

GLOBAL INVARIANTS OF OBJECTS IN TWO-DIMENSIONAL MINKOWSKI SPACE

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ABSTRACT. In this paper, "the concept of T-object" is defined in two dimensional Minkowski space. By this concept, the complete systems of invariants of m-point, paths and vector fields are investigated as a single theory. Using this approach, we obtain the complete systems of invariants of T-object and the equivalence conditions of two T-objects are given in terms of their global invariants.

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1. INTRODUCTION

For a system consisting of points, paths and surfaces, finding the complete system of invariants is one of the fundamental problems in the invariant theory and it has many applications in computational geometry, computer vision theory, image processing, etc. The goal of the applications such as robot motions, point set registrations, pattern recognition, etc. is the automatic identification of objects. (see some references [26, 28, 29, 30, 31, 32, 33, 34, 35, 36] and references therein). For the identification, it is necessary to find the corresponding relationship between two m -point systems, two paths or two surfaces and obtain the transformation which transforming one into the other. This problem is known in the mathematics literature as the equivalence problem of two m -points, two curves and two surfaces.

Up to now, the complete systems of invariants are investigated separately for m -point and separately for paths in two dimensional Euclidean, pseudo-Euclidean, projective spaces, etc.

From the points of view of a general theory of m -point invariants, for example, a generator system of points (or vectors) is given for the orthogonal group in n -dimensional Euclidean space and similar problem for the Lorentz group is suggested as a problem in [25].

Later on Berger proved that the orbit of m vectors for the orthogonal groups in the Euclidean space is characterized by distances between m -vectors. Moreover, a complete system of relations between the elements of this complete system of invariants is given in [1]. The orbit of m vectors for the orthogonal groups in the Euclidean and pseudo-Euclidean spaces is characterized by their Gram matrices and an additional subspace in [2]. Invariants of m -points for the orthogonal group in the Lorentz space are given in [3]. In [4], using the methods of invariant theory, generating systems of invariants of m -point for all the fundamental groups in Lorentzian geometry are obtained.

Invariants of m -point and equivalence problems for two m -points are also investigated (see some references [5, 6, 7, 11, 12, 13, 37]).

Meanwhile, there are two main methods to study the theory of curves. The first of them is the method of Frenet-Serret. It provides the curvature functions of a curve. But the curvatures give only conditions of local equivalences of curves. Besides,

the local equivalences can be investigated only for the group of rigid motions (For detailed informations, see [8, 10, 14, 15, 16, 17] and references therein).

The other method is a global approach to the theory of curves by using invariant theory. Using this method, the conditions of equivalence for two curves are obtained not only for this rigid motions but for all fundamental groups in the Euclidean, pseudo-Euclidean, similarity, affine spaces, etc.(see [14, 16, 17, 19, 20, 21, 22, 23, 24]).

So we address the problem from a perspective that differ in two ways and we consider the identification problem of two objects under pseudo-Euclidean transformations as a purely mathematical question.

In this paper, firstly, "the concept of T-object" is defined similarly to definition in [10]. By this concept, complete systems of m -point and curves are given as a single theory. Moreover, this approach will obtain the Lorentz transformation which transform one into the other.

Moreover, since the concept of a T-object allows us to cover a very wide class of geometry(a T-object can be a system consisting of points, paths, and surfaces), the complete system of invariants of a T-object are investigated as globally for Euclidean, pseudo-Euclidean spaces, etc.

The concept is following:

Let T be a set such that it has at least two elements. Let $\mathcal{M}$ be two dimensional Minkowski space. Denote by $\Omega(T, \mathcal{M})$ the set of all mappings of the set T to $\mathcal{M}$. An element of $\Omega(T, \mathcal{M})$ will be called a T -object in the space $\mathcal{M}$.

For example, let $\mathbb{N}$ be the set of all natural numbers and $m \in \mathbb{N}, m \geq 1$. Put $T = \{j \in \mathbb{N} \mid 1 \leq j \leq m\}$.

Then, a mapping $\alpha : T \rightarrow \mathcal{M}$ will be called an m -tuple in $\mathcal{M}$. Denote it in the following form $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$. Clearly, if T is a finite set with m elements, then a T-object is called m -tuple (m -point or m vectors).

For second example, let T is an interval (a, b) of $\mathbb{R}$. Then a T-object is called a T-path. A some class of T-paths is called a curve.

For third example, let T is an open subset of $\mathcal{M}$. Then a T- object is called a vector field in $\mathcal{M}$.

These examples show that the paper defines a very broad concept of T-object. It contains all the currently known concepts of geometric objects and shapes (see

some references [8], [18] and [39]). Moreover it contains T-objects, which have not been defined until now.

