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SPINOR REPRESENTATIONS OF FRAMED CURVES IN THE THREE-DIMENSIONAL LIE GROUPS

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Abstract. In this study, we determine the spinor representations of special singular curves (framed curves) in the three-dimensional Lie groups with a bi-invariant metric. Also, we construct spinor framed equations for some special cases and obtain the relations between the spinor representations of the general frame and adapted frame along the framed curves in the three-dimensional Lie groups. Then, we give some geometric properties and results with respect to them.

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1. Introduction

The theory of curves is a fundamental and substantial area of classical differential geometry and forms an important part of it. From the discovery of the Frenet-Serret frame, which is a pioneer and attractive moving frame, moving frames interested several researchers. Lots of studies have been completed and are ongoing with respect to the Frenet-Serret frame for regular curves in the existing literature. The cause why this frame is called the Frenet-Serret frame is that the researchers J. F. Frenet [25] and J. A. Serret [50] studied this concept at different times unaware of each other in the 19th century. Inspired by this popular moving frame, several types new moving frames are established such as Darboux frame was constructed on regular surfaces by J. G. Darboux [12] and Bishop frame (type-1, type-2, type-3) [3, 52, 61]. It is well known that, if space curves have non-regular points (namely singular points), then we can not establish the Frenet-Serret frame. To put it more clearly, while the Frenet-Serret frame is constructed for only regular curves with the well-known condition the curvature κ never vanishes along this curve, in that case this frame is not established if this curve has singular points. Recently, in order to establish the Frenet-Serret frame for non-regular curves, two researchers S. Honda and M. Takahashi [30] contribute to the literature the new and interesting notions i.e. framed curves and framed base curves. Honda and Takahashi determined the framed curves as they are smooth curves with a moving frame that might have singular points [16, 30]. Indeed, framed curves include both regular curves and singular curves, and they are a generalization of regular curves with linear independent conditions and Legendre curves in unit tangent bundles [26, 53]. As it will be clear from the reasons we have mentioned above, many researchers have worked on the framed curves and these studies continue. In the existing literature, several studies have been completed related to framed curves such as; existence conditions of framed curves [26], Bertrand and Mannheim curves of framed curves [31], framed rectifying and normal curves [16, 17, 59], the surfaces related to the framed curves and framed helices [29, 32, 33, 40, 60].

On the other hand, spinor was studied by É Cartan who examined the spinor in 1913, but P. Ehrenfest had earned the notion spinor to the literature in the 1920s [57]. Spinors are one of the attractive concepts for several workframes from mathematics to physics. While according to physicists, spinors are multilinear transformations and via this attribute, spinors are mathematical structure. However, according to mathematicians this multilinear property does not matter and spinors are a vectorial concept [20, 22, 24]. Also, spinors have been used in Clifford algebra (or called as geometric algebra by W. K. Clifford) [8, 57]. Physicists argued that the notion spin to explain some features of quantum particles which seemed in the course of several experiments. When these features were considered quantitatively, the new mathematical notion was determined which was called spinor [28]. Studying the angular momentum of an electron and spin of the non-relavistic electron, expressing the spin of non-relativistic electron, other spin-1/2 particles, characterizing the state of relativistic many-particle systems in quantum field theory are given to some usage areas where the spinor is used [22, 37, 47]. Pauli matrices, which be said the spin matrices and described the spin in quantum theory were studied by W. Pauli [20, 49]. Spin can be represented by using the Pauli matrices in the quantum theory. In 1927, Pauli expressed that the wave function of an electron can be represented via a vector with two complex components. The name of this vector is spinor [28]. On the other hand, Cartan scrutinize the mathematical concepts of spinors when examining the representation of groups in 1913. The concept of spinors is expressed that the spinors supply a linear representation of the groups of rotations of a space of any dimension by Cartan. Therefore, the relationship of spinor with physics can also be established with geometry [28]. In the pioneer study [7], the geometric construction of spinor representations is examined by Cartan. The set of isotropic vectors constructs a two-dimensional surface in the space $\mathbb{C}^2$. Conversely, these vectors in $\mathbb{C}^2$, represent the same isotropic vectors. These complex vectors are as two-dimensional in the space $\mathbb{C}^2$ by Cartan [7, 20]. On the other hand, spinors are studied by M. D. Vivarelli in [58]. Vivarelli established relations between the spinors and quaternions, and constructed the spinor representation of rotations in $\mathbb{E}^3$ by the relations between the rotations in $\mathbb{E}^3$ and quaternions [20, 58].

Several studies have been done with respect to the spinors in mathematics especially geometric sense and Clifford algebra. del Castillo and Barrales constructed a study which includes the relationships between differential geometry and spinor [14]. Inspired by this important and pioneering study, several researchers examined the relations between the spinors and moving frames and curves (see [20, 22, 23, 37, 51, 55, 56]). These studies are related to regular curves. Fibonacci spinors are investigated by utilizing the study [58] by Erişir and Güngör in [24]. In [19, 34], spinor representations of framed Mannheim and framed Bertrand curves were introduced. These studies [2, 21, 35, 36] can be given in order to find the relations and representations of hyperbolic spinors and differential geometry. In addition to these, [5, 13, 14, 28, 41, 42, 57, 58] can be examined related to spinors.

The degenerate semi-Riemannian geometry of Lie group was examined by Çöken and Çiftçi in [9]. The generalization of Lancrets theorem and general helices were introduced by Çiftçi in [11]. Also, slant helices, general helices of AW(k)-type, Bertrand and Mannheim curve couples in the three-dimensional Lie group with a bi-invariant metric were studied in [27, 43, 44, 62]. Then, the characterization of rectifying, osculating ad normal curves in three-dimensional compact Lie groups [4] and inextensible flows of curves in three-dimensional Lie groups were scrutinized [45]. In [46], Smarandache curves in the three-dimensional Lie groups were given. Moreover, generalized Bertrand and Mannheim curves in 3D Lie groups were presented [48]. Okuyucu et al. investigate the spinor Frenet equations in the three-dimensional Lie groups [47]. Also, Bertrand curves of AW(k)-type in three-dimensional Lie groups and spherical indicatrices of them are determined in [6, 38]. These studies were studied on regular curves, then Doğan Yazıcı et al. introduced the framed curves which might have singular points in the three-dimensional Lie groups [18].

