

RATIONAL PERIODIC SOLUTIONS ON SOME GENERALIZED ABEL EQUATIONS

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ABSTRACT. In this paper we deal with the equations $a(x)dy/dx = A(x)y^2 + B(x)y^3$, where $a(x)$, $A(x)$ and $B(x)$ are complex polynomials with $a(x)B(x) \not\equiv 0$ and $a(x)$ non-constant. First we show that the unique rational limit cycles that these equations can have are of the form $y = 1/p(x)$ being $p(x)$ some polynomial. Second we provide an upper bound on the number of these rational limit cycles. Moreover, we prove that if $\deg(B(x)) - \deg(a(x)) + 1$ is odd, or $\deg(A) > (\deg(B(x)) + \deg(a(x)) - 1)/2$, then these Abel equations have at most two rational limit cycles and we provide examples of these Abel equations with three nontrivial rational periodic solutions.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

We study the generalized Abel equations of the form

$$(1) \quad a(x) \frac{dy}{dx} = A(x)y^2 + B(x)y^3,$$

where x, y are complex variables and $a(x)$, $A(x)$ and $B(x)$ are polynomials in $\mathbb{C}[x]$ (we recall that $\mathbb{C}[x]$ denotes the set of all real polynomials in the variable x). Moreover $a(x)B(x) \not\equiv 0$ and $a(x)$ non constant. The periodic solutions of these equations have been intensively investigated mainly when $a(x)$ is constant and the functions $A(x)$ and $B(x)$ are periodic, specially when they are real polynomials (see for instance [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 26, 27, 28, 29]). In this paper we are interested in the rational periodic solutions of equation (1) when $a(x), A(x), B(x) \in \mathbb{C}[x]$ with $a(x)B(x) \not\equiv 0$.

A *periodic solution* of equation (1) is a solution $y(x)$ defined in the closed interval $[0, 1]$ such that $y(0) = y(1)$ (without loss of generality we are assuming that the period is 1).

The periodic solution is called a *polynomial periodic solution* if $y(x)$ is a polynomial in the variable x . In particular the authors of [15] proved that any polynomial periodic solution of system (1) when $a(x) = 1$ is of the form $y = c$ with $c \in \mathbb{C}$, and that if a polynomial periodic solution exists with $c \neq 0$, then no other polynomial periodic solutions can exist.

In this paper we want to consider the existence of *rational periodic solutions* for system (1), i.e. we want to consider periodic solutions of the form $y(x) = q(x)/p(x)$ where $p, q \in \mathbb{C}[x]$ and $(p(x), q(x)) = 1$. Here we will study only the rational periodic solutions that are not polynomial periodic solutions.

When $a(x)$ is constant (that without loss of generality we can take $a(x) = 1$) this equation was studied in [19, 9]. In [9] the authors, following some ideas in [19], they obtain an upper bound on the number of rational solutions that an Abel equation of the form (1) with $a(x) \in \mathbb{C}$ can have, and show that in some cases this upper bound is two. In [19] the authors provide many classes of Abel equations (with $a(x) \in \mathbb{C}$) in which the maximum number of rational isolated periodic solutions is two providing examples of these systems. However although in [19] the authors state that all Abel equations with $a(x) \in \mathbb{C}$ have at most two rational isolated periodic solutions, this is not correct since there is a mistake in the proof causing that in fact the authors do not study all the possible situations.

In the present paper we restrict to the case in which $a(x) \in \mathbb{C}[x] \setminus \mathbb{C}$ and we follow closely the ideas in [19] and [9].

Our main theorem is the following one.

Theorem 1.1. *The following holds for system (1) with $a(x) \in \mathbb{C}[x] \setminus \mathbb{C}$.*

- (i) *All rational limit cycles are of the form $y = 1/p(x)$ with $p \in \mathbb{C}[x] \setminus \mathbb{C}$;*
- (ii) *If $\deg(B(x)) - \deg(a(x)) + 1$ is odd, or $\deg(A) > (\deg(B(x)) + \deg(a(x)) - 1)/2$ it has at most two rational limit cycles;*
- (iii) *Otherwise, the maximum number of rational limit cycles that can have is*

$$\left(\frac{\deg(B(x))}{\deg(B(x)) + \deg(a(x)) + 1} \right) + 1.$$

The proof of Theorem 1.1 is given in section 2. Similar results and proofs to Theorem 1.1 can be done for other equations such as generalized Bernoulli polynomial equations and Abel trigonometric polynomial equations (see [24, 25], respectively).

