

ON APPROXIMATION OF RELATIVE ENTROPIES

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ABSTRACT. We analyze when symbolic dynamical systems have intermediate relative entropies for two symbolic spaces (X, σ_1) and (Y, σ_2) . By this is understood to study the denseness of relative measure-theoretic entropies in the interval $[0, h(\nu, \sigma_1)]$, where $h(\nu, \sigma_1)$ is a relative topological entropy of the shift σ_1 with respect to a σ_2 -invariant measure ν in Y . We prove that for irreducible shifts with the (strong) specification property as introduced by Bowen, the set of the relative measure-theoretic entropies is dense in the above mentioned sense. To this, we use the decomposition of factor codes of Yoo and a variational principle for relative entropies given by Barral and Feng.

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1. INTRODUCTION

Let (X, f) be a dynamical system with X a compact metric space and f be a continuous map. Let denote by $\mathcal{M}(X, f)$ the set of f -invariant probability measures on X and by $\mathcal{M}_{erg}$ the set of ergodic measures in (X, f) . The variational principle states that the topological entropy of f can be expressed, in a variational way, as

the supremum over all the measures $\mu \in \mathcal{M}(X, f)$ of the measure-theoretic entropy $h_\mu(f)$. The measures at which the supremum is attained are called measures of maximal entropy. Then the problem of “intermediate entropies” can be described as to analyze the denseness of the set

$$(1.1) \quad \mathcal{H}(X, f) = \{h_\mu(f) : \mu \in \mathcal{M}(X, f)\}$$

in the interval $[0, h(f)]$ with $h(f)$ the topological entropy of f .

A first result of this class of problems was obtained by Katok (1980), who proved that for $C^{1+\alpha}$ diffeomorphisms holds

$$(1.2) \quad h_\mu(f) \leq \max(0, \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f))$$

where $P_n(f)$ is the set of n -periodic points of f . More precisely it was proved (Katok, 1980) that such class of maps verify

$$(1.3) \quad h(f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f).$$

Therefore, for the class of diffeomorphisms, we have

$$(1.4) \quad [0, h(f)] \subset \mathcal{H}(X, f).$$

Several results about the Katok conjecture on the validity of this kind of results for intermediate entropies were further obtained. For instance Guan, Sun and Wu (2017) proved the Katok conjecture, for homogeneous dynamical systems consistent in quotient spaces of a connected semisimple Lie group by a cocompact lattice. The dynamics are induced by left translation on the quotient space. The result was obtained under properties of almost weak-specification and asymptotically entropy expansiveness. In its turn, Sun (2020a) proved that *USC*-systems have the intermediate entropy property. By *USC*-systems is understood those for which the measure-theoretic entropy function is upper semi-continuous. More recently, the intermediate property for entropies was verified for star vector fields with positive entropy by Li and coworkers (2020). Sun also considered a more general problem of denseness of intermediate pressures and proved that the set

$$(1.5) \quad \mathcal{P}(X, f, \varphi) = \{h_\mu(f) + \int \varphi d\mu : \mu \in \mathcal{M}(X, f)\}$$

is dense in an interval of the form $[A, P(\varphi)]$ where $P(\varphi)$ is the topological pressure of the potential φ (Sun, 2020b). The systems for which the result is valid are those which have the Climenhaga-Thompson (2016) structure. This property ensures the existence and uniqueness of equilibrium states under conditions of weak-specification and almost expansiveness.

The objective of this article is to obtain similar results for intermediate relative entropies working within the following scheme: let (X, σ_1) and (Y, σ_2) two Bernoulli shifts, with finite respective alphabets Ω_1, Ω_2 , i.e. each space consists of all infinite sequences in the respective alphabets. The maps $\sigma_1 : X \rightarrow X$ and $\sigma_2 : Y \rightarrow Y$, are the Bernoulli maps for shifts. Finite sequences in the symbols of the alphabet are called words. A map $\pi : X \rightarrow Y$ is called a *factor map*, or *factor code*, if it is continuous, surjective and $\pi \circ \sigma_1 = \sigma_2 \circ \pi$. In this case says that (Y, σ_2) is a *factor* of (X, σ_1) .