One of the most important workframe of mathematics is Lie groups which have three types different mathematical concepts such as abelian Lie groups, S^3, and $SO(3)$. Additionally, there are several types usage areas of Lie groups with respect to the theory and application in mechanics and physics. In addition to these, incompressible inviscid fluid motion and rigid body motion correspond to the geodesic flow of left or right invariant metric have been defined on a Lie group [1, 10, 39].

At the beginning of this paper, the question whack to our curiosity: What kind of geometric interpretations and results do we attain if we bring together the spinors and framed curves in three-dimensional Lie groups?. In order to get an answer with respect to this question, we aim to study this concept. Thanks to the studies [47] and [18], we determine and examine the spinor representations of framed curves in three-dimensional Lie groups with bi-invariant metric. We believe that this paper contributes to the existing literature because of the fact that this study includes both the spinor representation of regular curves in three-dimensional Lie groups and spinor representation of singular curves in three-dimensional Lie groups.

In this study, we concentrate on the spinor representations of special singular curves in the three-dimensional Lie groups. In Section 1 and Section 2, we write and remind some fundamental notations and notions related to the spinors, framed curves, Lie groups, and framed curves in the three-dimensional Lie groups. In Section 3, we introduce the spinor equations and find spinor components of them. By using the adapted frame and general frame of the framed curves in three-dimensional Lie groups, we give the spinor equations of them and then we get some relations with respect to them. Also, we obtain the spinor framed equations in the special cases of the three-dimensional Lie groups.

2. Basic Concepts

In this section, some fundamental and required concepts and notations are reminded related to the spinors, framed curves, Lie groups, and framed curves in three-dimensional Lie groups.

2.1. **Spinors.** A spinor is determined as two-dimensional complex vectors

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

with the help of the vectors $\chi, \vartheta, \upsilon \in \mathbb{R}^3$ such that

$$\chi + i\vartheta = \Phi^t \sigma \Phi, \tag{1}$$

$$\upsilon = -\widehat{\Phi}^t \sigma \Phi, \tag{2}$$

where t represents the transposition, $\widehat{\Phi}$ is the conjugation [7] (or mate [13]) of Φ, $\overline{\Phi}$ is the complex conjugation of Φ. In that case, the followings are written [14]:

$$\widehat{\Phi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\Phi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\Phi}_1 \\ \overline{\Phi}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\Phi}_2 \\ \overline{\Phi}_1 \end{pmatrix}.$$

Also, Pauli matrices which are 2×2 are cartesian components for the vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are expressed as follows [14]:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Moreover, $\chi + i\vartheta$ is an isotropic vector and υ is a real vector. Let $\flat = (\flat_1, \flat_2, \flat_3) \in \mathbb{C}^3$ be an isotropic vector if $\langle \flat, \flat \rangle = 0$ where $\mathbb{C}^3$ is three-dimensional complex vector space. The isotropic vectors in $\mathbb{C}^3$ construct the two-dimensional surface in $\mathbb{C}^2$.

Provided that this surface is parametrized via the Φ_1 and Φ_2, then $\flat_1 = \delta_1^2 - \delta_2^2$, $\flat_2 = i(\delta_1^2 + \delta_2^2), \flat_3 = -2\delta_1\delta_2$ can be written. Hence, we get $\Phi_1 = \pm\sqrt{\frac{\flat_1 - i\flat_2}{2}}$ and $\Phi_2 = \pm\sqrt{\frac{-\flat_1 - i\flat_2}{2}}$ [7, 14, 20]. According to the equation $\chi + i\vartheta = (\flat_1, \flat_2, \flat_3)$ the followings can be expressed with the Eqs. (1) and (3):

$$\flat_1 = \Phi^t\sigma_1\Phi = \Phi_1^2 - \Phi_2^2, \quad \flat_2 = \Phi^t\sigma_2\Phi = i(\Phi_1^2 + \Phi_2^2), \quad \flat_3 = \Phi^t\sigma_3\Phi = -2\Phi_1\Phi_2$$

and

$$\begin{aligned} \chi + i\vartheta &= \left(\Phi_1^2 - \Phi_2^2, i(\Phi_1^2 + \Phi_2^2), -2\Phi_1\Phi_2\right), \\ \upsilon &= \left(\Phi_1\overline{\Phi}_2 + \overline{\Phi}_1\Phi_2, i(\Phi_1\overline{\Phi}_2 - \overline{\Phi}_1\Phi_2), |\Phi_1|^2 - |\Phi_2|^2\right). \end{aligned} \tag{4}$$

