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SOME RESULTS ON THE OPEN SUBSET INCLUSION GRAPH OF A PRODUCT TOPOLOGICAL SPACE

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Abstract. In the recent paper, R. A. Muneshwar et al., introduced a graph structure, called open subset inclusion graph $\jmath(\tau)$ of (X, τ) . They also discussed a basic properties such as connectedness, diameter, girth, clique number, chromatic number, domination number, etc., of an open subset inclusion graph of a topological space. In this paper, we study some important results of the graph $\jmath(\tau_\chi \times \tau_\gamma)$ of a product topological space $(X \times Y, \tau)$. We also determine relationship between diameter, girth, clique number, chromatic number, domination number, degree etc., of the graphs $\jmath(\tau_\chi)$, $\jmath(\tau_\gamma)$ and $\jmath(\tau_\chi \times \tau_\gamma)$.

AMS Classification: 05C69, 05C25, 05C12, 05C07.

Keywords: Discrete topology, Graph, Diameter, Clique number, Girth, Chromatic number, Degree, Domination number.

1. Introduction

The first instance of work of graph is due to Beck[2], who introduced the idea of zero divisor graph of a commutative ring with unity. In the recent decades, graph of the several algebraic structures were defined which can be found in [1, 3].

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Das A., introduced vector space graphs in [4, 5, 6], and he covered some of their characteristics. He also studied different properties of these graphs.

Some results on the graphs of a vector space were studied independently by Jafari N. R. et al.[7] and Talebi Y. et al.[11]. "Some properties on incomparability graphs of lattices were discussed by Wasadikar, M. and Survase P.[12, 13]. They prove that the complete graph K_n with two horns is realizable as incomparability graph of a lattice L and complete graph K_3 with three horns is not realizable as incomparability graph of a lattice L. Moreover, they also give a necessary and sufficient condition for a complete bipartite graph with one horn and with two horns to be realizable as incomparability graph of some lattice L. Muneshwar R. A. et al. [8, 9, 10], introduced new concept of graphs of topological space (χ, τ). In [8], authors introduced an open subset inclusion graph of a (χ, τ), which is defined as follows.

Definition 1.1. [8] ***Open Subset Inclusion Graph of a Topological Space:*** *Let τ be a topology defined on finite set χ then a graph $\jmath(\tau) = (\Theta, E)$ is called as open subset inclusion graph of a (χ, τ), where $\Theta = \{\mu \in \tau \mid \mu \neq \phi, \mu \neq \chi\}$ and $E = \{(\mu_1, \mu_2) \mid \mu_1 \subset \mu_2$ or $\mu_2 \subset \mu_1, \forall \mu_1, \mu_2 \in \Theta\}$*

1.1. **Note.** If two vertices μ_1, μ_2 are adjacent in $\jmath(\tau)$ then it is denoted by $\mu_1 \sim \mu_2$ or (μ_1, μ_2).

For undefined concepts and terms the reader can referred to [14].
Open subset inclusion graph of (χ, τ) with $|\chi| = 3, 4, 5$ are shown in figure 1, figure 2 and figure 3.

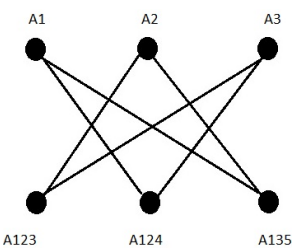


FIGURE 1. The Graph $\jmath(\tau)$ of a Discrete Topological Space (X, τ) with $|X| = 3$

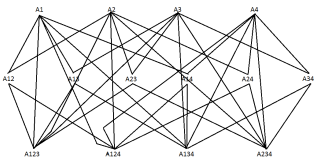


FIGURE 2. The Graph $\jmath(\tau)$ of a Discrete Topological Space (X, τ) with $|X| = 4$

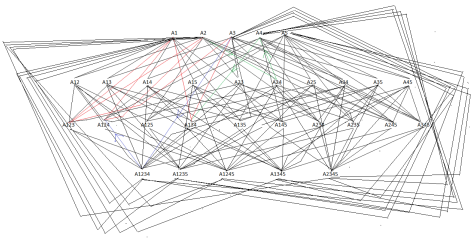


FIGURE 3. The Graph $\jmath(\tau)$ of a Discrete Topological Space (X, τ) with $|X| = 5$

2. Dimeter & Girth of Inclusion Graph of Product Topology on $\chi \times \gamma$

We examine the dimeter and girthof the inclusion graph of the product topology on $\chi \times \gamma$, in this section. In this section, we make the assumptions that, $\tau_\chi \times \tau_\gamma$ is a product topology on $\chi \times \gamma$.

Theorem 2.1. *If (χ, τ_χ) and (γ, τ_γ) are discrete topological spaces with $|\chi| = m$ and $|\gamma| = n$ then $\jmath(\tau_\chi \times \tau_\gamma)$ is connected & $diam(\jmath(\tau_\chi \times \tau_\gamma) = 4$.*

Proof. Let $|\chi| = m$ & $|\gamma| = n$, where $m > n$.

Claim: $\jmath(\tau_\chi \times \tau_\gamma)$ is connected graph.

For this, let $\mu_1 \times \vartheta_1$, $\mu_2 \times \vartheta_2 \in \jmath(\tau_\chi \times \tau_\gamma$ are open subsets of $\chi \times \gamma$. The following three situations might occur to demonstrate this theorem:

Case 1: If $(\mu_1 \times \vartheta_1) \cap (\mu_2 \times \vartheta_2) \neq \phi$, then $(\mu_1 \times \vartheta_1) \cap (\mu_2 \times \vartheta_2) \subset (\mu_1 \times \vartheta_1)$ and $(\mu_1 \times \vartheta_1) \cap (\mu_2 \times \vartheta_2) \subset (\mu_2 \times \vartheta_2)$. Therefore $(\mu_1 \times \vartheta_1) \sim (\mu_1 \times \vartheta_1) \cap (\mu_2 \times \vartheta_2) \sim (\mu_2 \times \vartheta_2)$ be a necessary pathway from $(\mu_1 \times \vartheta_1)$ to $(\mu_2 \times \vartheta_2)$ & therefore $dist((\mu_1 \times \vartheta_1), (\mu_2 \times \vartheta_2)) = 3$.

Case 2: If $(\mu_1 \times \vartheta_1) \cup (\mu_2 \times \vartheta_2) \neq \chi \times \gamma$, then $(\mu_1 \times \vartheta_1) \subset (\mu_1 \times \vartheta_1) \cup (\mu_2 \times \vartheta_2)$

and $(\mu_2 \times \vartheta_2) \subset (\mu_1 \times \vartheta_1) \cup (\mu_2 \times \vartheta_2)$. Therefore $(\mu_1 \times \vartheta_1) \sim (\mu_1 \times \vartheta_1) \cup (\mu_2 \times \vartheta_2) \sim (\mu_2 \times \vartheta_2)$ be a necessary pathway from $(\mu_1 \times \vartheta_1)$ to $(\mu_2 \times \vartheta_2)$ & therefore $dist((\mu_1 \times \vartheta_1), (\mu_2 \times \vartheta_2)) = 3$.