A *language* on a subshift is the set of words of any length, which are allowed or accepted. A subshift with a regular language is called a *sofic subshift*. A subshift is irreducible if given two words in the language of the shift can be connected by a third word such that the word resulting by the concatenation of the three sequences is also accepted by the language. We shall consider Bernoulli shifts, i.e. full shifts so this condition is satisfied since all sequences are allowed. Besides the system satisfy the property of expansiveness.

Given two Bernoulli shifts (X, σ_1) , (Y, σ_2) and a measure $\nu \in \mathcal{M}(Y, \sigma_2)$, the *relative measure-theoretic entropy of the measure $\mu \in \mathcal{M}(X, \sigma_1)$ with respect to ν , and the shifts σ_1, σ_2* is

$$(1.6) \quad \mathcal{F}_\nu(\mu) = \mathcal{F}_\nu(\mu, \sigma_1, \sigma_2) = h_\mu(\sigma_1) - h_\nu(\sigma_2)$$

where are the respective measure-theoretic entropies of the shifts σ_1, σ_2 . A measure $\mu \in \mathcal{M}(X, \sigma_1)$ is a *measure of maximal relative entropy with respect to ν* , or over ν , if μ maximizes the map $\mathcal{F}_\nu(\mu)$ subject to the restriction $\pi_*(\mu) = \nu$ where $\pi_*(\mu)$ is the push-forward of the measure μ , i.e. $\pi_*(\mu)(E) = \mu(\pi^{-1}(E))$, for any measurable E .

There is a variational principle for relative entropies and potentials. A first version is due to Ledrappier and Walters (1977). Then Feng (2011) and Barral-Feng (2012) extended the variational principle for additive potentials in the context of subshifts. The relative variational principle, relativized to a measure $\nu \in \mathcal{M}(Y, \sigma_2)$ is stated as

$$(1.7) \quad \sup \left\{ h_\mu(\sigma_1) - h_\nu(\sigma_2) + \int \varphi d\mu : \mu \in \mathcal{M}(X, \sigma_1), \pi_*(\mu) = \nu \right\} = P(\nu, \sigma_1, \varphi)$$

where $P(\nu, \sigma_1, \varphi)$ is the relative variational pressure, for the potential $\varphi : X \rightarrow \mathbb{R}$ and the shift σ_1 relativized to the measure ν , in the sense of Ledrappier and Walters (1977). For $\varphi \equiv 0$ is obtained the *relative topological entropy relativized to the measure ν* . We denote $h(\nu, \sigma_1) = P(\nu, \sigma_1, 0)$. Thus the relative variational principle relativised to a measure $\nu \in \mathcal{M}(Y, \sigma_2)$ is stated in this case as

$$(1.8) \quad \sup \{ h_\mu(\sigma_1) - h_\nu(\sigma_2) : \mu \in \mathcal{M}(X, \sigma_1), \pi_*(\mu) = \nu \} = h(\nu, \sigma_1)$$

We will consider the case $\varphi \equiv 0$ and the definition of the relative entropy will be displayed in the next section.

The measures $\mu \in \mathcal{M}(X, \sigma_1)$, for which the supremum in (1.7) is attained are *the relative equilibrium states for the potential φ relative to the measure ν* . When $\varphi \equiv 0$, as we pointed out, the measures for which the supremum is attained are the *measures of maximal entropy relative to ν* .

To ensure the uniqueness of relative equilibrium states can be used the decomposition of factor codes given by Yoo (2018) according to which a factor code map, with X a finite type irreducible subshift and Y sofic, can be written as the composition of a finite-to-one code and a code with degree class equal 1.

Our main result reads:

Theorem 1: Let (X, σ_1) be an irreducible shift with the specification property, and let (Y, σ_2) be a sofic shift. Let us consider a factor code $\pi : X \rightarrow Y$ and let us fix a measure $\nu \in \mathcal{M}(Y, \sigma_2)$. Then the set

$$(1.9) \quad \mathcal{G}(\nu, \sigma_1, \sigma_2) := \{ \mathcal{F}_\nu(\mu) = h_\mu(\sigma_1) - h_\nu(\sigma_2) : \mu \in \mathcal{M}(X, \sigma_1), \pi_*(\mu) = \nu \}$$

is dense in the interval $[0, h(\nu, \sigma_1)]$.