Since $\chi + i\vartheta \in \mathbb{C}^3$ is an isotropic vector, the vectors $\chi, \vartheta, \upsilon$ are mutually orthogonal and $|\chi| = |\vartheta| = |\upsilon| = \overline{\delta}^t\delta$. Therefore, $\langle \chi \times \vartheta, \upsilon\rangle = \det(\chi, \vartheta, \upsilon) > 0$. Conversely, provided that the vectors $\chi, \vartheta, \upsilon \in \mathbb{R}^3$ are mutually orthogonal vectors of the same magnitude ($\det(\chi, \vartheta, \upsilon) > 0$), then there is a spinor which is determined in the equation (1) [7, 13, 14, 20, 22, 23, 37, 47, 51, 55]. Let Φ and Ψ be any two spinors and the following properties hold:

$$\Phi^t\sigma\Psi = \Psi^t\sigma\Phi, \tag{5}$$

$$\overline{\Phi^t\sigma\Psi} = -\widehat{\Phi}^t\sigma\widehat{\Psi}, \tag{6}$$

$$(\widehat{z_1\Phi + z_2\Psi}) = \overline{z}_1\widehat{\Phi} + \overline{z}_2\widehat{\Psi}, \tag{7}$$

where $z_1, z_2 \in \mathbb{C}$ [14]. Since the matrices $\sigma_1, \sigma_2, \sigma_3$ which are presented in the equation (3) are symmetric, the equation (5) holds. Since the spinors δ and $-\delta$ correspond to the same ordered orthogonal basis $\{\chi, \vartheta, \upsilon\}$ with $|\chi| = |\vartheta| = |\upsilon|$ and $\det(\chi, \vartheta, \upsilon) > 0$, the correspondence between the spinors and orthogonal bases which are given in the equation (1) is two-to-one. The ordered triads $\{\chi, \vartheta, \upsilon\}$, $\{\vartheta, \upsilon, \chi\}, \{\upsilon, \chi, \vartheta\}$ correspond to different spinors. Additionally, the set $\{\delta, \overline{\delta}\}$ is linearly independent if $\delta \neq 0$ [7, 13, 14, 20, 22, 23, 37, 47, 51, 55].

2.2. **Some fundamental notions and notations of Lie groups.** Let G be a Lie group with a bi-invariant metric $\langle , \rangle$ and D be the Levi-Civita connection of Lie group G. Provided that the notation $\mathfrak{g}$ denotes the Lie algebra of Lie group G, then we can write that $\mathfrak{g}$ is isomorphic to T_eG where e is the neutral element of Lie group G. If $\langle , \rangle$ is a bi-invariant metric on Lie group G, we get as follows:

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle$$

and

$$D_X Y = \frac{1}{2}[X, Y]$$

for every $X, Y, Z \in \mathfrak{g}$. In addition to these, Lie bracket of any two vector fields W and Z is written as follows:

$$[W, Z] = \sum_{i=1}^{n} w_i z_i \left[X_i, X_j\right]$$

where $W = \sum_{i=1}^{n} w_i X_i$ and $Z = \sum_{i=1}^{n} z_i X_i$ with the orthonormal basis $\{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$ and also $w_i = I \to \mathbb{R}$ and $z_i = I \to \mathbb{R}$. If $\phi : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve, then the covariant derivative of the vector field W along the curve ϕ which is denoted by the notation $D_{\phi'} W$ is obtained:

$$D_{\phi'} W = D_T W = \dot{W} + \frac{1}{2}[T, W]$$

where T is the unit tangent vector of the curve ϕ and $\dot{W} = \sum_{i=1}^{n} \frac{dw}{dt} X_i$. Moreover, provided that the vector field W is the left invariant vector field of the curve ϕ, then $\dot{W} = 0$ is written [10, 18, 47].

2.3. **Framed curves in the three-dimensional Lie groups.** The framed curves, which were established in order to overcome the difficulties of constructing the moving frame for the non-regular curves in Euclidean space, were examined by Honda and Takahashi [30]. To get more detailed information related to the framed curves, we can refer to the studies [16–19, 26, 29–34, 40, 53, 59, 60]. Framed curves in the three-dimensional Lie groups were investigated by Doğan Yazıcı et al. in [18] by inspiring the study [30].

Definition 2.1 ([18])**.** *$(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ is a framed curve in the three-dimensional Lie group G if $\langle \gamma'(s), \eta_1(s)\rangle = 0$ and $\langle \gamma'(s), \eta_2(s)\rangle = 0$ for every $s \in I$ where*

$$\Delta_G = \{\eta = (\eta_1, \eta_2) \in G \times G : \langle \eta_i, \eta_j \rangle = \delta_{ij},\ i, j = 1, 2\}.$$

Also, $\nu(s) = \eta_1(s) \times \eta_2(s)$ which is a unit vector. Additionally, if $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ is a framed curve, then the covariant derivative of the vector field W along the framed curve (γ, η_1, η_2) by using the unit vector ν by assumption as follows:

$$D_\nu W = \dot{W} + \frac{1}{2}\left[\nu, W\right].$$

Theorem 2.2 ([18]). *Let $(\gamma,\eta_1,\eta_2): I \to G \times \Delta_G$ be a framed curve in the three-dimensional Lie group G. The following Frenet-Serret type formulas of the framed curves in the three-dimensional Lie group are written:*

$$\begin{pmatrix} \dot{\nu}(s) \\ \dot{\eta}_1(s) \\ \dot{\eta}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s)-\delta_G(s) \\ n(s) & -\left(l(s)-\delta_G(s)\right) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \eta_1(s) \\ \eta_2(s) \end{pmatrix} \tag{8}$$

where

$$m(s)=\langle \nu(s),\dot{\eta}_1(s)\rangle, \quad n(s)=\langle \nu(s),\dot{\eta}_2(s)\rangle, \quad l(s)=\langle \eta_2(s),\dot{\eta}_1(s)\rangle+\delta_G(s), \tag{9}$$

and

$$\delta_G(s)=\frac{1}{2}\langle [\nu(s),\eta_1(s)],\eta_2(s)\rangle$$

or

$$\begin{aligned}\delta_G(s)=&\frac{1}{2\left(m(s)n'(s)-m'(s)n(s)+l(s)\left(m^2(s)+n^2(s)\right)\right)}\langle \ddot{\nu}(s),[\nu(s),\dot{\nu}(s)]\rangle \\ &+\frac{1}{4\left(m(s)n'(s)-m'(s)n(s)+l(s)\left(m^2(s)+n^2(s)\right)\right)}\|[\nu(s),\dot{\nu}(s)]\|^2.\end{aligned}$$