Now we provide a generalized Abel polynomial equation (1) with three rational periodic solutions taking

$$a(x) = 18 - 8x + 9x^2 - 2x^3 + x^4,$$

and

$$\begin{aligned} A(x) &= -2(-1 + 2x)(3806 - 5874x + 9015x^2 - 6997x^3 + 5346x^4 \\ &\quad - 2385x^5 + 1075x^6 - 240x^7 + 60x^8), \\ B(x) &= 4(-1 + 2x)(1 - x + x^2)(4 - x + x^2)(7 - 8x + 8x^2)(26 - 18x \\ &\quad + 21x^2 - 6x^3 + 3x^4)(100 - 71x + 83x^2 - 24x^3 + 12x^4). \end{aligned}$$

It is straightforward to check that equation (1) has the three rational periodic solutions $y_i(x) = 1/p_i(x)$ for $i = 1, 2, 3$ with

$$\begin{aligned} p_1(x) &= (7 - 8x + 8x^2)(26 - 18x + 21x^2 - 6x^3 + 3x^4), \\ p_2(x) &= 2(1 - x + x^2)(4 - x + x^2)(7 - 8x + 8x^2), \\ p_3(x) &= 2(1 - x + x^2)(100 - 71x + 83x^2 - 24x^3 + 12x^4). \end{aligned}$$

Note that $p_i(0) = p_i(1)$ and $p_i(x) \neq 0$ for $x \in [0, 1]$. The generalized Abel polynomial system that we have constructed has three rational periodic solutions.

2. PROOF OF THEOREM 1.1

We start with three auxiliary results that will be used through the proof.

Proposition 2.1. *The rational function $y = q(x)/p(x)$ with $p(x)$ non-constant is a periodic solution of system (1) if and only if $q(x) = c$, with $c \in \mathbb{C} \setminus \{0\}$, $p(0) = p(1)$ and $p(x)$ has no zero in $[0, 1]$ and*

$$(2) \quad a(x)p'(x) + A(x) + c \frac{B(x)}{p(x)} = 0.$$

Proof. For the reverse implication we note that if $q(x) = c$ with $c \in \mathbb{C} \setminus \{0\}$, $p(0) = p(1)$, $p(x)$ has no zero in $[0, 1]$ and equality (2) holds then it is clear that the rational function $y = 1/p(x)$ is a periodic solution of system (1).

For the direct implication we note that if $y(x) = q(x)/p(x)$ is a periodic solution of system (1) then $p(x) \neq 0$ for $x \in [0, 1]$. Let $g(x, y) = p(x)y - q(x)$ and $h(x, y) = a(x)g(x, y)$. Then

$$\begin{aligned} 0 &= \frac{dh(x, y)}{dx} \Big|_{g(x, y)=0} = a(x)p'(x)y + a(x)p(x) \frac{dy}{dx} - a(x)q'(x) \\ &= a(x)p'(x)y + p(x)(A(x)y^2 + B(x)y^3) - a(x)q'(x). \end{aligned}$$

Note that $g(x, y)$ is irreducible, so there exists a polynomial $k(x, y)$ so that

$$(3) \quad a(x)p'(x)y + p(x)(A(x)y^2 + B(x)y^3) - a(x)q'(x) = k(x, y)g(x, y).$$

Since the highest degree in y in the left-hand side is 3 and the highest degree in y in $g(x, y)$ is 1 we get that the highest degree in y in $k(x, y)$ is 2 and so it can be written as $k(x, y) = k_0(x) + k_1(x)y + k_2(x)y^2$, where $k_0, k_1, k_2 \in CR[x]$. Comparing the coefficients of y^0 , y^1 , y^2 and y^3 in (3) we get

$$\begin{aligned} (4) \quad &a(x)q'(x) = k_0(x)q(x), \\ &a(x)p'(x) = k_0(x)p(x) - k_1(x)q(x), \\ &p(x)A(x) = k_1(x)p(x) - k_2(x)q(x), \\ &p(x)B(x) = k_2(x)p(x). \end{aligned}$$