Case 3: If $(\mu_1 \times \vartheta_1) \cap (\mu_2 \times \vartheta_2) = \phi$ and $(\mu_1 \times \vartheta_1) \cup (\mu_2 \times \vartheta_2) = \chi \times \gamma$. As $(\mu_1 \times \vartheta_1) \cap (\mu_2 \times \vartheta_2) = (\mu_1 \cap \mu_2) \times (\vartheta_1 \cap \vartheta_2) = \phi$ then $(\mu_1 \cap \mu_2) = \phi$ or $(\vartheta_1 \cap \vartheta_2) = \phi$. Also $(\mu_1 \times \vartheta_1) \cup (\mu_2 \times \vartheta_2) = (\mu_1 \cup \mu_2) \times (\vartheta_1 \cup \vartheta_2) = \chi \times \gamma$ then $(\mu_1 \cup \mu_2) = \chi$ and $(\vartheta_1 \cup \vartheta_2) = \gamma$. Now we will choose $(a_1, b_1) \in (\mu_1 \times \vartheta_1)$, $(a_2, b_2) \in (\mu_2 \times \vartheta_2)$ and denote $\mu_3 \times \vartheta_3 = \{(a_1, b_1)\}, \mu_4 \times \vartheta_4 = \{(a_2, b_2)\}$. Then clearly $\mu_3 \times \vartheta_3 \subset (\mu_1 \times \vartheta_1), (\mu_4 \times \vartheta_4) \subset (\mu_2 \times \vartheta_2)$ and $(\mu_3 \times \vartheta_3) \subset (\mu_3 \times \vartheta_3) \cup (\mu_4 \times \vartheta_4) \supset (\mu_4 \times \vartheta_4)$. Thus $(\mu_1 \times \vartheta_1) \sim (\mu_3 \times \vartheta_3) \sim (\mu_3 \times \vartheta_3) \cup (\mu_4 \times \vartheta_4) \sim (\mu_4 \times \vartheta_4) \sim (\mu_2 \times \vartheta_2)$ & therefore $dist(\mu_1 \times \vartheta_1, \mu_2 \times \vartheta_2) = 4$.

Thus the graph $\jmath(\tau_\chi \times \tau_\gamma)$ is connected & $diam(\tau_\chi \times \tau_\gamma) = 4$. □

Theorem 2.2. *If τ_χ & τ_γ are discrete topologies on χ & γ resp. with $|\chi| = m, |\gamma| = n$ resp. then following are hold.*

1. *If $m = 2, n = 1$ then $girth(\jmath(\tau))) = \infty$.*
2. *If $m = 2, n = 3$ then $girth(\jmath(\tau))) = 3$.*
3. *If $m = n = 3$ then $girth(\jmath(\tau_\chi \times \tau_\gamma)) = 3$.*
4. *If $m = 2, n = 2$ then $girth(\jmath(\tau_\chi \times \tau_\gamma)) = 8$.*
5. *If $m \geq 4$ or $m \geq 4$ then $girth(\jmath(\tau_\chi \times \tau_\gamma)) = 3$.*

Proof. Let $\chi = \{a_1, a_2, ..., a_m\}$ and $\gamma = \{b, b_2, ..., b_n\}$, where $m > n$. Let $\mu_i = \{a_1, a_2, ..., a_i\} \in \tau_\chi$, for $i = 1, 2, ...m - 1$ & $\vartheta_j = \{b_1, b_2, ..., b_j\} \in \tau_\gamma$, for $j = 1, 2, ...n - 1$.

The following three cases might occur to demonstrate this theorem:

Case I: If $m = 2, n = 1$ then by Corollary 3.3[8], graph $\jmath(\tau)$ is a null graph and hence $girth(\jmath(\tau))) = \infty$.

Case II: As $m = n = 3$, & let $a_1, a_2, a_3 \in \chi$. Let $\mu_1 = \{a_1\}, \mu_2 = \{a_1, a_2\}$ are two open subset of χ. Let b_1, b_2, b_3 are three distinct elements in γ. Let $\vartheta_1 = \{b_1\}, \vartheta_2 = \{b_1, b_2\}$ are two open subset of γ. Then we have $\mu_1 \times \vartheta_1 \subset \mu_1 \times \vartheta_2 \subset \mu_2 \times \vartheta_2 \supset \mu_1 \times \vartheta_1$ and therefore $\mu_1 \times \vartheta_1 \sim \mu_1 \times \vartheta_2 \sim \mu_2 \times \vartheta_2 \times \mu_1 \times \vartheta_1$ is a triangle. Hence in this case, $girth(\jmath(\tau))) = 3$.

Case III: If $m = 2, n = 3$, let $\chi = \{a_1, a_2\}$ and let $\mu_1 = \{a_1\}, \mu_2 = \{a_2\}$ are two open subsets of χ. Let $\gamma = \{b_1, b_2, b_3\}$ and let $\vartheta_1 = \{b_1\}, \vartheta_2 = \{b_1, b_2\}$ are two open subsets of γ. Then $\mu_1 \times \vartheta_1 \subset \mu_1 \times \vartheta_2 \subset \chi \times \vartheta_2 \supset \mu_1 \times \vartheta_1$ and hence

$\mu_1 \times \vartheta_1 \sim \mu_1 \times \vartheta_2 \sim \chi \times \vartheta_2 \sim \mu_1 \times \vartheta_1$ is a triangle and hence in this case $girth(\jmath(\tau))) = 3$.

Case IV: If $m \geq 4$ or $n \geq 4$. For our convenience suppose $m \geq 4$.

Consider the three independent subcases listed below to demonstrate this theorem in the case.

Sub case i: If $|\mu_1| = 1$ and since $|\chi| \geq 4$, $\exists\ \mu_2\ ,\ \mu_3 \in \tau_\chi$ with $|\mu_2| = 2, |\mu_3| = 3$ and $\mu_1 \subset \mu_2 \subset \mu_3$. Then there exist open subsets $\mu_1 \times \gamma, \mu_2 \times \gamma, \mu_3 \times \gamma$ in $\chi \times \gamma$ such that $\mu_1 \times \gamma \subset \mu_2 \times \gamma \subset \mu_3 \times \gamma$. Hence $\mu_1 \times \gamma \sim \mu_2 \times \gamma \sim \mu_3 \times \gamma \sim \mu_1 \times \gamma$, which is triangle.

Sub case ii: If $|\mu_1| = 2$ and since $|\chi| \geq 4$, $\exists\ \mu_2\ ,\ \mu_3 \in \tau_\chi$ with $|\mu_2| = 1, |\mu_3| = 3$ and $\mu_2 \subset \mu_1 \subset \mu_3$. Then there exist open subsets $\mu_1 \times \gamma, \mu_2 \times \gamma, \mu_3 \times \gamma$ in $\chi \times \gamma$ such that $\mu_2 \times \gamma \subset \mu_1 \times \gamma \subset \mu_3 \times \gamma$. Hence $\mu_2 \times \gamma \sim \mu_1 \times \gamma \sim \mu_3 \times \gamma \sim \mu_1 \times \gamma$, which is triangle.

Sub case iii: As $|\mu_1| = k \geq 3$ and since $|\chi| \geq 4$, $\exists\ \mu_2\ , \mu_3 \in \tau_\chi$ with $|\mu_2| = k-1, |\mu_3| = k+1$ such that $\mu_2 \subset \mu_1 \subset \mu_3$. Then there exist open subsets $\mu_2 \times \gamma, \mu_1 \times \gamma, \mu_3 \times \gamma$ in $\chi \times \gamma$ such that $\mu_2 \times \gamma \subset \mu_1 \times \gamma \subset \mu_3 \times \gamma \supset \mu_1 \times \gamma$. Hence $\mu_1 \times \gamma \sim \mu_2 \times \gamma \sim \mu_3 \times \gamma \sim \mu_1 \times \gamma$, which is triangle.