In addition to these, adapted frames of framed curves in the three-dimensional Lie groups are determined in [18]. Let $(\gamma,\eta_1,\eta_2): I \to G \times \Delta_G$ be a framed curve in the three-dimensional Lie groups and $(\overline{\eta}_1,\overline{\eta}_2) \in \Delta_G$ are determined by:

$$\begin{pmatrix} \overline{\eta}_1(s) \\ \overline{\eta}_2(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \eta_1(s) \\ \eta_2(s) \end{pmatrix} \tag{10}$$

where $\theta(s)$ is a smooth function on the range I. In that case, $(\gamma,\overline{\eta}_1,\overline{\eta}_2): I \to G\times\Delta_G$ is a framed curve. There are two special cases as follows [18]:

(1) Suppose that $m(s)\sin\theta(s)+n(s)\cos\theta(s)=0$. Let $m(s)=-p(s)\cos\theta(s)$ and $n(s)=p(s)\sin\theta(s)$. Then, we get:

$$\begin{pmatrix} \dot{\nu}(s) \\ \dot{\overline{\eta}}_1(s) \\ \dot{\overline{\eta}}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & p(s) & 0 \\ -p(s) & 0 & q(s)-\delta_G(s) \\ 0 & -\left(q(s)-\delta_G(s)\right) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \overline{\eta}_1(s) \\ \overline{\eta}_2(s) \end{pmatrix} \tag{11}$$

where $q(s)=l(s)-\theta'(s)$. If $\delta_G(s)=0$, then the given frame in the equation (11) corresponds to the Frenet-type frame of framed curves in the three-dimensional Euclidean space [18, 59].

(2) Assume that $l(s) - \theta'(s) - \delta_G(s) = 0$, then we get:

$$\begin{pmatrix} \dot{\nu}(s) \\ \dot{\overline{\eta}}_1(s) \\ \dot{\overline{\eta}}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{m}(s) & -\widetilde{n}(s) \\ \widetilde{m}(s) & 0 & 0 \\ \widetilde{n}(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \overline{\eta}_1(s) \\ \overline{\eta}_2(s) \end{pmatrix} \tag{12}$$

where

$$\begin{pmatrix} \widetilde{m}(s) \\ \widetilde{n}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} m(s) \\ n(s) \end{pmatrix} \tag{13}$$

If $\delta_G(s) = 0$, then the frame which is written in the equation (12) corresponds to the Bishop frame of framed curves in the three-dimensional Euclidean space [18, 30].

Proposition 2.3 ([18])**.** *Let $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ with the triad $\{\nu, \eta_1, \eta_2\}$ and $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ with the triad of adapted frame $\{\overline{\nu}, \overline{\eta}_1, \overline{\eta}_2\}$ be framed curves in the three-dimensional Lie group G. In that case, we get:*

$$\delta_G = \frac{1}{2}\langle[\nu, \eta_1], \eta_2\rangle = \frac{1}{2}\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_2\rangle.$$

Remark 2.4. *The following mathematical expressions are satisfied:*

(1) Provided that $\langle \nu, [\eta_1, \eta_2]\rangle = \langle \nu\, [\eta_2, \eta_1]\rangle$ (namely, abelian), then $\delta_G = 0$.

(2) Provided that G is a SO^3 then $\delta_G = \frac{1}{2}$.

(3) Provided that G is a S^3 then $\delta_G = 1$.

where G is a Lie group with a bi-invariant metric $\langle , \rangle$ (cf. [11, 15, 18, 47]).

3. The Spinor Representations of Framed Curves in the Three-Dimensional Lie Groups

In this section, we investigate the spinor representations of framed curves in the three-dimensional Lie group G and then we give the spinor representations of adapted frames along the framed curve in G. Also, we find some relations with respect to the general frame and adapted frame of the framed curve. Additionally, we present the spinor equations of them in the three types special cases of G. Then, some geometric interpretations and results are written related to them.

Definition 3.1. *Let $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ be a framed curve in the three-dimensional Lie groups and the spinor ψ corresponds to the triad $\{\eta_1, \eta_2, \nu\}$. Then,* *the spinor representations of the framed curve in the three-dimensional Lie groups are obtained as follows:*

$$\eta_1 + i\eta_2 = \psi^t \sigma \psi, \tag{14}$$

$$\nu = -\widehat{\psi}^t \sigma \psi, \tag{15}$$

where $\overline{\psi}^t \psi = 1$.

Theorem 3.2. *Let* $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ *be a framed curve in the three-dimensional Lie group* G *and the spinor* ψ *corresponds to the triad* $\{\eta_1, \eta_2, \nu\}$. *Then, the following single spinor equation is given as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\,(l - \delta_G)\,\psi + (m + in)\,\widehat{\psi} \right]. \tag{16}$$

Proof. Let us take the derivative of the equation (14) according to the parameter s, then we get:

$$\dot{\eta}_1 + i\dot{\eta}_2 = \left(\frac{d\psi}{ds} \right)^t \sigma \psi + \psi^t \sigma \frac{d\psi}{ds}. \tag{17}$$

Since $\{\psi, \widehat{\psi}\}$ is a basis for spinors, then the spinor $\dfrac{d\psi}{ds}$ can be written as follows:

$$\frac{d\psi}{ds} = \rho \psi + \varrho \widehat{\psi} \tag{18}$$

where ρ and ϱ are any complex valued functions. With the help of the equations (8), (17) and (18), we have:

$$\begin{aligned} m\,\nu + (l - \delta_G)\,\eta_2 + i\,(n\nu - (l - \delta_G)\,\eta_1) &= \left(\rho\psi + \varrho\widehat{\psi} \right)^t \sigma\psi + \psi^t \sigma \left(\rho\psi + \varrho\widehat{\psi} \right) \\ &= 2\rho\psi^t\sigma\psi + \varrho\widehat{\psi}^t\sigma\psi + \varrho\psi^t\sigma\widehat{\psi}. \end{aligned}$$