From the first relation in (4) we get that

$$q(x) = \kappa \exp \left(\int \frac{k_0(x)}{a(x)} dx \right) \quad \text{where } \kappa \in \mathbb{C}.$$

If $\deg(k_0(x)) \geq \deg(a(x))$, then we make the Euclidean division and we get

$$k_0(x) = \alpha(x)a(x) + \beta(x)$$

where $\deg(\beta(x)) \leq \deg(a(x))$. Therefore we have

$$\frac{k_0(x)}{a(x)} = \alpha(x) + \frac{\beta(x)}{a(x)}.$$

Hence

$$(5) \quad q(x) = \kappa \exp(\alpha(x)) \exp\left(\int \frac{\beta(x)}{a(x)} dx\right).$$

The first factor in (5) cannot cancel with the second factor of (5) and this gives a contradiction because $q(x)$ is a polynomial. So, $\deg(k_0(x)) < \deg(a(x))$. Now we write $H(x) = (k_0(x), a(x))$. Then

$$\frac{k_0(x)}{a(x)} = \frac{H(x)\bar{k}_0(x)}{H(x)\bar{a}(x)} = \frac{\bar{k}_0(x)}{\bar{a}(x)}.$$

Assume first that $\bar{a}(x)$ is not-square free. Using the affine transformation $x \mapsto x + \alpha$ with $\alpha \in \mathbb{C}$ (if necessary) we can write $\bar{a}(x) = x^\mu r(x)$ where $\mu > 1$ and $r(0) \neq 0$. Moreover, $\bar{k}_0(0) \neq 0$ because $\bar{k}_0(x)$ and $\bar{a}(x)$ are coprime. If we develop $\bar{k}_0(x)/\bar{a}(x)$ in simple fractions of x we obtain

$$\frac{\bar{k}_0(x)}{\bar{a}(x)} = \frac{c_\mu}{x^\mu} + \frac{c_{\mu-1}}{x^{\mu-1}} + \cdots + \frac{c_1}{x} + \frac{\alpha(x)}{r(x)}$$

where $\alpha(x)$ is a polynomial with $\deg(\alpha(x)) < \deg(r(x))$ and $c_i \in \mathbb{C}$ for $i = 1, \dots, \mu$. Note that $c_\mu = \bar{k}_0(0)/\bar{a}(0) \neq 0$. Integrating we get

$$\begin{aligned} q(x) &= \kappa_2 \frac{1}{a(x)} \exp\left(\frac{c_\mu}{1-\mu} \frac{1}{x^{\mu-1}}\right) \\ &\quad \times \exp\left(\int \left(\frac{c_{\mu-1}}{x^{\mu-1}} + \cdots + \frac{c_1}{x} + \frac{\alpha(x)}{r(x)}\right) dx\right). \end{aligned}$$

The first exponential factor cannot cancel with any part of the second exponential factor and we get to a contradiction with the fact that $q(x)$ is a polynomial. So, $\bar{a}(x)$ is square-free. Then we have that

$$\int \frac{\bar{k}_0(s)}{\bar{a}(s)} ds = \log h(x), \quad h(x) \in \mathbb{C}[x] \setminus \{0\}.$$

Therefore,

$$\frac{\bar{k}_0(x)}{\bar{a}(x)} = \frac{h'(x)}{h(x)},$$

for some polynomial $h(x)$. Write

$$h = \prod_{i=1}^n f_j(x)^{m_i}$$

for some $f_i(x)$ being coprime and irreducible polynomials and m_i non-negative integers. In this case we have

$$\frac{h'(x)}{h(x)} = \frac{\sum_{i=1}^n m_i f'_i(x) \prod_{j \neq i} f_j(x)}{\prod_{i=1}^n f_i(x)}$$

and so

$$\bar{a}(x) = \prod_{i=1}^n f_i(x), \quad \bar{k}_0(x) = \sum_{i=1}^n m_i f'_i(x) \prod_{j \neq i} f_j(x)$$

implying that

$$\begin{aligned} a(x) &= H(x) \prod_{i=1}^n f_i(x), \\ k_0(x) &= H(x) \sum_{i=1}^n m_i f'_i(x) \prod_{j \neq i} f_j(x), \\ q(x) &= \kappa h(x) = \kappa \prod_{i=1}^n f_j(x)^{m_i}. \end{aligned}$$

Note that since $a(x)$ is square-free, no f_i can divide $H(x)$ and since $H(x) = (a(x), k_0(x))$ we also have that no f_i divides $\sum_{i=1}^n m_i f'_i(x) \prod_{j \neq i} f_j(x)$.