For the code map $\pi : X \longrightarrow Y$ we introduce the following metric in X :

$$(1.10) \quad d(x, y) = \max[\exp(-|x \wedge y|), \exp(-|\pi(x) \wedge \pi(y)|)]$$

where if $x = x_1x_2\dots$ and $y = y_1y_2\dots$ then

$$(1.11) \quad |x \wedge y| := \max\{n : x_i = y_i, i = 1, 2, \dots, n\}$$

2. BASIC DEFINITIONS

Let $\{X, \sigma_1\}, (Y, \sigma_2)\}$ be a system of finite type subshifts with alphabets Ω_1, Ω_2 , i.e. each space X, Y , consists in infinite sequences $u = u_1u_2\dots$, with $u_i \in \Omega_i$. Let $\pi : X \longrightarrow Y$ be a factor code map. The Bernoulli shift $\sigma = \sigma_i$ on X, Y is defined as

$$(2.1) \quad (\sigma u)_n = u_{n+1} \text{ for any sequence } u = u_1u_2\dots \text{ belonging to } X \text{ or } Y$$

For any $n \in \mathbb{N}$ the “dynamical metric” in X

$$(2.2) \quad d_n(x, y) = \max\{d(\sigma_1^i x, \sigma_1^i y), i = 1, 2, \dots, n\}.$$

can be defined.

Let now $n \in \mathbb{N}$, $\varepsilon > 0$ and $y \in Y$. Define the number $Sep(n, \varepsilon, y)$ by

$$(2.3) \quad Sep(n, \varepsilon, y) = \max\{m : \text{there are } m \text{ points } x_1, x_2, \dots, x_m \in \pi^{-1}(y) \}$$

with $d_n(u, v) > \varepsilon$.

This is the maximal number of points in each fiber $\pi^{-1}(y)$ which are ε -separated by the metric d_n . In this case we say that the points are separated relative to $\pi^{-1}(y)$.

Now set

$$H(n, \varepsilon, y, \sigma_1) = Sep(n, \varepsilon, y)$$

and

$$H(y, \varepsilon, \sigma_1) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log H(n, \varepsilon, y, \sigma_1),$$

so

$$H(\varepsilon, \sigma_1, \nu) = \int_{X_2} H(y, \varepsilon, \sigma_1) d_\nu(y)$$

and the *entropy of the shift σ_1 relative to the measure ν* is defined as

$$(2.4) \quad h(\nu, \sigma_1) = \lim_{\varepsilon \rightarrow 0} H(\varepsilon, \sigma_1, \nu).$$

If $E \subset X$, we call $Sep(E, n, \varepsilon, y)$ the maximal number of points in $E \cap \pi^{-1}(y)$ which are ε -separated or, also, the points in E relative to the fiber $\pi^{-1}(y)$. Thus, setting

$$H(E, n, \varepsilon, y, \sigma_1) = Sep(E, n, \varepsilon, y)$$

and

$$H(E, y, \varepsilon, \sigma_1) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log H(E, n, \varepsilon, y, \sigma_1),$$

we have

$$H(E, \varepsilon, \sigma_1, \nu) = \int_{X_2} H(E, y, \varepsilon, \sigma_1) d_\nu(y).$$

The *entropy of the set E* , relative to the the measure ν , is the defined

$$(2.5) \quad h(E, \nu, \sigma_1) = \lim_{\varepsilon \rightarrow 0} H(E, \varepsilon, \sigma_1, \nu).$$

If β is the constant of expansiveness for the shift σ_1 then, by a property of entropy of shifts, holds

$$H(E, \varepsilon, \sigma_1, \nu) = h(E, \nu, \sigma_1) \quad \text{for } \varepsilon \in (0, \beta/2).$$

Let $u = u_1 u_2 \dots$ be a word of length n in X or Y , the corresponding *cylinder of length n* is

$$\mathcal{C}(u_1 u_2 \dots u_n) = \{v : u_i = v_i : i = 1, 2, \dots, n\}$$

A word $u_1 u_2 \dots u_n$ is called *admissible* when $\mathcal{C}(u_1 u_2 \dots u_n) \neq \emptyset$. Let us denote by $\mathcal{L}_n(X)$ the set of admissible words of length n on X . A language on X is the set

$$\mathcal{L}(X) = \bigcup_{n \geq 1} \mathcal{L}_n(X).$$

In the same way is defined $\mathcal{L}(Y)$ as a language on Y .

A *regular language* is that which can be recognized by a finite automaton. A subshift with a regular language is called a sofic subshift.

