be bounded, if there exists a constant $K > 0$ such that $D(p, q, r) \leq K$ for all $p, q, r \in O(a)$. The constant K is called a D-bound of $O(a)$. A D-metric space A is said to be g-orbitally bounded if $O(a)$ is bounded for each $a \in A$. A sequence $\{a_n\} \subset A$ is said to be D-Cauchy if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that, for all $m, n, \rho \geq n_0$, $D(a_m, a_n, a_\rho) < \varepsilon$. A sequence $\{a_n\} \subset$A is said to be D-convergent to a point $a \in$ A if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that, $\forall\, m, n \geq n_0$, $D(a_m, a_n, a) < \varepsilon$. An orbit $O(a)$ is called g-orbitally complete if every D-cauchy sequence in $O(a)$ converges to a point in A.

Lemma 1.1 [3]: Let (A, D) be a D-metric space and $\{x_n\} \subset A$ be a bounded sequence with D-bound K satisfying

$$D(x_n, x_{n+1}, x_m) \quad \leq \quad \lambda^n\, K. \tag{1}$$

for all positive integers $m > n$, and some $0 \leq \lambda < 1$. Then $\{x_n\}\, is D - Cauchy$.

Definition 1.3: Let (A, D) be a D-metric space and $f : A \to A$. Then f is said to continuous at $a \in A$ if $lim_{n\to\infty} f(a_n) = f(a)$, whenever$\{a_n\}$ is a sequence in A with $\lim_{n\to\infty} a_n = a$.

Note that every convergent sequence in a D- metric space is Cauchy and any subsequence converges to the limit of the sequence.

Definition 1.4: For a subset S of a D- metric space (A, D), we define $\overline{S} = D$-closure of $S = \{a \in A$: there exists a Cauchy sequence in S which converges to $a\}$

2. Main Result

Theorem 2.1 *Let $(A, d)_1)$ and (B, d_2) be two D-metric spaces f, g be self maps on A and B respectively. Suppose that there exist $a_0 \in A,\ b_0 \in B$ such that $O(a_0)$ and $O(b_0)$ are D-bounded and f, g orbitally complete. Suppose also that f and g satisfies*

$$(2) \qquad d_1(f(a), f(r), f(s)) \leq \lambda\ max\{d_1(a, r, t), d_1(a, f(a), t), d_1(a, r, f(r))\} \quad for \quad a, r, s \in \overline{O(a_0)}$$

$$(3) \qquad d_2(g(b), g(s), g(v)) \leq q\ max\{d_2(b, s, v), d_2(b, g(b), v), d_2(b, s, g(s))\} \quad for \quad b, s, v \in \overline{O(b_0)}$$

for some $0 \leq\ \lambda,\ q < 1$. Then f and g have unique fixed points in A and B respectively.

Proof: For $a_0 \in A$, $b_0 \in B$, let a_1=$f(a_0)$, b_1=$g(b_0)$ and for $n \in N$, a_n=$f(a_{n-1})$, b_n=$g(b_{n-1})$. Then orbit of f at a_0 is $O(a_0)$=$\{a_0, a_1, a_2, ...\}$ and orbit of g at b_0 is $O(b_0)$=$\{b_0, b_1, b_2, ...\}$. As $O(a_0)$ and $O(b_0)$ are D-bounded so there exist real numbers k , c such that for all $a, r, s \in O(a_0)$ and $b, u, v \in O(b_0)$;

$$(4) \qquad d_1(a, r, s) \leq\ k,\ \ d_2(b, u, v) \leq\ c$$

(i) If a_n=a_{n+1} for some $n \in N$ then $a_n \in A$ is a fixed point of f. let $a_n \neq a_{n+1}$ for all $n \in N$ and in this case we prove;

$$(5) \qquad d_1(a_n, a_{n+1}, a_m) \quad \leq \quad \lambda^n k, for, m > n$$

It is true for $n = 0$ by (4). By (2) for $m > 1$;

$$\begin{aligned} d_1(a_1, a_2, a_m) &\leq \lambda max\{d_1(a_0, a_1, a_{m-1}), d_1(a_0, a_1, a_{m-1}), d_1(a_0, a_1, a_2)\} \\ &\leq \lambda k \quad by\,(4). \end{aligned}$$