From the fourth relation we obtain $k_2 = B(x)$ and from the third relation we get

$$p(x)A(x) = k_1(x)p(x) - B(x)\kappa \prod_{i=1}^n f_j(x)^{m_i}, \quad k_1(x) = A(x) + \frac{B(x)}{p(x)}\kappa \prod_{i=1}^n f_j(x)^{m_i}.$$

Finally, the second relation becomes

$$\begin{aligned} (6) \quad H(x)p'(x) \prod_{i=1}^n f_i &= H(x)p(x) \sum_{i=1}^n m_i f'_i(x) \prod_{j \neq i} f_j(x) \\ &\quad - \left(A + \frac{B}{p(x)}\kappa \prod_{i=1}^n f_i(x)^{m_i} \right) \prod_{i=1}^n f_j(x)^{m_i}. \end{aligned}$$

Now we claim that

$$(7) \quad q(x) = c \in \mathbb{C}.$$

In order to prove the claim (7) we must show that $m_i = 0$ for $i = 1, \dots, n$. Assume that there exists $i \in \{1, \dots, n\}$ for which $m_i \neq 0$. Then f_i divides then it follows from (6) that f_i must divide $H(x)p(x) \sum_{i=1}^n m_i f'_i(x) \prod_{j \neq i} f_j(x)$. But we have already seen that f_i does not divide $H(x)$ and it also does not divide $\sum_{i=1}^n m_i f'_i(x) \prod_{j \neq i} f_j(x)$ and so it must divide $p(x)$ but this is a contradiction with the fact that $(q, p) = 1$ since f_i would divide both $q(x)$ and $p(x)$. So, $m_i = 0$. In

short we have proved the claim (7) and so $q(x) = c \in \mathbb{C} \setminus \{0\}$. Moreover (6) becomes equation (2) and $y(x) = q(x)/p(x) = c/p(x)$ is a periodic solution which implies that $p(0) = p(1)$ and $p(x)$ has no zero in $[0, 1]$. So, the direct inclusion is proved. This concludes the proof of the proposition. $\square$

In short all the rational limit cycles of (1) are of the form $y(x) = c/p(x)$ and without loss of generality we can assume that $c = 1$ and so all solutions are of the form $y(x) = 1/p(x)$ (that in the paper we write as $1 - p(x)y = 0$ and they satisfy $p(0) = p(1)$, $p(x) \neq 0$ for $x \in [0, 1]$ and (2) becomes

$$(8) \quad a(x)p'(x) + A(x) + \frac{B(x)}{p(x)} = 0.$$

From (8) we must have that $B(x)$ is multiple of $p(x)$ and so $B(x) = p(x)r(x)$ for some polynomial $r(x)$. Therefore, (2) becomes

$$(9) \quad r(x) + a(x)p'(x) + A(x) = 0 \quad \text{and so} \quad p(x) = \kappa - \int \frac{A(s) + r(s)}{a(s)} ds, \quad \kappa \in \mathbb{C}.$$

Assume that equation (1) has two rational limit cycles, $y(x) = 1/p_1(x)$ and $y(x) = 1/p_2(x)$ with $p_1(x), p_2(x) \in \mathbb{C}[x] \setminus \mathbb{C}$. Denote by $q(x) = (p_1(x), p_2(x))$, i.e. the maximum common divisor of the polynomials $p_1(x)$ and $p_2(x)$, and consequently

$$(10) \quad p_1(x) = q(x)s_1(x), \quad p_2(x) = q(x)s_2(x)$$

with $q(x), s_i(x) \in \mathbb{C}[x]$ and $(s_1(x), s_2(x)) = 1$. Note that in view of the above observation we must have that

$$(11) \quad B(x) = q(x)s_1(x)s_2(x)s_3(x)$$

for some $s_3(x) \in \mathbb{C}[x]$.

We take the notation $Q(x) = a(x)q(x)$ and write $Q(x) = \prod_{i=1}^n Q_i(x)^{\delta_i}$ the decomposition of $Q(x)$ into irreducible monic polynomials.