Case V: If $m = 2, n = 2$, let $\chi = \{a_1, a_2\}$ let $\mu_1 = \{a_1\}, \mu_2 = \{a_2\}$ are two open subsets of χ. Let $\gamma = \{b_1, b_2\}$ let $\vartheta_1 = \{b_1\}, \vartheta_2 = \{b_2\}$ are two open subsets of γ. Then $\mu_1 \times \vartheta_1 \subset \mu_1 \times \gamma \supset \mu_1 \times \vartheta_2 \subset \chi \times \vartheta_2 \supset \mu_2 \times \vartheta_2 \subset \mu_2 \times \gamma \supset \mu_2 \times \vartheta_1 \supset \chi \times \vartheta_1 \supset \mu_1 \times \vartheta_1$ and hence $\mu_1 \times \vartheta_1 \sim \mu_1 \times \gamma \sim \mu_1 \times \vartheta_2 \sim \chi \times \vartheta_2 \sim \mu_2 \times \vartheta_2 \sim \mu_2 \times \gamma \sim \mu_2 \times \vartheta_1 \sim \chi \times \vartheta_1 \sim \mu_1 \times \vartheta_1$ is a octagone and hence in this case girth of the graph $\jmath(\tau_\chi \times \tau_\gamma)$ is 8. □

Theorem 2.3. *Let τ_χ & τ_γ are discrete topologies on χ & γ resp. with $|\chi| = m, |\gamma| = n$. If $m \geq 4$, or $n \geq 4$ then $\jmath(\tau_\chi \times \tau_\gamma)$ is a triangulated graph.*

Proof. Let $\chi = \{a_1, a_2, ..., a_m\}$ and $\gamma = \{b, b_2, ..., b_n\}$, where $m \geq 2, m \geq 3$. Let $\mu_i = \{a_1, a_2, ..., a_i\}$, for $i = 1, 2, ...m-1$ is open subsets of χ, and $\vartheta_j = \{b_1, b_2, ..., b_j\}$, for $j = 1, 2, ...n-1$ is open subsets of χ. Let $\mu \times \vartheta \in \tau_\chi \times \tau_\gamma$ be subset of $\chi \times \gamma$. The following three distinct scenarios will be used to demonstrate the theorem.

Case I: If $|\mu_1| = 1$ and since $|\chi| \geq 4$, $\exists\ \mu_2$, $\mu_3 \in \tau_\chi$ with $|\mu_2| = 2, |\mu_3| = 3$ and $\mu_1 \subset \mu_2 \subset \mu_3$. Then there exist open subsets $\mu_1 \times \gamma, \mu_2 \times \gamma, \mu_3 \times \gamma$ in $\chi \times \gamma$ such that $\mu_1 \times \gamma \subset \mu_2 \times \gamma \subset \mu_3 \times \gamma \subset \mu_1 \times \gamma$. Hence $\mu_1 \times \gamma \sim \mu_2 \times \gamma \sim \mu_3 \times \gamma \sim \mu_1 \times \gamma$,

which is triangle.

Case II: If $|\mu_1| = 2$ and since $|\chi| \geq 4$, $\exists\ \mu_2,\ \mu_3 \in \tau_\chi$ with $|\mu_2| = 1, |\mu_3| = 3$ and $\mu_2 \subset \mu_1 \subset \mu_3$. Then there exist open subsets $\mu_1 \times \gamma, \mu_2 \times \gamma, \mu_3 \times \gamma$ in $\chi \times \gamma$ such that $\mu_2 \times \gamma \subset \mu_1 \times \gamma \subset \mu_3 \times \gamma \subset \mu_2 \times \gamma$. Hence $\mu_2 \times \gamma \sim \mu_1 \times \gamma \sim \mu_3 \times \gamma \sim \mu_1 \times \gamma$, which is triangle.

Case 3: If $|\mu_1| = k \geq 3$ and since $|\chi| \geq 4$, there exist two open subsets μ_2 and μ_3 of χ with $|\mu_2| = k-1, |\mu_3| = k+1$ such that $\mu_2 \subset \mu_1 \subset \mu_3$. Therefore, there exist open subsets $\mu_1 \times \gamma, \mu_2 \times \gamma, \mu_3 \times \gamma$ in $\chi \times \gamma$ such that $\mu_2 \times \gamma \subset \mu_1 \times \gamma \subset \mu_3 \times \gamma \supset \mu_2 \times \gamma$. Hence $\mu_1 \times \gamma \sim \mu_2 \times \gamma \sim \mu_3 \times \gamma \sim \mu_1 \times \gamma$, is a triangle.

Thus, the graph $\jmath(\tau_\chi \times \tau_\gamma)$ is a triangulated. □

3. Clique & Chromatic Number of Inclusion Graph of Product Topology on $\chi \times \gamma$

In this section, we investigate the clique & chromatic number of the graph $\jmath(\tau_\chi \times \tau_\gamma)$.

Theorem 3.1. *If τ_χ is the discrete topology on χ with $|\chi| = n$ and Z be a clique in $\jmath(\tau_\chi \times \tau_\chi)$, then Z is a ascending or descending chain of non-trivial proper open subsets of $(\chi \times \chi, \tau)$, where τ is product topology on $\chi \times \chi$.*

Proof. As Z is a clique in $(\chi \times \chi, \tau)$, for any two $\mu_1 \times \vartheta_1, \mu_2 \times \vartheta_2 \in Z$ then either $\mu_1 \times \vartheta_1 \subset \mu_2 \times \vartheta_2$ or $\mu_2 \times \vartheta_2 \subset \mu_1 \times \vartheta_1$. The theorem follows from the fact that any two elements in Z may be compared. □

Theorem 3.2. *As (χ, τ_χ) and (γ, τ_γ) are discrete spaces with $|\chi| = m$, and $|\gamma| = n$ then $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m + n - 2) = \omega(\jmath(\tau_\chi)) + \omega(\jmath(\tau_\gamma))$, $\forall\ m, n$.*

Proof. Let $\chi = \{a_1, a_2, ..., a_m\}$ and $\gamma = \{b_1, b_2, ..., b_n\}$. Let $\mu_i = \{a_1, a_2, ..., a_i\}$, for $i = 1, 2, ...m-1$, is open subsets of χ and, $\vartheta_j = \{b_1, b_2, ..., b_j\}$, for $j = 1, 2, ...n-1$, is open subsets of γ.

Case I: If $m = n$, then suppose $Z_1 = \{\mu_1 \times \vartheta_1 \subset \mu_2 \times \vartheta_1 \subset \mu_2 \times \vartheta_2 \subset \mu_3 \times \vartheta_2 \subset \mu_3 \times \vartheta_3 \subset \mu_4 \times \vartheta_3 \subset \mu_4 \times \vartheta_4 \subset ... \subset \mu_{n-2} \times \vartheta_{n-3} \subset \mu_{n-2} \times \vartheta_{n-2} \subset \mu_{n-1} \times \vartheta_{n-2} \subset \mu_{n-1} \times \vartheta_{n-1} \subset \chi \times \vartheta_{n-1}\}$ be a ascending chain of open subsets of $(\chi \times \gamma, \tau)$. Then clearly Z_1 is an another clique in $\jmath(\tau_\chi \times \tau_\chi)$.