Definition: The shift space (X, σ_1) is *irreducible* if given two words $x_1^1 x_2^1 \dots x_n^1$ and $x_1^2 x_2^2 \dots x_m^2 \in \mathcal{L}(X)$ there is third word $w_1 w_2 \dots w_k$ such that the concatenation $x_1^1 x_2^1 \dots x_n^1 w_1 w_2 \dots w_k x_1^2 x_2^2 \dots x_m^2 \in \mathcal{L}(X)$.

Definition: A shift space (X, σ_1) satisfies the *weak-specification property* if for some natural p holds the following: given two words $x_1^1 x_2^1 \dots x_n^1$ and $x_1^2 x_2^2 \dots x_m^2 \in \mathcal{L}(X)$ there is a word $w_1 w_2 \dots w_k \in \cup_{i=0}^p \mathcal{L}_i(X)$ such that the concatenation $x_1^1 x_2^1 \dots x_n^1 w_1 w_2 \dots w_k x_1^2 x_2^2 \dots x_m^2 \in \mathcal{L}(X)$.

In particular irreducible shifts verify the weak-specification property.

Definition: A dynamical system (X, f) , with (X, d) a compact metric space, satisfies the *specification property* (introduced by Bowen) if for any $\delta > 0$ there is an integer $m = m(\delta)$ such that for any sequence of integer intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2], \dots, I_r = [a_r, b_r], \dots$ with $\text{dist}(I_r, I_s) > m = m(\delta)$ and for any points $x_1, x_2, \dots, x_r, \dots \in X$ there exists a point $z \in X$ such that $d(f^{s+a_r}(z), f^{a_r}(x_r)) < \delta$ with $s = 0, 1, \dots, |I_r|$ for any r .

A factor code $\pi : X \rightarrow Y$ is *finite-to-one* if $\text{card} \pi^{-1}(y) < \infty$. A point $y \in Y$ is *double transitive* if both orbits, back and forward, of y are dense in Y .

The degree of π finite-to-one, denoted by $\deg(\pi)$, is the unique number d , such that $\text{card} \pi^{-1}(y) = d$, for any double transitive point $y \in Y$.

Denote $x \rightarrow \bar{x}$ when $\pi(x) = \pi(\bar{x}) = y$, for some $y \in Y$, and such that for any natural n there is a $\bar{\bar{x}} \in \pi^{-1}(y)$ with $\bar{\bar{x}}_i = x_i$, for $i < n$ and $\bar{\bar{x}}_j = x_j$ for $j > n$. We say that x and $\bar{x}$ are equivalent, denoted by $x \sim \bar{x}$ if $x \rightarrow \bar{x}$ and $\bar{x} \rightarrow x$. The equivalence classes for this relation on X are called the *transition classes*. This concept was introduced by Allahbakhshi and Quas (2013). The transition classes over y are the transition classes in $\pi^{-1}(y)$. Since the number of transition classes over y is finite for each y , it can be defined the so called *class degree* of a factor code $\pi : X \rightarrow Y$ by the number

$$C_\pi = \min_{y \in Y} \text{number of transition classes over } y.$$

By the decomposition theorem of Yoo (2018), for each factor code $\pi : X \rightarrow Y$ with X an irreducible finite type subshift and Y a sofic one, there exists an irreducible subshift $(\bar{Y}, \bar{\sigma})$ such that π can be decomposed as $\pi = \pi_1 \circ \pi_2$, with $\pi_1 : X \rightarrow \bar{Y}$, $\pi_2 : \bar{Y} \rightarrow Y$ and where $C_{\pi_1} = 1$ and π_2 a finite-to-one factor code with $C_{\pi_2} = C_\pi$.

Yoo theorem is a key argument to prove the existence and uniqueness of measures of maximal entropy relative, as we will see in the next section, outlining the original proof (Yoo, 2018).

3. PROOF OF THE THEOREM 1

Let $\mathcal{M}_{rel}(X, \sigma_1, \nu)$ be the subset of $\mathcal{M}(X, \sigma_1)$ of the measures of maximal entropy relative to ν . Since the subshift X is irreducible, the map $\mu \mapsto h_\mu(\sigma_1)$ is upper-semicontinuous and the fiber $\pi_*(\nu)$ is compact, we have that $\mathcal{M}_{rel}(X, \sigma_1, \nu) \neq \emptyset$.