Thus (5) is true for $n = 0, 1$.
Consider $n \geqslant 1$ where (5) is true (induction hypothesis). Then by(2),

$$\begin{aligned} d_1(a_{n+1}, a_{n+2}, a_m) &= d_1(f(a_n), f(a_{n+1}), f(a_{m-1})), \quad for\, m > n+1 \\ &\leq \lambda max\{d_1(a_n, a_{n+1}, a_{m-1}), d_1(a_n, a_{n+1}, a_{m-1}), \\ &\quad d_1(a_n, a_{n+1}, a_{n+2})\} \\ &\leq \lambda\lambda^n k. \end{aligned}$$

by induction hypothesis. Hence (5) is true for $n + 1$ in place of n.
Thus by the principle of mathematical induction (5) follows for all $n, m \in N \cup 0$ where $m > n$. Therefore by lemma 1.1, $\{a_n\}$ is a D-Cauchy sequence in A and as f is orbitally complete, so this sequence converges to a point $p \in A$, say and so $p \in \overline{O(a_0)} \subseteq A$
Thus $\forall \epsilon > 0$, $\exists$ $n_0 \in N$ such that $m, n \geq n_0 \Rightarrow d_1(a_m, a_n, p) < \epsilon$, which gives $d_1(a_m, a_n, p) \to 0$ as $m, n \to \infty$; $d_1(a_n, a_{n+1}, p) \to 0$ as $n \to \infty$.
Now

$$\begin{aligned} d_1(a_{n+1}, a_{n+2}, f(p)) &= d_1(f(a_n), f(a_{n+1}, f(p)) \\ &\leq \lambda max\{d_1(a_n, a_{n+1}, p), d_1(a_n, a_{n+1}, a_{n+2}), by\,(2) \\ &\quad \to 0 \; as \; n \to \infty, \end{aligned}$$

since

$$\lim_{n\to\infty} a_n = p$$

and $\{a_n\}$ is a D-Cauchy sequence.

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = f(p)$$

$$i.e. \lim_{n\to\infty} f(a_n) \quad = \quad f(p)$$

where

$$\lim_{n\to\infty} a_n \quad = \quad p.$$

Hence f is continuous at p. Now $\{a_{n+1}\}$ is a subsequence of $\{a_n\}$, so

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} a_n \\ &= p \end{aligned}$$

and thus $f(p) = p$,
i.e. f has a fixed point $p \in A$.
Uniqueness: Let p,q be fixed points of f in A. Then

$$\begin{aligned} d_1(p,p,q) &= d_1(f(p), f(p), f(q)) \\ &\leq \lambda max\{d_1(p,p,q), d_1(p,p,p)\} \quad by(2). \end{aligned}$$

i.e.

$$\begin{aligned} 0 &\leq d_1(p,p,q) \\ &\leq \lambda d_1(p,p,q) \quad where 0 \leq \lambda < 1. \end{aligned}$$

$$\Rightarrow \quad d_1(p,p,q) = 0 \quad i.e. p = q$$

And uniqueness of fixed point of f follows.
(ii) If $b_w = b_{w+1}$ for some $w \in N$ then $b_w \in B$ is a fixed point of f. Let $b_w \neq b_{w+1}$ for all $w \in N$ and in this case it follows that

$$d_2(b_w, b_{w+1}, b_x) \quad \leq \quad q^w c \quad for\, x > w$$

This follows replacing d_1 by d_2, a_n by b_w, m by x in (i) etc. and the result g has a unique fixed point in $\overline{O(b_0)} \subseteq B$ follows.

Corollary 2.1:[2] Let(X, D) be a D-metric space, f a self map of X. Suppose that there exists an $x_0 \in X$ such that $O(x_0)$ is D-bounded and f orbitally complete. Suppose also that f satisfies

$$\text{(6)} \qquad \begin{aligned} D(f(x), f(y), f(z)) &\leq \lambda\, max\{D(x,y,z), D(x, f(x), z)\} \\ for \quad x,y,z &\in \overline{O(x_0)} \end{aligned}$$

for some $0 \leq \lambda < 1$. Then f has a unique fixed point in X.
Proof of Corollary 2.1 follows by Theorem 2.1, since

$$\begin{aligned} D(f(x), f(y), f(z)) &\leq \lambda\, max\{D(x,y,z), D(x,f(x),z)\} \\ &\leq \lambda\, max\{D(x,y,z), D(x,f(x),z), D(x,y,f(y))\} \end{aligned}$$

as hypothesis (2) (or (3)) of Theorem 2.1 satisfies.
Corollary 2.2:[2] Let f be a self map f a complete and bounded D-metric space (X, D) satisfying.

$$D(f(x), f(y), f(z)) \leq \lambda D(x,y,z) \tag{7}$$

for all $x, y, z \in$ X, for some $0 \leq \lambda < 1$. Then f has a unique fixed point p, and f is continuous at p.
Proof of Corollary 2.2 similarly follows by Theorem 2.1, since

$$\begin{aligned} D(f(x), f(y), f(z)) &\leq \lambda\, max\{D(x,y,z))\} \\ &\leq \lambda\, max\{D(x,y,z), D(x,f(x),z), D(x,y,f(y))\}\, etc. \end{aligned}$$

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