If at all feasible, let $Z_2 = Z_1 \cup \{\mu \times \vartheta\}$ be a clique, where μ and ϑ are open subsets besides μ_i^s and ϑ_i^s resp. Thus by Theorem 3.1, there exists $k \in \{1, 2, ...n-2\}$ such

that $\mu_k \times \vartheta_i \subset \mu \times \vartheta \subset \mu_{k+1} \times \vartheta_i$ or $\mu_j \times \vartheta_k \subset \mu \times \vartheta \subset \mu_j \times \vartheta_{k+1}$. Then we have $\mu_k \subset \mu \subset \mu_{k+1}$ or $\vartheta_k \subset \vartheta \subset \vartheta_{k+1}$ As every inclusion is appropriate and χ and γ is of finite cardinality, we have $|\mu_k| < |\mu| < |\mu_{k+1}|$ that is, $k < |\mu| < k+1$, or $|\vartheta_k| < |\vartheta| < |\vartheta_{k+1}|$ that is, $k < |\vartheta| < k+1$, in both cases we get a contradiction. Therefore, Z_1 is a maximal clique of size $\{1 + 2(n-2) + 1 = 2n-2\}$. Hence, by Theorem 5.2[8], we have, the clique number of $\jmath(\tau_\chi \times \tau_\gamma)$ is $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = 2n-2 = 2(n-1) = 2\omega(\jmath(\tau_\chi) = 2\omega(\jmath(\tau_\gamma)$.

For the converse, let $\omega(\jmath(\tau_\chi \times \tau_\chi)) = 2n-2$. If possible, let $|\chi| = m$. Then by above arguments, $\omega(\jmath(\tau_\chi \times \tau_\chi)) = 2m-2$. Thus $m = n$ and hence the proof.

Case II: If $m < n$, then suppose $Z = \{\mu_1 \times \vartheta_1 \subset \mu_2 \times \vartheta_1 \subset \mu_2 \times \vartheta_2 \subset \mu_3 \times \vartheta_2 \subset \mu_3 \times \vartheta_3 \subset \mu_4 \times \vartheta_3 \subset \mu_4 \times \vartheta_4 \subset ... \subset \mu_{m-2} \times \vartheta_{m-2} \subset \mu_{m-1} \times \vartheta_{m-2} \subset \mu_{m-1} \times \vartheta_{m-1} \subset \chi \times \vartheta_{m-1} \subset \chi \times \vartheta_m \subset \chi \times \vartheta_{m+1} \subset \chi \times \vartheta_{m+2} \subset ... \subset \chi \times \vartheta_{n-2} \subset \chi \times \vartheta_{n-1}\}$ be a ascending chain of open subsets of $(\chi \times \gamma, \tau)$. Then clearly Z is a clique in $\jmath(\tau_\chi \times \tau_\gamma)$. If possible, let $Z_1 = Z \cup \{\mu \times \vartheta\}$ be a clique, where μ and V are open subsets besides μ_i^s and ϑ_i^s resp. Thus by Theorem 3.1, there exists $k \in \{1, 2, ...n-2\}$ such that $\mu_k \times \vartheta_i \subset \mu \times \vartheta \subset \mu_{k+1} \times \vartheta_i$ or $\mu_j \times \vartheta_k \subset \mu \times \vartheta \subset \mu_j \times \vartheta_{k+1}$. Then we have $\mu_k \subset \mu \subset \mu_{k+1}$ or $\vartheta_k \subset \vartheta \subset \vartheta_{k+1}$. As every inclusion is appropriate and χ and γ are of finite cardinality, then $|\mu_k| < |\mu| < |\mu_{k+1}|$ that is, $k < |\mu| < k+1$, or $|\vartheta_k| < |\vartheta| < |\vartheta_{k+1}|$ that is, $k < |\vartheta| < k+1$, in both cases we get a contradiction. Therefore Z_1 is not a clique and hence Z is a maximal clique of size $\{(1 + 2(m-2)) + (n-m+1)\} = (m+n-2)$. Hence, by Theorem 5.2[8], we have, the clique number of the graph $\jmath(\tau_\chi \times \tau_\chi)$ is $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m+n-2) = (m-1) + (n-1) = \omega(\jmath(\tau_\chi)) + \omega(\jmath(\tau_\gamma))$.

Case III: If $m > n$, then suppose $Z = \{\mu_1 \times \vartheta_1 \subset \mu_2 \times \vartheta_1 \subset \mu_2 \times \vartheta_2 \subset \mu_3 \times \vartheta_2 \subset \mu_3 \times \vartheta_3 \subset \mu_4 \times \vartheta_3 \subset \mu_4 \times \vartheta_4 \subset ... \subset \mu_{n-2} \times \vartheta_{n-2} \subset \mu_{n-1} \times \vartheta_{n-2} \subset \mu_{n-1} \times \vartheta_{n-1} \subset \mu_{n-1} \times \gamma \subset \mu_n \times \gamma \subset \mu_{n+1} \times \gamma \subset \mu_{n+2} \times \gamma \subset ... \subset \mu_{m-2} \times \gamma \subset \mu_{m-1} \times \gamma\}$ be a ascending chain of open subsets of $(\chi \times \gamma, \tau)$. Then clearly Z is a clique in $\jmath(\tau_\chi \times \tau_\gamma)$.

If possible, let $Z_1 = Z \cup \{\mu \times \vartheta\}$ be a clique, where μ and ϑ is a non-trivial proper open subsets other than μ_i^s and ϑ_i^s resp. Thus by Theorem 3.1, there exists $k \in \{1, 2, ...n-2\}$ such that $\mu_k \times \vartheta_i \subset \mu \times \vartheta \subset \mu_{k+1} \times \vartheta_i$ or $\mu_j \times \vartheta_k \subset \mu \times \vartheta \subset \mu_j \times \vartheta_{k+1}$. Then we have $\mu_k \subset \mu \subset \mu_{k+1}$ or $\vartheta_k \subset \vartheta \subset \vartheta_{k+1}$. As every inclusion is appropriate and χ, γ are of finite cardinality, then $|\mu_k| < |\mu| < |\mu_{k+1}|$ that is, $k < |\mu| < k+1$, or $|\vartheta_k| < |\vartheta| < |\vartheta_{k+1}|$ that is, $k < |\vartheta| < k+1$, in both cases we get

a contradiction. Therefore Z_1 is a clique and hence Z_1 is a maximal clique of size $(2(n-2)+1)+(m-n+1) = (m+n-2)$. Hence, by Theorem 5.2[8], we have, the clique number of the graph $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m+n-2) = \omega(\jmath(\tau_\chi)) + \omega(\jmath(\tau_\gamma))$. Thus, finally we conclude that, $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m+n-2) = \omega(\jmath(\tau_\chi)) + \omega(\jmath(\tau_\gamma))$, $\forall\ m, n$. □

Theorem 3.3. *If (χ, τ_χ) and (γ, τ_γ) are discrete space with $|\chi| = m$ and $|\gamma| = n$ resp. then $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = \ell(\jmath(\tau_\chi \times \tau_\gamma))$.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$ resp. & Z be a clique in $\jmath(\tau_\chi \times \tau_\gamma)$. Obviously, every chain of open subsets corresponds to a clique in $\jmath(\tau_\chi \times \tau_\gamma)$. Now, suppose that H is a largest clique in $\jmath(\tau_\chi \times \tau_\gamma)$. Something worth mentioning is that one may buy a tournament for H. In light of this, H possesses a Hamiltonian path that is equivalent to a chain of *chi*. □