Here we indistinctly denote the metric in X or Y by d . Let $\pi : X \rightarrow Y$ be a finite-to-one factor code, if ν is the unique measure of maximal entropy for (Y, σ_2) , then there is an unique measure $\mu \in \mathcal{M}_{rel}(X, \sigma_1, \nu)$.

Let also $\bar{\nu} = \pi_{1*}(\mu)$, we have that $\nu = \pi_*(\mu) = \pi_{2*}(\pi_{1*})(\mu) = \pi_{2*}(\bar{\nu})$. Since $\bar{\nu}$ projects on ν by π_2 and this code is finite-to-one, $\bar{\nu}$ is the unique measure in Y relative to ν . Therefore

$$(3.1) \quad \mu \in \mathcal{M}_{rel}(X, \sigma_1, \nu) \text{ if and only if } \mu \in \mathcal{M}_{rel}(X, \sigma_1, \bar{\nu}).$$

The uniqueness is ensured because $C_{\pi_1} = 1$.

Step 1

The first part of the proof of the theorem consists in showing that: for any $\alpha_0 > 0$ and for any $t \in (0, h(\nu, \sigma_1))$ there is a compact invariant set $\Lambda \subset X$ such that

$$(3.2) \quad H(\Lambda, \gamma, \sigma_1, \nu) - t > \alpha_0 \text{ for } \gamma > 0.$$

We assume that $0 < t - \alpha_0 < t + \alpha_0 < h(\nu, \sigma_1)$. Let: $\alpha = \{\alpha_0/5, t/5\}$, $\delta > 0$, $m = m(\delta)$ the integer of the specification property of the shift (X, σ_1) and $\gamma > 0$ with $\delta < \gamma/2 < \beta/4$, where β is the constant of expansiveness for the shift σ_1 .

By the definition of relative entropy there is a constant C_0 and an integer N_1 such that

$$(3.3) \quad C_0 \exp(H(n, 2\gamma, y, \sigma_1)) > Sep(n, 2\gamma, y) \text{ for any } n > N_1 \text{ and } y \in Y.$$

For a natural number τ let us consider the symbolic space

$$S_\tau = \{(r_l) : r_l \in \{1, 2, \dots, \tau\}\}$$

We shall assume that the systems considered satisfy specification property with $(r_l = |I_l|) \in S_\tau$ for some natural τ , and where the I_l are the intervals given for the specification definition. Also we may assume that $|I_l| > N_0$, for some integer N_0 .

Let us choose an integer $N > \max(N_0, N_1)$, and such that be valid

$$C_0 \exp(N\alpha) > 1; \quad C_0 \exp(2N\alpha) > 2$$

and

$$N > \frac{t\tau}{\alpha} > \tau; \quad N(t - \alpha) > (n + \tau)(t - 2\alpha).$$

Let $F \subset X$ with points which are 2γ -separated, thus for any $y \in Y$ holds

$$(3.4) \quad \text{card}(F \cap \pi^{-1}(y)) > C_0 \exp(N(t + \alpha_0)) > \exp(N(t + \alpha)).$$

Then it can be found a set $E \subset F$ such that

$$(3.5) \quad \exp(N(t - \alpha)) < \text{card}(E \cap \pi^{-1}(y)) < \exp(N(t + \alpha)).$$

Now, for a sequence (I_l) of integer intervals, take

$$a_l = (l + 1)N + \sum_{i=1}^{l-1} |I_i|.$$

Let $Z((x_l), (r_l))$ be the set formed by all points $z = z((x_l), (r_l)) \in X$ given by the specification property for the points x_l and the intervals I_l of length r_l with constant of specification δ . Let us consider the symbolic space

$$S_E = \{(x_l) : x_l \in E\}$$

and define

$$Z = \bigcup \{Z((x_l), (r_l)) : (r_l) \in S_\tau; (x_l) \in S_E\}.$$

We need to prove that $Z \subset X$ is compact. For this let $z \in \overline{Z}$ (the closure), then there is sequence $(z_j) \subset Z$ such that $\lim_{j \rightarrow \infty} (z_j) = z$.