Proposition 2.2. *The following equalities hold*

$$(12) \quad s_1(x) - s_2(x) = \kappa_2 \prod_{i=1}^n Q_i(x)^{m_i}$$

for some constant $\kappa_2 \in \mathbb{C} \setminus \{0\}$ and non-negative integers m_i , for $i = 1, \dots, n$ and

$$(13) \quad s_3(x) = a(x)q'(x) + \prod_{i=1}^n Q_i(x)^{\delta_i-1} \sum_{i=1}^n m_i Q_i'(x) \prod_{j \neq i} Q_j(x).$$

Proof. Note that in view of (9) we have

$$\begin{aligned} q(x)s_1(x) &= \kappa_0 - \int \frac{A(s) + s_2(s)s_3(s)}{a(s)} ds, \\ q(x)s_2(x) &= \kappa_1 - \int \frac{A(s) + s_1(s)s_3(s)}{a(s)} ds, \end{aligned}$$

with $\kappa_0, \kappa_1 \in \mathbb{C}$. Hence,

$$\begin{aligned} a(x)q'(x)s_1(x) + a(x)q(x)s_1'(x) &= -A(x) - s_2(x)s_3(x), \\ a(x)q'(x)s_2(x) + a(x)q(x)s_2'(x) &= -A(x) - s_1(x)s_3(x), \end{aligned}$$

and so

$$\begin{aligned} a(x)q'(x)s_1(x) + a(x)q(x)s_1'(x) - a(x)q'(x)s_2(x) + a(x)q(x)s_2'(x) \\ = -s_2(x)s_3(x) + s_1(x)s_3(x) = (s_1(x) - s_2(x))s_3(x), \end{aligned}$$

which gives

$$a(x)q'(x)(s_1(x) - s_2(x)) + a(x)q(x)(s_1(x) - s_2(x))' = (s_1(x) - s_2(x))s_3(x),$$

that is

$$a(x)q(x)(s_1(x) - s_2(x))' = (s_1(x) - s_2(x))(s_3(x) - a(x)q'(x)).$$

We write $Q(x) = a(x)q(x)$ and so

$$(14) \quad Q(x)(s_1(x) - s_2(x))' = (s_1(x) - s_2(x))(s_3(x) - a(x)q'(x))$$

In particular $s_1(x) - s_2(x)$ divides $Q(x)(s_1(x) - s_2(x))'$. One can write $s_1(x) - s_2(x) = \tilde{Q}(x)h(x)$ where

$$\tilde{Q}(x) = \prod_{i=1}^n Q_i(x)^{m_i}$$

for some $Q_i(x)$ coprime irreducible monic factors of $Q(x)$ and m_i non-negative integers. Note that in particular none of the irreducible factors of $h(x)$ divides $Q(x)$ and so $h(x)$ divides $s_1'(x) - s_2'(x)$.

Since $s_1'(x) - s_2'(x) = \tilde{Q}'(x)h(x) + \tilde{Q}(x)h'(x)$, we have that $h'(x)$ divides $\tilde{Q}(x)h'(x)$. As none of the irreducible factors of $h(x)$ divides $Q(x)$ and $Q(x)$ is a product of some (possibly repeated) factors of $Q(x)$ we have that $h(x)$ divides $h'(x)$ which is only possible if $h(x) = c \in \mathbb{C}$. Then

$$s_1(x) = s_2(x) + c \prod_{i=1}^n Q_i(x)^{m_i}$$

for some non-negative integers m_i for $i = 1, \dots, l$. Moreover as $s_1(x) \neq s_2(x)$ we must have that $c \in \mathbb{C} \setminus \{0\}$.