Theorem 3.4. *If (χ, τ_χ) is discrete space with $|\chi| = n$ then $\chi(\jmath(\tau_\chi \times \tau_\chi)) = 2(n-1)$.*

Proof. Let $\chi = \{a_1, a_2, ..., a_n\}$ & $\mu_i = \{a_1, a_2, ..., a_i\} \in \tau_\chi$, for $i = 1, 2, ...n-1$. Let $Z = \{\mu_1 \times V_1 \subset \mu_2 \times \vartheta_1 \subset \mu_3 \times \vartheta_1 \subset ... \subset \mu_{n-1} \times \vartheta_1 \subset \chi \times \vartheta_1 \subset \chi \times \vartheta_2 \subset \chi \times \vartheta_3 \subset ... \subset \chi \times \vartheta_{n-1}\}$ be a ascending chain of open subsets of $(\chi \times \gamma, \tau)$. Then clearly Z is clique in $\jmath(\tau_\chi \times \tau_\gamma)$. As clique number of $\jmath(\tau_\chi \times \tau_\chi)$ is $2n-2$, then we have $\chi(\jmath(\tau_\chi \times \tau_\gamma)) \geq 2n-2$.

Claim: We demonstrate a $2n-2$ coloring of $\jmath(\tau_\chi \times \tau_\gamma)$:

Now label the vertex $\mu_i \times \vartheta_1$ and $\mu_{n-1} \times \vartheta_j$ by the color C_{i1} and $C_{n-1,j}$ respectively. To demonstrate a $2n-2$ coloring of $\jmath(\tau_\chi \times \tau_\gamma)$, we start with any vertex $\mu \times \vartheta$ of $\jmath(\tau_\chi \times \tau_\gamma)$. As $\mu \times \vartheta$ is non empty proper open subsets of $\chi \times \chi$, then following cases may arises.

Case I: If $\mu \subset \chi$ and $\vartheta = \gamma$ then $|\mu| = k$, where $1 < k < (n-1)$. Hence $\exists$ $\mu \times \vartheta = \mu \times \gamma$ be an open subset such that $\mu \times \vartheta \not\sim \mu_k \times \gamma$. Hence vertex $\mu \times \vartheta$ is determined with color $C_{k,n}$. Thus there is no need of new color to label $\mu \times \vartheta$ such that adjacent vertices gets different colors.

Case II: If $\mu = \chi$ and $\vartheta \subset \gamma$ then $|\vartheta| = q$, where $1 < q < (n-1)$. Hence $\exists$ $\mu \times \vartheta = \chi \times \vartheta$ be an open subset such that $\mu \times \vartheta \not\sim \chi \times \vartheta_q$. Hence vertex $\mu \times \vartheta$ is determined with color $C_{m,q}$. Thus there is no need of new color to label $\mu \times \vartheta$ such that adjacent vertices gets different colors.

Case III: If $\mu \subset \chi$ and $\vartheta \subset \gamma$ then $\exists$ μ, ϑ be an open subset such that $|\mu| = p$ and

$|\vartheta| = q$, where $1 < p < (m-1)$, $1 < q < (n-1)$. As $\mu \subset \chi$ then there exist $a_i \notin \mu$ such that $\mu \times \vartheta \subset \mu_i \times \vartheta_1$ and hence $\mu \times \vartheta \not\sim \mu_i \times \vartheta_1$. Also as $\upsilon \subset \gamma$ then there exist $b_j \notin \vartheta$ such that $\mu \times \vartheta \subset \chi \times \vartheta_j$ and hence $\mu \times \vartheta \not\sim \chi \times \vartheta_j$. Thus vertex $\mu \times \vartheta$ is determined with color $C_{m,q}$. Thus there is no need of new color to label $\mu \times \vartheta$ such that adjacent vertices gets different colors. Hence we have, $\chi(\jmath(\tau_\chi \times \tau_\gamma)) \leq 2n-2$. □

Example 3.5. *If* $\chi = \{a_1, a_2\}$ *and* $\gamma = \{b_1, b_2\}$ *then vertex set of* $\jmath(\tau_\chi \times \tau_\gamma)$ *is* $\vartheta = \{A_1 = \{a_1\} \times \{b_1\}, A_2 = \{a_1\} \times \{b_2\}, A_3 = \{a_1\} \times \{b_1, b_2\}, A_4 = \{a_2\} \times \{b_1\}, A_5 = \{a_2\} \times \{b_2\}, A_6 = \{a_2\} \times \{b_1, b_2\}, A_7 = \{a_1, a_2\} \times \{b_1\}, A_8 = \{a_1, a_2\} \times \{b_2\}\}$, *which is also written as* $\vartheta = \{A_1 = \{(a_1, b_1)\}, A_2 = \{(a_1, b_2)\}, A_5 = \{(a_1, b_1), (a_1, b_2)\}, A_3 = \{(a_2, b_1)\}, A_4 = \{(a_2, b_2)\}, A_6 = \{(a_2, b_1), (a_2, b_2)\}, A_7 = \{(a_1, b_1), (a_2, b_1)\}, A_8 = \{(a_1, b_2), (a_2, b_2)\}$.

Then inclusion graph $\jmath(\tau_\chi \times \tau_\gamma)$ is shown in figure 4. The coloring of inclusion graph $\jmath(\tau_\chi \times \tau_\gamma)$ with $\chi(\jmath(\tau_\chi \times \tau_\gamma)) = 2 = \omega(\jmath(\tau_\chi \times \tau_\gamma))$ is shown in figure 5.

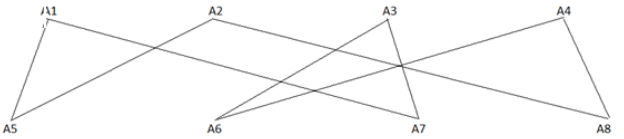


FIGURE 4. Iinclusion graph $\jmath(\tau_X \times \tau_Y)$ with $|X| = 2$ and $|Y| = 2$.

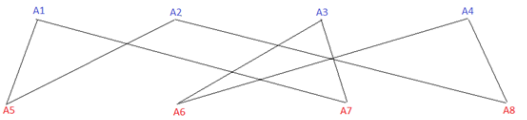


FIGURE 5. The coloring of inclusion graph $\jmath(\tau_X \times \tau_Y)$ with $\chi(\jmath(\tau_X \times \tau_Y)) = 2$

Theorem 3.6. *If (χ, τ_χ) and (γ, τ_γ) are discrete topological space with $|\chi| = n$ and $|\gamma| = n$, where $m < n \geq 4$. then $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m+n-2) = \chi(\jmath(\tau_\chi)) + \chi(\jmath(\tau_\gamma)) = \chi(\jmath(\tau_\chi \times \tau_\gamma))$.*

Proof. Let (χ, τ_χ) and (γ, τ_γ) are discrete topological space with $|\chi| = n$ and $|\gamma| = n$, where $m < n \geq 4$.

Claim: To prove that $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = \chi(\jmath(\tau_\chi \times \tau_\gamma))$.

For this it is sufficient to prove that the graph $\jmath(\tau_\chi \times \tau_\gamma)$ and $\jmath(\tau_\chi \times \tau_\gamma)^c$ do not include an induced odd cycle that is at least 5 in duration.