The are sequences $(r_l)_j$ in S_τ and $(x_l)_j$ in S_E with $z = z((x_l)_j, (r_l)_j)$. The spaces S_τ and S_E are symbolic spaces in finite alphabets so they are compact in the product discrete topology, in this way there is a sequence (j_m) such that

$$\lim_{m \rightarrow \infty} (x_l)_{j_m} = (\overline{x_l}) \quad \text{and} \quad \lim_{m \rightarrow \infty} (r_l)_{j_m} = (\overline{r_l}).$$

Thus we have, for any l

$$(3.6) \quad d(\sigma_1^{s+a_l}(z), \sigma(\overline{x_l})) = \lim_{m \rightarrow \infty} d(\sigma_1^{s+a_l}(z_{j_m}), \sigma((\overline{x_l})_{j_m})) < \delta.$$

Hence $z \in Z$ and thus Z is closed. Since X is compact Z is compact too. Then set

$$\Lambda = \bigcup_{i=0}^{N+\tau-1} \sigma_1^{-i}(Z)$$

Now we consider sequences (x_l) and $(\overline{x}_l)$ of points of E with $x_l \neq \overline{x}_l$. We shall see the associated points $z((x_l))$ and $z((\overline{x}_l))$ are γ - separated in the metric $d_{l(N+\tau)}$ for each l . We have that

$$a_l + N \leq (l-1)(N+\tau) + N < lN + \tau$$

and, by the specification property,

$$(3.7) \quad d_N(\sigma_1^{a_l}(z((x_l))), x_l) < \delta \quad \text{and} \quad d_N(\sigma_1^{a_l}(z((\overline{x}_l))), \overline{x}_l) < \delta.$$

Recall also that any pair of points $x_l, \overline{x}_l$ is 2γ -separated in the metric d_N .

Let $(y_1, y_2, \dots, y_k) \in E \cap \pi^{-1}(y)$, $(s_1, s_2, \dots, s_k) \in \mathbb{Z}^k$, $s_i < \tau$, $y \in Y$ and let

$$C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)) = \{((x_l), (r_l)) : x_l = y_l, r_l = s_l\}$$

where $l = 1, 2, \dots, k$. Thus we put

$$\begin{aligned} & Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))) \\ &= \bigcup \{Z((x_l), (r_l)) : ((x_l), (r_l)) \in C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))\}. \end{aligned}$$

Let

$$Y_k = \{((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)) : Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))) \neq \emptyset\},$$

so that

$$\begin{aligned} \text{card}(Y_k) &\geq \int_Y \frac{\text{card}\{(y_1, y_2, \dots, y_k) \in (E \cap \pi^{-1}(y))\}}{\tau^{k-1}} d\nu(y) \\ &\geq \int_Y \frac{\text{card}\{(E \cap \pi^{-1}(y))\}}{\tau^{k-1}} d\nu(y) \\ (3.8) \quad &\geq \exp\left(\frac{k(t-\alpha)}{\tau^{k-1}}\right). \end{aligned}$$

Recall that the points $z((x_l))$ and $z((\overline{x}_l))$ are γ - separated in the metric $d_{l(N+\tau)}$ for each l . Thus

$$\begin{aligned} H(Z, \gamma, \sigma_1, \nu) &= \int_Y \limsup_{k \rightarrow \infty} \frac{1}{k((N+\tau))} \log H(Z, k((N+\tau)), \gamma, y, \sigma_1) d\nu(y) \\ &\geq \frac{1}{k((N+\tau))} [N(t-\alpha) - N\alpha - \log t] \\ (3.9) \quad &> t - 2\alpha > t - \alpha_0. \end{aligned}$$

By the construction of Λ and Z we obtain

$$(3.10) \quad h(\Lambda, \nu, \sigma_1) - t > \alpha_0.$$

Step 2

We shall prove that for $\alpha_0 > 0$, $\delta > 0$ and for any $t \in (0, h(\nu, \sigma_1))$ holds eq.(3.2):

$$H(\Lambda, 2\delta, \sigma_1, \nu) - t < \alpha_0.$$

Let $\delta > 0$ and let $E \subset X$ be a set with points δ -separated in the metric d_τ . Since X is compact, E is finite. Set

$$(3.11) \quad T(\delta, \tau, y) = \text{Sep}(\delta, \tau, y)$$

and

$$(3.12) \quad T(E, \delta, \tau, y) = \text{Sep}(E, \delta, \tau, y).$$

If $B_{\delta, \tau}(x)$ is the ball of center x and radius δ in the metric d_τ , then we can express