Write $Q(x) = \prod_{i=1}^n Q_i(x)^{\delta_i}$ as being the decomposition of $Q(x)$ into irreducible monic polynomials. Substituting in (14) and solving in $s_3(x)$ we get

$$\begin{aligned} s_3(x) &= a(x)q'(x) + \prod_{i=1}^n Q_i(x)^{\delta_i} \frac{c \sum_{i=1}^n m_i Q_i'(x) Q_i(x)^{m_i-1} \prod_{j \neq i} Q_j(x)^{m_j}}{c \prod_{i=1}^n Q_i(x)^{m_i}} \\ &= a(x)q'(x) + \prod_{i=1}^n Q_i(x)^{\delta_i-1} \sum_{i=1}^n m_i Q_i'(x) \prod_{j \neq i} Q_j(x). \end{aligned}$$

The proof of the proposition is complete. $\square$

Proposition 2.3. *Equation (1) has at most two invariant curves whose coefficients in the variable y are proportional.*

Proof. Assume that equation (1) has two invariant curves $1 + p(x)y = 0$ and $1 + \kappa p(x)y = 0$ with $\kappa \in \mathbb{C} \setminus \{1\}$. We first show that in this case

$$(15) \quad A(x) = (\kappa + 1)a(x)p'(x) \quad \text{and} \quad B(x) = \kappa a(x)p'(x).$$

Indeed, it follows from (9) that the curve $1 + p(x)y = 0$ is invariant if and only if there exists $r \in \mathbb{C}[x]$ such that

$$(16) \quad B(x) = p(x)r(x) \quad \text{and} \quad A(x) = -r(x) - a(x)p'(x).$$

If $1 + \kappa p(x)y = 0$ is invariant then there exists $\hat{r}(x) \in \mathbb{C}[x]$ such that

$$(17) \quad B(x) = \kappa p(x)\hat{r}(x) \quad \text{and} \quad A(x) = -\hat{r}(x) - a(x)\kappa p'(x).$$

Equating the first identity in (16) and (17) we get $r = \kappa \hat{r}$ and introducing this in the second identities in (16) and (17) and equating them we get that $\hat{r} = a(x)p'(x)$ and so $r(x) = \kappa a(x)p'(x)$.

Conversely, if

$$B(x) = \kappa a(x)p(x)p'(x), \quad A(x) = -(\kappa + 1)a(x)p'(x)$$

then taking $r(x) = \kappa a(x)p'(x)$ one has $B(x) = r(x)p(x)$ and $A(x) = -r(x) - a(x)p'(x)$ and in view of Proposition 2.1 we get that $1 + p(x)y = 0$. On the other hand, taking $r(x) = ap'(x)$ one has $B(x) = \kappa r(x)p(x)$ and $A(x) = -r(x) - \kappa a(x)p'(x)$ and so again in view of Proposition 2.1 we get that $1 + \kappa p(x)y = 0$

also an invariant curve. This completes the proof of (15) and it follows from (15) that there is no $\hat{\kappa} \notin \{1, \kappa\}$ such that $1 + \hat{\kappa}p(x)y = 0$.

Furthermore, if $\kappa \neq -1$ it follows from (15) that

$$p(x) = -\frac{(\kappa + 1)A(x)}{\kappa B(x)}.$$

Therefore, if $1 + \hat{p}(x)y = 0$ and $1 + \hat{\kappa}\hat{p}(x)y = 0$ are two invariant curves of equation (1) then

$$\hat{p}(x) = -\frac{(\kappa + 1)A(x)}{\kappa B(x)} = \left(\frac{(\kappa + 1)\hat{\kappa}}{(\hat{\kappa} + 1)\kappa}\right)p(x).$$

Hence

$$\frac{\hat{\kappa} + 1}{\kappa + 1} = \frac{\kappa(\hat{\kappa} + 1)}{\hat{\kappa}(\kappa + 1)}$$

which yields $\hat{\kappa} = \kappa$ or $\hat{\kappa} = 1/\kappa$ and consequently either $\hat{p}(x) = p(x)$ or $\hat{p}(x) = \kappa p(x)$.

On the other hand, if $\kappa = -1$ then $A(x) = 0$ and $B(x) = -a(x)p'(x)p(x)$ and equation (1) becomes

$$a(x)y' = a(x)p(x)p'(x)y^3 \quad \text{that is} \quad y' = p(x)p'(x)y^3$$

which has the rational first integral

$$\frac{1}{y^2} + p(x)^2 = \left(\frac{1}{y} + p(x)\right)\left(\frac{1}{y} - p(x)\right),$$

and so the unique invariant curves are $\frac{1}{y} + p(x) = 0$ (that is $1 + p(x)y = 0$) and $\frac{1}{y} - p(x) = 0$ (that is $1 - p(x)y = 0$). This completes the proof of the proposition. $\square$

2.1. Proof of Theorem 1.1. Statement (i) of the theorem follows directly from Proposition 2.1.

To prove statements (ii) and (iii) we first claim that equation (1) has more than two invariant curves if and only if the coefficients of y have degree $(\deg(B(x)) - \deg(a(x)) + 1)/2$.