If possible, let $H = \{\omega_{1,1} = \mu_1 \times \vartheta_1, \omega_{2,1} = \mu_2 \times \vartheta_1, \omega_{3,1} = \mu_3 \times \vartheta_1, ..., \omega_{2k,1} = \mu_{2k} \times \vartheta_1, \omega_{2k+1,1} = \mu_{2k+1} \times \vartheta_1\}$ be induced odd cycle of length at least 5, where $k \geq 2$. Since $\omega_{1,1} \sim \omega_{2,1}$, then without loss of generality, we suppose that $\omega_{1,1} \subset \omega_{2,1}$. Now, since $\omega_{2,1} \sim \omega_{3,1}$ then we have, $\omega_{2,1} \subset \omega_{3,1}$ or $\omega_{3,1} \subset \omega_{2,1}$. However, if $\omega_{2,1} \subset \omega_{3,1}$ holds, then by transitivity of set inclusion, we have $\omega_{1,1} \subset \omega_{3,1}$. Hence $\omega_{1,1} \sim \omega_{3,1}$, a contradiction to the fact that C is an induced cycle of length $2k+1$, with $k \geq 2$ in $\jmath(\tau)$ and therefore, $\omega_{3,1} \subset \omega_{2,1}$. Proceeding in a similar manner finally we get $\omega_{1,1} \subset \omega_{2,1} \supset \omega_{3,1} \subset \omega_{4,1} \supset \omega_{5,1} \subset, ..., \supset \omega_{2k+1,1} \subset \omega_{1,1}$. However, this implies that $\omega_{2k+1,1} \subset \omega_{1,1} \subset \omega_{2,1}$, which is a cycle of length 3, a contradiction to the fact that C is an induced cycle of length $2k+1$, with $k \geq 2$ in $\jmath(\tau_\chi \times \tau_\gamma)$. Hence the graph $\jmath(\tau_\chi \times \tau_\gamma)$ do not contain induced odd cycle of length at least 5.

If possible, let $C : \omega_{1,1} \sim \omega_{2,1} \sim \omega_{3,1} \sim, ..., \sim \omega_{2k+1,1} \sim \omega_{1,1}$ be an induced cycle in $\jmath(\tau_\chi \times \tau_\gamma)^c$ with $k \geq 2$. For sake of definiteness and for further use, let $\omega_{2k+1,1}$ be the final vertex in the induced cycle C. Since $\omega_{1,1} \sim \omega_{2,1}$, therefore we have $\omega_{1,1} \not\subset \omega_{2,1}$ and $\omega_{2,1} \not\subset \omega_{1,1}$. On the other hand, as $\omega_{1,1} \not\sim \omega_{3,1}$, without loss of generality, let us suppose that $\omega_{1,1} \subset \omega_{3,1}$. As $\omega_{1,1}$ is also not adjacent to vertex $\omega_{4,1}$,then we have either $\omega_{1,1} \subset \omega_{4,1}$ or $\omega_{4,1} \subset \omega_{1,1}$. But, if $\omega_{4,1} \subset \omega_{1,1}$ is holds, then by transitivity of set inclusion, we have $\omega_{4,1} \subset \omega_{3,1}$ and hence $\omega_{3,1} \not\sim \omega_{4,1}$ in $\jmath(\tau_\chi \times \tau_\gamma)^c$, a contradiction. Thus,we have $\omega_{1,1} \subset \omega_{4,1}$. Again $\omega_{2,1} \not\sim \omega_{4,1}$. then we have, either $\omega_{2,1} \subset \omega_{4,1}$ or $\omega_{4,1} \subset \omega_{2,1}$. But, if $\omega_{4,1} \subset \omega_{2,1}$, then by transitivity of set inclusion, we get $\omega_{1,1} \subset \omega_{2,1}$. Hence $\omega_{1,1} \not\sim \omega_{2,1}$ in $\jmath(\tau)^c$, a contradiction and therefore $\omega_{2,1} \subset \omega_{4,1}$. Also, since $\omega_{2,1} \not\sim \omega_{5,1}$ then we have, either $\omega_{2,1} \subset \omega_{5,1}$ or $\omega_{5,1} \subset \omega_{2,1}$. But, if $\omega_{5,1} \subset \omega_{2,1}$, then by transitivity of set inclusion, we have $\omega_{5,1} \subset \omega_{4,1}$ and hence $\omega_{5,1} \not\sim \omega_{4,1}$ in $\jmath(\tau_\chi \times \tau_\gamma)^c$, a contradiction. Thus, $\omega_{2,1} \subset \omega_{5,1}$. Finally, as $\omega_{3,1} \not\sim \omega_{5,1}$, then we have either $\omega_{3,1} \subset \omega_{5,1}$ or $\omega_5 \subset \omega_{3,1}$. If $\omega_{5,1} \subset \omega_{3,1}$,

then by transitivity of set inclusion, we get $\omega_{2,1} \subset \omega_{3,1}$. Hence $\omega_{2,1} \not\sim \omega_{3,1}$ in $\jmath(\tau_\chi \times \tau_\gamma)$, a contradiction. On the other hand, if $\omega_{3,1} \subset \omega_{5,1}$, then by transitivity of set inclusion, $\omega_{1,1} \subset \omega_{5,1}$ and hence $\omega_{1,1} \not\sim \omega_{5,1}$ in $\jmath(\tau_\chi \times \tau_\gamma)^c$, therefore $\omega_{5,1}$ is not the final vertex in C. Thus, there exists at least two more vertices $\omega_{6,1}$ and $\omega_{7,1}$ in the induced cycle C in $\jmath(\tau_\chi \times \tau_\gamma)^c$. However, continuing in the same manner, we can show that $\omega_1 \subset \omega_7$. Hence $\omega_{1,1} \not\sim \omega_{7,1}$ in $\jmath(\tau_\chi \times \tau_\gamma)^c$, and we conclude that $\omega_{7,1}$ is not the final vertex in C. By the same argument as above, the induced odd cycle C has two more vertices $\omega_{8,1}$ and $\omega_{9,1}$. Be aware that this procedure never ends, preventing us from obtaining an induced subparagraph or a contradiction. Thus the graph $\jmath(\tau_\chi \times \tau_\gamma)^c$ has no triggered cycle longer than three. Thus, by Theorem 5.4[8], we have, $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m + n - 2) = \chi(\jmath(\tau_\chi)) + \chi(\jmath(\tau_\gamma)) = \chi(\jmath(\tau_\chi \times \tau_\gamma))$. □