By using the equations (5), (14) and (15), we obtain:

$$-i\,(l - \delta_G)\,(\eta_1 + i\eta_2) + (m + i\,n)\,\nu = 2\rho\,(\eta_1 + i\,\eta_2) - 2\varrho\nu.$$

Then, we get:

$$\rho = \frac{-\,i\;(l - \delta_G)}{2} \quad \text{and} \quad \varrho = -\frac{m + i\,n}{2}. \tag{19}$$

In that case, if we substitute the equation (19) in the equation (18), then we have:

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\left(l-\delta_G\right)\psi + (m+in)\,\widehat{\psi}\right].$$

□

By using the Remark 2.4, we can get the following Corollary 3.3-3.5.

Corollary 3.3. *Let $(\gamma,\eta_1,\eta_2) : I \to G\times\Delta_G$ be a framed curve in abelian Lie group and the spinor ψ corresponds to the triad $\{\eta_1,\eta_2,\nu\}$. In that case, the following single spinor equation is written as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[il\psi + (m+in)\,\widehat{\psi}\right].$$

Corollary 3.4. *Let $(\gamma,\eta_1,\eta_2) : I \to G\times\Delta_G$ be a framed curve in Lie group SO^3 and the spinor ψ corresponds to the triad $\{\eta_1,\eta_2,\nu\}$. In that case, the following single spinor equation is written as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\left(l-\frac{1}{2}\right)\psi + (m+in)\,\widehat{\psi}\right].$$

Corollary 3.5. *Let $(\gamma,\eta_1,\eta_2) : I \to G\times\Delta_G$ be a framed curve in Lie group S^3 and the spinor ψ corresponds to the triad $\{\eta_1,\eta_2,\nu\}$. In that case, the following single spinor equation is written as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\left(l-1\right)\psi + (m+in)\,\widehat{\psi}\right].$$

Theorem 3.6. *Let $(\gamma,\eta_1,\eta_2) : I \to G\times\Delta_G$ be a framed curve in the three-dimensional Lie groups and the spinor ψ corresponds to the triad $\{\eta_1,\eta_2,\nu\}$. Then, the spinor equations of the framed vectors in the three-dimensional Lie groups are presented as follows:*

$$\eta_1 = \frac{1}{2}\left(\psi^t\sigma\psi - \widehat{\psi}^t\sigma\widehat{\psi}\right), \tag{20}$$

$$\eta_2 = -\frac{i}{2}\left(\psi^t\sigma\psi + \widehat{\psi}^t\sigma\widehat{\psi}\right), \tag{21}$$

$$\nu = -\widehat{\psi}^t\sigma\psi. \tag{22}$$

Proof. Let the spinor ψ corresponds to the triad $\{\eta_1, \eta_2, \nu\}$ of the framed curve (γ, ν_1, ν_2) in the three-dimensional Lie group. According to the equation (14), we have $\eta_1 = Re(\psi^t\sigma\psi)$ and $\eta_2 = Im(\psi^t\sigma\psi)$. By using the properties of complex numbers, we obtain:

$$\begin{aligned}
\eta_1 &= \frac{1}{2}\left(\psi^t\sigma\psi + \overline{\psi^t\sigma\psi}\right),\\
\eta_2 &= -\frac{i}{2}\left(\psi^t\sigma\psi - \overline{\psi^t\sigma\psi}\right).
\end{aligned}$$

Via the equation (6), we get the equations (20) and (21). Also, we know $\nu = -\widehat{\psi}^t\sigma\psi$ from the equation (15). □

Corollary 3.7. *Let $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ be a framed curve in the three-dimensional Lie groups and the spinor ψ corresponds to the triad $\{\eta_1, \eta_2, \nu\}$. Then, the spinor components of framed vectors in the three-dimensional Lie groups are given as:*

$$\begin{aligned}
\eta_1 &= \frac{1}{2}\left(\psi_1^2 - \psi_2^2 + \overline{\psi}_1^2 - \overline{\psi}_2^2, i\left(\psi_1^2 + \psi_2^2 - \overline{\psi}_1^2 - \overline{\psi}_2^2\right), -2\left(\psi_1\psi_2 + \overline{\psi}_1\overline{\psi}_2\right)\right),\\
\eta_2 &= -\frac{i}{2}\left(\psi_1^2 - \psi_2^2 - \overline{\psi}_1^2 + \overline{\psi}_2^2, i\left(\psi_1^2 + \psi_2^2 + \overline{\psi}_1^2 + \overline{\psi}_2^2\right), 2\left(\overline{\psi}_1\overline{\psi}_2 - \psi_1\psi_2\right)\right),\\
\nu &= \left(\psi_1\overline{\psi}_2 + \overline{\psi}_1\psi_2, i\left(\psi_1\overline{\psi}_2 - \overline{\psi}_1\psi_2\right), |\psi_1|^2 - |\psi_2|^2\right).
\end{aligned}$$

Proof. For the spinor expression $\psi^t\sigma\psi$, we can write:

$$\begin{cases}
\psi^t\sigma_1\psi = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \psi_1^2 - \psi_2^2,\\
\psi^t\sigma_2\psi = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = i(\psi_1^2 + \psi_2^2),\\
\psi^t\sigma_3\psi = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = -2\psi_1\psi_2.
\end{cases} \tag{23}$$