$$X = \bigcup_{x \in E} B_{\delta, \tau}(x).$$

For given $(y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)$ let consider a set $Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)))$ and define

$$\Omega_i = \begin{cases} \sigma_1^{i-t_l}(y_l) & \text{if } 0 < i - t_l < N \quad \text{for some } l \\ \sigma_1^{i-t_l-N}(E) & \text{if } t_l + N < i < t_{l+1} \quad \text{for some } l, \end{cases}$$

where $t_l = \sum_{i=1}^{l-1} (N + s_i)$, and

$$\Omega = \{(z_1, z_2, \dots, z_{t_l+N-1}) : z_i \in \Omega_i, i = 1, 2, \dots, t_l + N - 1\}.$$

We have $i - t_l - N < \tau$ since $t_{l+1} = t_l + N + s_l$.

Let F be a set with points 2δ -separated in the metric d_{t_l+N} and

$$F \subset Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))).$$

Since X is covered by balls $B_{\delta, \tau}$, it holds that for any $l = 1, 2, \dots, k-1$ and for any $w \in F$ there is a $\nu_l \in F$ such that $d(\sigma_1^{t_l+j}(w), \nu_l) < \delta$, for any $j = 1, 2, \dots, k-1$.

Now define a map $g : F \rightarrow \Omega$ by the condition $d(\sigma_1^{t_l+j}(w), g(w)) < \delta$, $j = 1, 2, \dots, t_l + N - 1$. Since F has points 2δ -separated in the metric d_{t_l+N} , this map is injective. Then $F \cap \pi^{-1}(y)$ has points in the metric d_{t_l+N} in $Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)))$ for any $y \in Y$. Hence

$$(3.13) \quad \text{card}(F \cap \pi^{-1}(y)) \leq \text{card}\Omega \leq T(\delta, \tau, y)^{k-1},$$

for $t_l + N > lN$.

Thus, the number of points of X in $Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)))$ which are 2δ -separated in the metric d_{kN} for any k -tuples $(y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)$ is at most $T(\delta, \tau, y)^{k-1}$.

Let $S_k = \left\lfloor \frac{(k-4)N}{N+\tau} \right\rfloor$ and let F be a set with points which are 2δ -separated in the metric $d_{(k-3)N+r}$, $r < N$, relative to $\pi^{-1}(y)$ and

$F \subset \sigma_1^s(Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))))$, for given k -uples $(y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)$ and $s = 1, 2, \dots, N + \tau - 1$. Then $\sigma_1^{-s}(F) \cap \pi^{-1}(y)$ contains points 2δ -separated in the metric $d_{(k-3)N+r+s}$ in $\sigma_1^s(Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))))$. We have $(k-3)N + r + s \leq kN$, thus

$$T(\sigma_1^s(Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))))), 2\delta, (k-3)N + r, y) \leq T(\delta, \tau, y)^{k-1}.$$

Let $u \in \sigma_1^s(Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))))$, so there is a point $z \in Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k)))$ such that $\sigma_1^s(u) = z$ and since, by the definition S_k , holds

$$t_{S_k} + N \leq (k-3)N + r \quad \text{and} \quad k - S_k \leq k + \frac{k\tau}{N}.$$

Then we have

$$\begin{aligned} & H(\sigma_1^s(Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))))), (k-3)N + r, 2\delta, y, \sigma_1) \\ (3.15) \quad & \leq T(\delta, \tau, y)^{k-1} \exp(2kN\alpha). \end{aligned}$$

Therefore

$$\begin{aligned} & H(\Lambda, (k-3)N + r, 2\delta, y, \sigma_1) \\ & \leq \sum_{s=0}^{N+\tau-1} \sum H(\sigma_1^s(Z(C((y_1, y_2, \dots, y_k), (s_1, s_2, \dots, s_k))))), (k-3)N + r, 2\delta, y, \sigma_1) \\ (3.16) \quad & \leq (N + \tau) \tau^{k-1} T(\delta, \tau, y)^{k-1} \exp[2kNt + (k-2)N(t + \alpha)]. \end{aligned}$$

where the second sum is over $(y_1, y_2, \dots, y_k) \in E \cap \pi^{-1}(y)$ and $(s_1, s_2, \dots, s_k) \in \mathbb{Z}^k$.