Theorem 3.7. *If (χ, τ_χ) and (γ, τ_γ) are discrete space with $|\chi| = m$ and $|\gamma| = n$, where $m > n$. then $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m+n-2) = \chi(\jmath(\tau_\chi)) + \chi(\jmath(\tau_\gamma)) = \chi(\jmath(\tau_\chi \times \tau_\gamma))$.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$, where $m < n \geq 4$.
Claim: To prove that $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m + n - 2) = \chi(\jmath(\tau_\chi \times \tau_\gamma))$.
For this it is sufficient to prove that the graph $\jmath(\tau_\chi \times \tau_\gamma)$ and $\jmath(\tau_\chi \times \tau_\gamma)^c$ don't include any caused odd cycles longer than 5.
Instead, start by assuming that $H = \{\omega_{1,1} = \mu_1 \times \vartheta_1, \omega_{2,1} = \mu_2 \times \vartheta_1, \omega_{3,1} = \mu_3 \times \vartheta_1, ..., \omega_{2k,1} = \mu_{2k} \times \vartheta_1, \omega_{2k+1,1} = \mu_{2k+1} \times \vartheta_1\}$ is an induced subgraph of G and $k \geq 2$. It is evident that there are three open subsets, (for simplicity consider) $\omega_{i,1}, \omega_{j,1}, \omega_{k,1}$ such that $\omega_{i,1} \subset \omega_{j,1} \subset \omega_{k,1}$, where $1 \leq i, j, k \leq 2m + 1$. So, $\omega_{i,1} \sim \omega_{j,1} \sim \omega_{k,1} \sim \omega_{i,1}$, G forms a triangle, a contradiction. Imagine that now $H = \{\omega_{1,1} = \mu_1 \times \vartheta_1, \omega_{m,2} = \chi \times \vartheta_2, \omega_{2m+1,1} = \mu_{2m+1} \times \vartheta_1, \omega_{m,1} = \mu_m \times \vartheta_1, ..., \omega_{2,1} = \mu_2 \times \vartheta_1, \omega_{m+2,1} = \mu_{m+2} \times V_3, \omega_{1,1} = \mu_1 \times \vartheta_1\}G$ be t he complement of the above cycle. Consider the cycle $\omega_{1,1} \sim \omega_{m,2} \sim \omega_{2m+1,1} \sim \omega_{m,1} \sim ... \sim \omega_{2,1} \sim \omega_{m+2}, \sim \omega_{1,1}$ as a subgraph of H. Likewise, it is evident that there are three open subsets $\omega_{i,1}, \omega_{j,1}, \omega_{k,1}, \omega_{i,1}$ such that $\omega_{i,1} \sim \omega_{j,1} \sim \omega_{k,1} \sim \omega_{i,1}$, forms a triangle, a contradiction, where $1 \leq i \leq 2m + 1, j = i + m(mod 2m + 1)$ and $k = j + m = i - 1(mod 2m + 1)$. So, $\omega_{i,1} \sim \omega_{j,1} \sim \omega_{k,1} \sim \omega_{i,1}$ is a triangle and $\omega_{i,1} \sim \omega_{k,1}$, a contradiction, and hence the result. □

Theorem 3.8. *Let (χ, τ_χ) and (γ, τ_γ) are discrete spaces with $|\chi| = m$ and $|\gamma| = n$. If $m = n \geq 4$ or $m > n \geq 3$ or $n > m \geq 3$ then the graph $\jmath(\tau_\chi \times \tau_\gamma)$ is not planer graph.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$, where $m > n \geq 4$.
Case I: If $m = n \geq 4$ then by Theorem 3.2, the clique number of the graph $\jmath(\tau_\chi \times \tau_\chi)$, is $\omega(\jmath(\tau_\chi \times \tau_\chi)) = 2(n-1) = 2\omega(\jmath(\tau_\chi)$. As $m = n \geq 4$, then the graph $\jmath(\tau_\chi)$ has a clique of size at least 5, i.e., a complete graph K_5 as a sub graph of the graph $\jmath(\tau_\chi \times \tau_\chi)$. Thus, it is not planar.
Case II: If $m > n \geq 3$ or $n > m \geq 3$ then by Theorem 3.2, the clique number of the graph $\jmath(\tau_\chi \times \tau_\gamma)$ is $,\omega(\jmath(\tau_\chi \times \tau_\gamma)) = (m + n - 2) = \omega(\jmath(\tau_\chi) + \omega(\jmath(\tau_\gamma)$. As $m \geq 4, n \geq 3$, or $n \geq 4, m \geq 3$ then the graph $\jmath(\tau_\chi \times \tau_\gamma)$ has a clique of size at least 5, Thus, the graph $\jmath(\tau_\chi \times \tau_\gamma)$ is not planar. □

Theorem 3.9. *If (χ, τ_χ) and (γ, τ_γ) are discrete topological space with $|\chi| = m$ and $|\gamma| = n$. If $Z = \{\mu_1 \times \gamma, \mu_2 \times \gamma, ..., \mu_m \times \gamma, \chi \times \vartheta_1, \chi \times \vartheta_2, ..., \chi \times \vartheta_n\}$ is a collection of open subsets of $\chi \times \gamma$ having cardinality $(m-1)n$ or $m(n-1)$ then Z is dominating set, where $|\mu_i| = (m-1)$ and $|\vartheta_j| = (n-1)$.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$ resp. & $Z = \{\mu_1 \times \gamma, \mu_2 \times \gamma, ..., \mu_m \times \gamma, \chi \times \vartheta_1, \chi \times \vartheta_2, ..., \chi \times \vartheta_n\}$ is a collection of open subsets of $\chi \times \gamma$ having cardinality $(m-1)n$ or $m(n-1)$, where $|\mu_i| = (m-1)$ and $|\vartheta_j| = (n-1)$. If $\mu \times \vartheta = \{a_1, a_2, ...a_p\} \times \{b_1, b_2, ...b_q\} \in \tau, \forall p < m, q < n$, be any open subset of $\chi \times \gamma$, after that, at least one element exists $\mu_1 \times \gamma$ or $\chi \times \vartheta_1$ of Z, such that $\mu \times \vartheta \subset \mu_1 \times \gamma$ or $\mu \times \vartheta \subset \chi \times \vartheta_1$ and hence $\mu \times \vartheta \sim \mu_1 \times \gamma$ or $\mu \times \vartheta \sim \chi \times \vartheta_1$. By Theorem 3.2[8], any two open subsets from Z are not adjacent. Thus Z is dominating set. □

Theorem 3.10. *If (χ, τ_χ) and (γ, τ_γ) are discrete topological space with $|\chi| = m$ and $|\gamma| = n$. If $Z = \{\mu_1 \times \gamma, \mu_2 \times \gamma, ..., \mu_m \times \gamma, \chi \times \vartheta_1, \chi \times \vartheta_2, ..., \chi \times \vartheta_n\}$ is a collection of open subsets of $\chi \times \gamma$ having cardinality $(m-1)n$ or $m(n-1)$ then Z is minimal dominating set.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$ & $Z = \{\mu_1 \times \gamma, \mu_2 \times \gamma, ..., \mu_m \times \gamma, \chi \times \vartheta_1, \chi \times \vartheta_2, ..., \chi \times \vartheta_n\}$ is a collection of open subsets of $\chi \times \gamma$ having cardinality $(m-1)n$ or $m(n-1)$. Then by Theorem 3.9, Z is dominating set.
Let $Z_i = Z - \{\mu_i \times \gamma\}; \forall i$ or $Z = Z - \{\chi \times \vartheta_j\}; \forall j$ thereafter, an open subset exists

$\mu \times \vartheta = \mu_i \times \gamma$ of $\chi \times \gamma$ or $\mu \times \vartheta = \chi \times \vartheta_j; \forall i, j$ of $\chi \times \gamma$ such that $\mu \times \vartheta \nsubseteq \mu_i \times \gamma$ or $\mu \times \vartheta \nsubseteq \chi \times \vartheta_j; \forall i, j$ and hence $\mu \times \vartheta \not\sim \mu_i \times \gamma$ or $\mu \times \vartheta \not\sim \chi \times \vartheta_j, \forall i, j$. Thus Z is minimal dominating set. □