Moreover, for the spinor expression $\widehat{\psi}^t\sigma\widehat{\psi}$, we get:

$$
\begin{cases}
\widehat{\psi}^t\sigma_1\widehat{\psi} = \begin{pmatrix}-\overline{\psi}_2 & \overline{\psi}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\overline{\psi}_2\\ \overline{\psi}_1\end{pmatrix} = \overline{\psi}_2^2 - \overline{\psi}_1^2,\\
\widehat{\psi}^t\sigma_2\widehat{\psi} = \begin{pmatrix}-\overline{\psi}_2 & \overline{\psi}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}-\overline{\psi}_2\\ \overline{\psi}_1\end{pmatrix} = i\left(\overline{\psi}_2^2 + \overline{\psi}_1^2\right),\\
\widehat{\psi}^t\sigma_3\widehat{\psi} = \begin{pmatrix}-\overline{\psi}_2 & \overline{\psi}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-\overline{\psi}_2\\ \overline{\psi}_1\end{pmatrix} = 2\overline{\psi}_1\overline{\psi}_2.
\end{cases} \tag{24}
$$

Furthermore, for the spinor expression $\widehat{\psi}^t\sigma\psi$, we obtain:

$$
\begin{cases}
\widehat{\psi}^t\sigma_1\psi = \begin{pmatrix}-\overline{\psi}_2 & \overline{\psi}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = -\overline{\psi}_2\psi_1 - \overline{\psi}_1\psi_2,\\
\widehat{\psi}^t\sigma_2\psi = \begin{pmatrix}-\overline{\psi}_2 & \overline{\psi}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = i\left(-\overline{\psi}_2\psi_1 + \overline{\psi}_1\psi_2\right),\\
\widehat{\psi}^t\sigma_3\psi = \begin{pmatrix}-\overline{\psi}_2 & \overline{\psi}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = -|\overline{\psi}_1|^2 + |\overline{\psi}_2|^2.
\end{cases} \tag{25}
$$

Substituting the equations (23), (24) and (25) in the equations (20), (21) and (22), we get desired results. □

Let us give the spinor representations of the adapted frame related to the framed curves in the three-dimensional Lie groups:

Definition 3.8. *Let $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be framed curve of adapted frame in the three-dimensional Lie groups and the spinor Υ corresponds to the triad $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$. In that case, the spinor representations of the adapted frame along the framed curve in the three-dimensional Lie groups are written as follows:*

$$\overline{\eta}_1 + i\,\overline{\eta}_2 = \Upsilon^t\sigma\Upsilon, \tag{26}$$

$$\overline{\nu} = -\widehat{\Upsilon}^t\sigma\Upsilon, \tag{27}$$

where $\overline{\Upsilon}^t\Upsilon = 1$.

Theorem 3.9. *Let $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be framed curve of adapted frame and the spinor ψ corresponds to the triad $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$. The followings are satisfied:*

(1) The single spinor equation is given as follows according to the first special case of the adapted frame of the framed curves in the three-dimensional Lie groups (see the adapted frame in the equation (11)*):*

$$\frac{d\Upsilon}{ds} = -\frac{1}{2}\left[i\left(q-\delta_G\right)\Upsilon + p\widehat{\Upsilon}\right].$$

(2) The single spinor equation is given as follows according to the second special case of the adapted frame of the framed curves in the three-dimensional Lie groups (see the adapted frame in the equation (12) *and* (13)*):*

$$\frac{d\Upsilon}{ds} = -\frac{1}{2}\left(\widetilde{m} + i\,\widetilde{n}\right)\widehat{\Upsilon}.$$

Proof. Using the similar method used in the Theorem 3.2 (for the first case: by using the equations (11), (26) and (27), and for the second case: by using the equations (12), (26) and (27)), we can complete the proof of this Theorem 3.9. For the sake of brevity, we skip this proof. □

With the help of the Remark 2.4, we have the following Corollary 3.10-3.12.

Corollary 3.10. *Let $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be framed curve of adapted frame in abelian Lie group and the spinor Υ corresponds to the triad $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$. In that case, the following single spinor equation is written as follows:*

$$\frac{d\Upsilon}{ds} = -\frac{1}{2}\left[iq\Upsilon + p\widehat{\Upsilon}\right].$$

Corollary 3.11. *Let $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be framed curve of adapted frame in the Lie group SO^3 and the spinor Υ corresponds to the triad $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$. In that case, the following single spinor equation is written as follows:*

$$\frac{d\Upsilon}{ds} = -\frac{1}{2}\left[i\left(q-\frac{1}{2}\right)\Upsilon + p\widehat{\Upsilon}\right].$$

Corollary 3.12. *Let $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be framed curve of adapted frame in the Lie group S^3 and the spinor Υ corresponds to the triad $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$. In that case, the following single spinor equation is written as follows:*

$$\frac{d\Upsilon}{ds} = -\frac{1}{2}\left[i\left(q-1\right)\Upsilon + p\widehat{\Upsilon}\right].$$

Now, let us examine the spinor relations between the general frame and adapted frame with respect to the framed curves in the three-dimensional Lie groups:

Theorem 3.13. *Let $(\gamma,\eta_1,\eta_2)\,,\ (\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. Then, the following relations between these spinors hold:*

$$\Upsilon^t\sigma\Upsilon = e^{i\theta}\left(\psi^t\sigma\psi\right), \tag{28}$$

$$\nu = \nu, \tag{29}$$

where $\nu=\overline{\nu}$.