Now we prove the claim. Assume that (1) has two rational solutions $1 + p_1(x)y = 0$ and $1 + p_2(x)y = 0$. Then as it follows from (11) that if we write $p_1(x) = q(x)s_1(x)$ and $p_2(x) = q(x)s_2(x)$, we get $B(x) = q(x)s_1(x)s_2(x)s_3(x)$. Therefore,

$$\begin{aligned} \deg(B(x)) &= \deg(q(x)) + \deg(s_1(x)) + \deg(s_2(x)) + \deg(s_3(x)) \\ &= \deg(q(x)) + \deg(p_1(x)) - \deg(q(x)) + \deg(p_2(x)) \\ &\quad - \deg(q(x)) + \deg(s_3(x)) \\ &= \deg(p_1(x)) + \deg(p_2(x)) - \deg(q(x)) + \deg(s_3(x)). \end{aligned}$$

It follows from (12), that $Q(x) = a(x)q(x)$ and the fact that m_i are non-negative integers, we get that $\deg(s_3(x)) = \deg(a(x)) + \deg(q(x)) - 1$ and so,

$$\deg(B(x)) = \deg(p_1(x)) + \deg(p_2(x)) + \deg(a(x)) - 1$$

yielding

$$(18) \quad \deg(B(x)) - \deg(a(x)) + 1 = \deg(p_1(x)) + \deg(p_2(x)).$$

Assume now that equation (1) has three invariant curves $1 + p_i(x)y = 0$ with $i = 1, 2, 3$. Then it follows from (18) that

$$\deg(p_1(x)) + \deg(p_2(x)) = \deg(p_1(x)) + \deg(p_3(x)) = \deg(p_2(x)) + \deg(p_3(x))$$

yielding

$$\deg(p_1(x)) = \deg(p_2(x)) = \deg(p_3(x)) = (\deg(B(x)) - \deg(a(x)) + 1)/2.$$

Thus, the claim is proved. It follows readily from the claim that if $\deg(B(x)) - \deg(a(x)) + 1$ is odd then equation (1) has at most two invariant curves.

Note that $\deg(p(x)) \neq (\deg(B(x)) - \deg(a(x)) + 1)/2$ is equivalent to either $\deg(p'(x)) > (\deg(B(x)) - \deg(a(x)) + 1)/2$ or $\deg(B(x)) - \deg(p(x)) > (\deg(B(x)) - \deg(a(x)) + 1)/2$ and in view of (9) this is equivalent to

$$\deg(A(x)) > (\deg(B(x)) + \deg(a(x)) - 1)/2.$$

This fact together with the claim yields that if $\deg(A(x)) > (\deg(B(x)) + \deg(a(x)) - 1)/2$ then the degree of the polynomial coefficient of y of no invariant curve of equation (1) can be $(\deg(B(x)) - \deg(a(x)) + 1)/2$ yielding that equation (1) has at most two invariant curves. This completes the proof of statement (ii) in the theorem.

Assume now that $\deg(A(x)) \leq (\deg(B(x)) + \deg(a(x)) - 1)/2$ and $\deg(B(x)) - \deg(a(x)) + 1$ is odd. In this case it follows from the explanation above that for any rational solution $1 + p(x)y = 0$ we have $\deg(p(x)) = (\deg(B(x)) - \deg(a(x)) + 1)/2$. Moreover, in view of (9) we have that $p(x)$ must divide $B(x)$ and so the maximum number of possible rational solutions $1 + p(x)y = 0$ are the possible divisors of $B(x)$ and so there are at most

$$\left(\frac{\deg(B(x))}{\deg(B(x)) + \deg(a(x)) + 1} \right) + 1.$$

up to proportionality. It follows from Proposition 2.3 that there is at most one invariant curve that has a proportional polynomial coefficient of y . Taking this into account and that $p(x)$ is determined in terms of $A(x)$, $B(x)$ and $a(x)$, the proof of the statement (iii) in the theorem is complete. So, the proof of the theorem is also complete.

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