Theorem 3.11. *If (χ, τ_χ) and (γ, τ_γ) are discrete space with $|\chi| = m$ and $|\gamma| = n$ then domination number of $\jmath(\tau_\chi \times \tau_\gamma)$ is $m + n$.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$ resp. & $Z = \{\mu_1 \times \gamma, \mu_2 \times \gamma, ..., \mu_m \times \gamma, \chi \times \vartheta_1, \chi \times \vartheta_2, ..., \chi \times \vartheta_n\}$ is a collection of open subsets of $\chi \times \gamma$ having cardinality $(m-1)n$ or $m(n-1)$ then by Theorem 3.10, Z is minimal dominating set & therefore domination number of" $\jmath(\tau_\chi \times \tau_\gamma)$ is $m + n$. □

Theorem 3.12. *If (χ, τ_χ) and (γ, τ_γ) are discrete topological space with $|\chi| = m$ and $|\gamma| = n$. If $Z = \{\mu_1 \times \gamma, \mu_2 \times \gamma, ..., \mu_m \times \gamma, \chi \times \vartheta_1, \chi \times \vartheta_2, ..., \chi \times \vartheta_n\}$ is a collection of open subsets of $\chi \times \gamma$ having cardinality $(m-1)n$ or $m(n-1)$ then Z is not totally domination set.*

Proof. Proof fallows from Theorem 3.10, and Theorem 3.2[8]. □

4. Degree of vertex and Inclusion Graph of Product Topology on $\chi \times \gamma$

In this section we investigate the degree of vertex in the graph $\jmath(\tau_\chi \times \tau_\gamma)$.

Theorem 4.1. *Let (χ, τ_χ) and (γ, τ_γ) are discrete topological space with $|\chi| = m$ and $|\gamma| = n$, where $m > n$. If $|\mu \times \vartheta| = pq$, where $|\mu| = p$ and $|\vartheta| = q$ then number of vertices of $deg(\mu \times \vartheta) = (2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2) + (2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)$, is equal to $(m!/(m-p)!p!)(n!/(n-q)!q!)$.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$, where $m > n \geq 4$. Let $\mu = \{a_1, a_2, ...a_p\}$ and $\vartheta = \{b_1, b_2, ...b_q\}$ are any open subset of χ and γ resp.. Let $\mu \times \vartheta$ be any open subset of $\chi \times \gamma$. If $\mu_1 \times \vartheta_1 \sim \mu \times \vartheta$ then we have $\mu_1 \times \vartheta_1 \subset \mu \times \vartheta$ or $\mu_1 \times \vartheta_1 \supset \mu \times \vartheta$.
Case 1: If $\mu_1 \times \vartheta_1 \subset \mu \times \vartheta$ is any proper subset of $\mu \times \vartheta$ then, clearly $\mu_1 \subset \mu$ and $\vartheta = \vartheta$ or $\mu = \mu$ and $\vartheta_1 \subset \vartheta$ or $\mu_1 \subset \mu$ and $\vartheta_1 \subset \vartheta$. To find degree of $\mu \times \vartheta$ consider following three cases.
Sub Case I: If $\mu_1 \subset \mu$ and $\vartheta = \vartheta$ then number of such appropriate nonempty proper subsets μ_1 of μ is equal to $(2^p - 2)$ and hence number open subsets of $\mu \times \vartheta$ is equal to $(2^p - 2)$.

Sub Case II: If $\mu = \mu_1$ and $\vartheta_1 \subset \vartheta$ then number of such nonempty proper subsets ϑ_1 of ϑ is equal to $(2^q - 2)$ and hence number of open subsets of $\mu \times \vartheta$ is equal to $(2^q - 2)$.

Sub Case III: If $\mu_1 \subset \mu$ and $\vartheta_1 \subset \vartheta$. As $\mu_1 \subset \mu$ then number of such nonempty proper subsets μ_1 of μ is equal to $(2^p - 2)$. Also as $\vartheta_1 \subset \vartheta$ then number of such open subsets ϑ_1 of ϑ is equal to $(2^q - 2)$. Hence number of open subsets of $\mu \times \vartheta$ is equal to $(2^p - 2)(2^q - 2)$.

Thus number of open subsets of $\mu \times \vartheta$ is equal to $(2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2)$.

Case 2: If $\mu_1 \times \vartheta_1 \neq \chi \times \gamma$, is any nonempty proper superset of $\mu \times \vartheta$ then, clearly $\mu \subset \mu_1$ and $\vartheta = \gamma$ or $\mu = \chi$ and $\vartheta \subset \vartheta_1$ or $\mu \subset \mu_1$ and $\vartheta \subset \vartheta_1$. To find degree of $\mu \times \vartheta$ consider following three cases.

Sub Case I: If $\mu \subset \mu_1$ and $\vartheta = \gamma$ then number of such open supersets μ_1 of μ is equal to $(2^{m-p} - 2)$ and hence number of open subsets of $\mu \times V$ is equal to $(2^{m-p} - 2)$.

Sub Case II: If $\mu_1 = \chi$ and $\vartheta \subset \vartheta_1$ then number of such open subsets ϑ_1 of ϑ is equal to $(2^{n-q} - 2)$ and hence number of open subsets of $\mu \times \vartheta$ is equal to $(2^{n-q} - 2)$.

Sub Case III: If $\mu \subset \mu_1$ and $\vartheta \subset \vartheta_1$. As $\mu \subset \mu_1$ then number of such open supersets μ_1 of μ is equal to $(2^{m-p} - 2)$. Also as $\vartheta \subset \vartheta_1$ then number of such subsets ϑ_1 of ϑ is equal to $(2^{n-q}2)$. Hence number of supersets of $\mu \times \vartheta$ is equal to $(2^{m-p} - 2)(2^{n-q} - 2)$. Thus Number of nonempty proper supersets of $\mu \times \vartheta$ is equal to $(2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)$.

Hence $deg(\mu \times \vartheta) = (2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2) + (2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)$. □

Theorem 4.2. *As (χ, τ_χ) and (γ, τ_γ) are discrete space with $|\chi| = m$ and $|\gamma| = n$ then $deg(\jmath(\tau_\chi \times \tau_\gamma)) = \Sigma_1^{(n-1)}(m!/((m-p)!p!))(n!/((n-q)!q!))\{(2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2) + (2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)\}$.*

Proof. Let $|\chi| = m$ and $|\gamma| = n$ & $|\mu \times \vartheta| = pq$, where $|\mu| = p$ and $|\vartheta| = q$ then by Theorem 4.1, we have $deg(U \times V) = (2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2) + (2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)$ and and number of such subsets is equal to $m!/(m-p)!p!n!/(n-q)!kq!$. Thus $deg(\jmath(\tau_\chi \times \tau_\gamma)) = \Sigma_1^{(m-1)}\Sigma_1^{(n-1)}(m!/((m-p)!p!))(n!/((n-q)!q!))\{(2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2) + (2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)\}$. □

5. Conclusion

In this essay, we looked at a few significant findings of the graph $\jmath(\tau_\chi \times \tau_\gamma)$. We also derived relationship between diameter, girth, clique number, chromatic number, domination number, etc., of an open subset inclusion graphs $\jmath(\tau_\chi)$, $\jmath(\tau_\gamma)$ and $\jmath(\tau_\chi \times \tau_\gamma)$. Moreover, we proved that, if (χ, τ_χ) and (γ, τ_γ) are discrete topological space with $|\chi| = m$ and $|\gamma| = n$, then $\omega(\jmath(\tau_\chi \times \tau_\gamma)) = \omega(\jmath(\tau_\chi)) + \omega(\jmath(\tau_\gamma))$ and $\chi(\jmath(\tau_\chi \times \tau_\gamma)) = \chi(\jmath(\tau_\chi)) + \chi(\jmath(\tau_\gamma))$.

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