If $b > 10N$ then there is a $k > 10$ and $r \in \{0, 1, \dots, N-1\}$ such that b can be expressed as $b = k - 3N + r$. Thus, since

$$\exp(N(t - \alpha)) < \text{card}(E \cap \pi^{-1}(y)) < \exp(N(t + \alpha)),$$

we get

$$\begin{aligned}
 H(\Lambda, 2\delta, \sigma_1, \nu) &= \int_Y \limsup_{b \rightarrow \infty} \frac{1}{b} \log H(\Lambda, b, 2\delta, y, \sigma_1) d\nu(y) \\
 &\leq \int_Y \limsup_{k \rightarrow \infty} \frac{1}{(k - 3N + r)} \log H(\Lambda, (k - 3N + r), 2\delta, y, \sigma_1) d\nu(y) \\
 (3.17) \quad &\leq \frac{\log \tau}{N} + \frac{T(\delta, \tau, y)}{N} + 2t + (t + \alpha) < 3t + (t + \alpha) < t + \alpha_0.
 \end{aligned}$$

Finally to demonstrated the theorem we must prove that for any $\alpha_0 > 0$, any $t \in (0, h(\nu, \sigma_1))$ and a fixed measure $\nu \in \mathcal{M}(Y, \sigma_2)$, there is a measure $\mu \in \mathcal{M}(X, \sigma_1)$ with $\pi_*(\mu) = \nu$ such that $|\mathcal{F}_\nu(\mu) - t| < \alpha_0$.

In the steps 1 and 2 we have obtained that for any $\alpha_0 > 0$ and any $t \in (0, h(\nu, \sigma_1))$ there is a compact invariant set $\Lambda \subset X$ such that

$$(3.18) \quad -\alpha_0 < H(\Lambda, 2\delta, \sigma_1, \nu) - t < \alpha_0.$$

Since $2\delta < \frac{\beta}{2}$ we have that

$$(3.19) \quad H(\Lambda, 2\delta, \sigma_1, \nu) = h(\Lambda, \nu, \sigma_1).$$

Let us consider an interval $(t - \alpha_0, t + \alpha_0) \subset [0, h(\nu, \sigma_1)]$, then we have by eqs.(3.18-3.19)

$$(3.20) \quad h(\Lambda, \nu, \sigma_1) \in (t - \alpha_0, t + \alpha_0).$$

By the relative variational principle there is a measure $\mu \in \mathcal{M}(X, \sigma_1)$ supported on Λ and such that

$$(3.21) \quad \mathcal{F}_\nu(\mu) = h(\Lambda, \nu, \sigma_1) \in (t - \alpha_0, t + \alpha_0),$$

and the theorem is proved □

REFERENCES

- [1] M. Allahbakhshi and A. Quas, Class degree and relative maximal entropy, Trans. Amer. Math. Soc. 365 (2013) 1347-1368.
- [2] J. Barral and D.J. Feng, Weighted thermodynamic formalism on subshifts and applications, Asian J. Math. 16 (2012) 319-352.
- [3] V. Climenhaga and D.J. Thompson. Unique equilibrium states for flows and homeomorphisms with non-uniform structure, Adv. Math. 303 (2016) 744-799.

- [4] D.J. Feng, Equilibrium states for factor maps between subshifts, *Adv. Math.* 226 (2011) 2370-2502.
- [5] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publ. Math. I.H.E.S.*, (1980) 137-173.
- [6] L. Guan, P. Sun and W. Wu, Measures of intermediate entropies and homogeneous dynamics, *Nonlinearity* , 30 (9) (2017) 3349-3361.
- [7] F. Ledrappier and P. Walters, A relativised variational principle for continuous transformations, *J. London Math. Soc* , 2 (16) (1977) 568-576.
- [8] M. Li, Y. Shi, S. Wang and X. Wang, Measures of intermediate entropies for star vector fields, *Israel Journal of Mathematics*, 240 (2020) 701-819.
- [9] P. Sun, Equilibrium states of intermediate entropies, *Dynam. Sys, An International Journal* (2020a) <https://doi.org/10.1080/14689367.2020.1823320>.
- [10] P. Sun, Denseness of intermediate pressures for systems with the Climenhaga-Thompson structures, *J. Math Anal Appl* 487 (2020b) 4-17.
- [11] J. Yoo, Decomposition of infinite-to-one factor codes and uniqueness of relative equilibrium states, *J. Mod. Dynam.*, 13 (2018) 271-284.