Proof. By using the equation (10), the followings can be written:

$$\begin{aligned}
\overline{\eta}_1 + i\,\overline{\eta}_2 &= \cos\theta\eta_1 - \sin\theta\eta_2 + i\left(\sin\theta\eta_1+\cos\theta\eta_2\right)\\
&= \left(\cos\theta + i\sin\theta\right)\eta_1 + i\left(-\sin\theta + i\cos\theta\right)\eta_2\\
&= \left(\cos\theta + i\sin\theta\right)\eta_1 + i\left(\cos\theta + i\sin\theta\right)\eta_2\\
&= e^{i\theta}\left(\eta_1 + i\eta_2\right). \qquad (30)
\end{aligned}$$

According to the equation (30), we have:

$$\overline{\eta}_1 + i\,\overline{\eta}_2 = e^{i\theta}\left(\eta_1+i\eta_2\right)$$

where θ is the Euclidean angle between the vectors $\overline{\eta}_1$ and η_1. Then, by using the equations (14) and (26), we get:

$$\Upsilon^t\sigma\Upsilon = e^{i\theta}\left(\psi^t\sigma\psi\right).$$

□

Corollary 3.14. *Let $(\gamma,\eta_1,\eta_2)\,,\ (\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. The following relation between the spinors Υ and ψ holds:*

$$\Upsilon = \pm e^{\frac{i\theta}{2}}\psi. \tag{31}$$

Proof. We know that the spinors ψ and $-\psi$ correspond to the same ordered orthonormal basis $\{\eta_1,\eta_2,\nu\}$, and the spinors Υ and $-\Upsilon$ correspond to the same ordered orthonormal basis $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$. With the help of the equation (4), (28) and (30), we get the desired result. □

Corollary 3.15. *Let $(\gamma, \eta_1, \eta_2)\,,\ (\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1, \eta_2, \nu\}$ and $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$, respectively. In that case, the angle between Υ and ψ is $\frac{\theta}{2}$.*

Theorem 3.16. *Let $(\gamma, \eta_1, \eta_2)\,,\ (\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1, \eta_2, \nu\}$ and $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$, respectively. The relation between these spinors is satisfied:*

$$\widehat{\Upsilon} = \pm e^{-i\frac{\theta}{2}}\widehat{\psi}. \tag{32}$$

Proof. Let us take the conjugate of both sides of the equation (31), then we have the equation $\widehat{\Upsilon} = \widehat{\pm e^{i\frac{\theta}{2}}\psi}$. Via using the equation (7), we get the desired equation $\widehat{\Upsilon} = \pm e^{\frac{\theta}{2}}\widehat{\psi}$. □

Corollary 3.17. *Let $(\gamma, \eta_1, \eta_2)\,,\ (\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1, \eta_2, \nu\}$ and $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$, respectively. While the spinor Υ makes a rotation with the angle $\theta/2$ to the spinor ψ, the spinor $\widehat{\Upsilon}$ makes an opposite rotation with the same angle value to the spinor $\widehat{\psi}$.*

Theorem 3.18. *Let $(\gamma, \eta_1, \eta_2)\,,\ (\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1, \eta_2, \nu\}$ and $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$, respectively. The derivative of the spinor ψ can be expressed as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\left(q + \theta' - \delta_G\right)\psi + e^{-i\theta}p\widehat{\psi}\right]. \tag{33}$$

Proof. By using the equations (11) and (16), we get:

$$\begin{aligned}\frac{d\psi}{ds} &= -\frac{1}{2}\left[i\,(l-\delta_G)\,\psi + (m+i\,n)\,\widehat{\psi}\right]\\ &= -\frac{1}{2}\left[i\,(q+\theta'-\delta_G)\,\psi + (-p\cos\theta + i\,p\sin\theta)\,\widehat{\psi}\right]\\ &= -\frac{1}{2}\left[i\,(q+\theta'-\delta_G)\,\psi - e^{-i\theta}p\widehat{\psi}\right].\end{aligned}$$

□

According to the Theorem 3.18, we get the following Corollary 3.19-3.21.

Corollary 3.19. *Let* $(\gamma,\eta_1,\eta_2)\,,\ (\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ *be the framed curves of the general frame and adapted frame in abelian Lie groups, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1,\eta_2,\nu\}$ *and* $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$*, respectively. The derivative of the spinor* ψ *can be given as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\,(q+\theta')\,\psi + e^{-i\theta}p\widehat{\psi}\right].$$

Corollary 3.20. *Let* $(\gamma,\eta_1,\eta_2)\,,\ (\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ *be the framed curves of the general frame and adapted frame in the Lie groups* SO^3*, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1,\eta_2,\nu\}$ *and* $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$*, respectively. The derivative of the spinor* ψ *can be written as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\left(q+\theta'-\frac{1}{2}\right)\psi + e^{-i\theta}p\widehat{\psi}\right].$$

Corollary 3.21. *Let* $(\gamma,\eta_1,\eta_2)\,,\ (\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ *be the framed curves of the general frame and adapted frame in the Lie groups* S^3*, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1,\eta_2,\nu\}$ *and* $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$*, respectively. The derivative of the spinor* ψ *can be obtained as follows:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\,(q+\theta'-1)\,\psi + e^{-i\theta}p\widehat{\psi}\right].$$

Corollary 3.22. *Let* $(\gamma,\eta_1,\eta_2)\,,\ (\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ *be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1,\eta_2,\nu\}$ *and* $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$*, respectively. If* *the angle between the* $\overline{\eta}_1$ *and* η_1 *is* θ*, then the angle between* $\dfrac{d\psi}{ds}$ *and* $\widehat{\psi}$ *is* $-\theta$ *on condition that* $\theta' = \delta_G - q$.

Corollary 3.23. *Let* (γ, η_1, η_2), $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ *be the framed curves of the general frame and adapted frame in abelian Lie groups, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1, \eta_2, \nu\}$ *and* $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$*, respectively. If the angle between the* $\overline{\eta}_1$ *and* η_1 *is* θ*, then the angle between* $\dfrac{d\psi}{ds}$ *and* $\widehat{\psi}$ *is* $-\theta$ *on condition that* $\theta' = -q$.

Corollary 3.24. *Let* (γ, η_1, η_2), $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ *be the framed curves of the general frame and adapted frame in Lie group* SO^3*, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1, \eta_2, \nu\}$ *and* $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$*, respectively. If the angle between the* $\overline{\eta}_1$ *and* η_1 *is* θ*, then the angle between* $\dfrac{d\psi}{ds}$ *and* $\widehat{\psi}$ *is* $-\theta$ *on condition that* $\theta' = \dfrac{1}{2} - q$.

Corollary 3.25. *Let* (γ, η_1, η_2), $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ *be the framed curves of the general frame and adapted frame in Lie group* S^3*, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1, \eta_2, \nu\}$ *and* $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$*, respectively. If the angle between the* $\overline{\eta}_1$ *and* η_1 *is* θ*, then the angle between* $\dfrac{d\psi}{ds}$ *and* $\widehat{\psi}$ *is* $-\theta$ *on condition that* $\theta' = 1 - q$.

Corollary 3.26. *Let* (γ, η_1, η_2), $(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$ *be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors* ψ *and* Υ *correspond to the triads* $\{\eta_1, \eta_2, \nu\}$ *and* $\{\overline{\eta}_1, \overline{\eta}_2, \overline{\nu}\}$*, respectively. Then, the following equation is satisfied:*

$$\frac{d\psi}{ds} = \pm\frac{1}{2}e^{-i\frac{\theta}{2}}\left[\left(q + \theta' - \delta_G\right)\Upsilon - p\widehat{\Upsilon}\right].$$

Proof. With the help of the equations (31), (32) and (33), we get:

$$\begin{aligned}\frac{d\psi}{ds} &= -\frac{1}{2}\left[i\left(q+\theta'-\delta_G\right)\psi - e^{-i\theta}p\widehat{\psi}\right]\\ &= -\frac{1}{2}\left[i\left(q+\theta'-\delta_G\right)\left(\pm e^{-i\frac{\theta}{2}}\Upsilon\right) - e^{-i\theta}p\left(\pm e^{i\frac{\theta}{2}}\widehat{\Upsilon}\right)\right]\\ &= \pm\frac{1}{2}e^{-i\frac{\theta}{2}}\left[\left(q+\theta'-\delta_G\right)\Upsilon - p\widehat{\Upsilon}\right].\end{aligned}$$

□

Corollary 3.27. *Let (γ,η_1,η_2), $(\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. The angle between the spinors $\frac{d\psi}{ds}$ and $\widehat{\Upsilon}$ is $-\frac{\theta}{2}$ provided that $\theta' = \delta_G - q$.*

Corollary 3.28. *Let (γ,η_1,η_2), $(\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of the general frame and adapted frame in abelian Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. The angle between the spinors $\frac{d\psi}{ds}$ and $\widehat{\Upsilon}$ is $-\frac{\theta}{2}$ provided that $\theta' = -q$.*

Corollary 3.29. *Let (γ,η_1,η_2), $(\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of the general frame and adapted frame in Lie group SO^3, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. The angle between the spinors $\frac{d\psi}{ds}$ and $\widehat{\Upsilon}$ is $-\frac{\theta}{2}$ provided that $\theta' = \frac{1}{2} - q$.*

Corollary 3.30. *Let (γ,η_1,η_2), $(\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of the general frame and adapted frame in Lie group S^3, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. The angle between the spinors $\frac{d\psi}{ds}$ and $\widehat{\Upsilon}$ is $-\frac{\theta}{2}$ provided that $\theta' = 1 - q$.*

Theorem 3.31. *Let (γ,η_1,η_2), $(\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. The derivative of the spinor ψ can be expressed as:*

$$\frac{d\psi}{ds} = -\frac{1}{2}\left[i\theta'\psi + e^{-i\theta}\left(\widetilde{m} + i\,\widetilde{n}\right)\widehat{\psi}\right]. \tag{34}$$

Proof. By using the equations (13) and (16), we get:

$$\begin{aligned}\frac{d\psi}{ds} &= -\frac{1}{2}\left[i\left(l-\delta_G\right)\psi + \left(m+i\,n\right)\widehat{\psi}\right]\\ &= -\frac{1}{2}\left[i\theta'\psi + e^{-i\theta}\left(\widetilde{m}+i\,\widetilde{n}\right)\widehat{\psi}\right].\end{aligned}$$

□

Corollary 3.32. *Let (γ,η_1,η_2), $(\gamma,\overline{\eta}_1,\overline{\eta}_2) : I \to G\times\Delta_G$ be the framed curves of the general frame and adapted frame in the three-dimensional Lie groups, and the spinors ψ and Υ correspond to the triads $\{\eta_1,\eta_2,\nu\}$ and $\{\overline{\eta}_1,\overline{\eta}_2,\overline{\nu}\}$, respectively. The following equation is satisfied:*

$$\frac{d\psi}{ds} = \pm\frac{1}{2}e^{-i\frac{\theta}{2}}\left[i\theta'\Upsilon + \left(\widetilde{m}+i\,\widetilde{n}\right)\widehat{\Upsilon}\right].$$

Proof. With the help of the equations (31), (32) and (34), we have:

$$\begin{aligned}\frac{d\psi}{ds} &= -\frac{1}{2}\left[i\theta'\psi + e^{-i\theta}\left(\widetilde{m}+i\,\widetilde{n}\right)\widehat{\psi}\right]\\ &= -\frac{1}{2}\left[i\theta'\left(\pm e^{-i\frac{\theta}{2}}\Upsilon\right) + e^{-i\theta}\left(\widetilde{m}+i\,\widetilde{n}\right)\left(\pm e^{i\frac{\theta}{2}}\widehat{\Upsilon}\right)\right]\\ &= \pm\frac{1}{2}e^{-i\frac{\theta}{2}}\left[i\theta'\Upsilon + \left(\widetilde{m}+i\,\widetilde{n}\right)\widehat{\Upsilon}\right].\end{aligned}$$

□

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