The structure of the paper is the following. Some known results on the linear representation of the algebra of hyperbolic numbers in two-dimensional Minkowski space are introduced in Section 2. Evident forms of elements of pseudo-orthogonal groups $SO(1, 1)$ and $O(1, 1)$ are given in Section 3. In Section 4, a complete systems of G-invariants of a T-object in $\mathcal{M}$ for the group $G = O^+(1, 1)$ and a complete system of relations between elements of the obtained complete system of invariants are given. In the other sections, similar problems in Section 4 are investigated for the groups $O(1, 1)$, $MO^+(1, 1)$, $MO(1, 1)$.

2. SOME PROPERTIES OF A LINEAR REPRESENTATION OF THE ALGEBRA OF HYPERBOLIC NUMBERS IN TWO-DIMENSIONAL REAL SPACE

Results of this section are known (see [8]).

Denote algebra of hyperbolic numbers by $\mathbb{H}$. Let $h = h_1 + jh_2 \in \mathbb{H}$. Denote by Λ_h the matrix of the form $\begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix}$. Denote by $\Lambda(\mathbb{H})$ the set $\{\Lambda_h | h \in \mathbb{H}\}$. We consider on the set $\Lambda(\mathbb{H})$ following matrix operations: the component-wise addition and the multiplication of matrices. Then $\Lambda(\mathbb{H})$ is an algebra with respect to these operations. In it the unit element is the unit matrix.

Proposition 2.1. *The mapping $\Lambda : \mathbb{H} \rightarrow \Lambda(\mathbb{H})$, where $\Lambda : h \rightarrow \Lambda_h, \forall h \in \mathbb{H}$, is an isomorphism of algebras $\mathbb{H}$ and $\Lambda(\mathbb{H})$.*

Proof. It is obvious. □

Let $a = a_1 + ja_2, b = b_1 + jb_2 \in \mathbb{H}$. Put $\langle a, b \rangle = a_1b_1 - a_2b_2$. Then $\langle a, b \rangle$ is a bilinear form on $\mathbb{R}^2$ and $\langle a, a \rangle = a_1^2 - a_2^2$ is a quadratic form on $\mathbb{R}^2$. For convenience, we denote by $Q(a)$ the quadratic form $\langle a, a \rangle$.

The following propositions are known.

Proposition 2.2. *The following equalities $Q(x) = \det(\Lambda_x)$ and $Q(xy) = Q(x)Q(y)$ hold for all $x = x_1 + jx_2, y = y_1 + jy_2 \in \mathbb{H}$.*

For $x = x_1 + jx_2 \in \mathbb{H}$, we set $\Delta(x) = \bar{x} = x_1 - jx_2$.

Proposition 2.3. *The mapping $x \rightarrow \bar{x}$ is an involution of the algebra $\mathbb{H}$ and the following equalities $x + \bar{x} = 2x_1$, $\langle x, x \rangle = x\bar{x} = \bar{x}x = x_1^2 - x_2^2$, $Q(x) = Q(\Delta(x)) = Q(\bar{x})$ hold for all $x = x_1 + jx_2 \in \mathbb{H}$.*

Proposition 2.4. *Let $x \in \mathbb{H}$. Then the element x^{-1} exists if and only if $Q(x) \neq 0$. In the case $Q(x) \neq 0$, the equalities $x^{-1} = \frac{\bar{x}}{Q(x)} = \frac{\Delta(x)}{Q(x)}$ and $Q(x^{-1}) = \frac{1}{Q(x)}$ hold.*

Put $\mathbb{H}^* = \{x \in \mathbb{H} \mid Q(x) \neq 0\}$. $\mathbb{H}^*$ is a group with respect to the multiplication operation in the algebra $\mathbb{H}$. Denote by $\Lambda(\mathbb{H}^*)$ the set of all matrices Λ_a , where $a \in \mathbb{H}^*$. Since $a \in \mathbb{H}^*$, we have $Q(a) = a_1^2 - a_2^2 \neq 0$ and $Q(a) = \det(\Lambda_a) \neq 0$. Consider elements $a = a_1 + ja_2 \in \mathbb{H}^*$ and $x = x_1 + jx_2 \in \mathbb{H}$ as column vectors $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then the equality $ax = (a_1 + ja_2)(x_1 + jx_2) = (a_1x_1 + a_2x_2) + j(a_1x_2 + a_2x_1)$ has the following form

$$(1) \quad ax = \begin{pmatrix} a_1x_1 + a_2x_2 \\ a_1x_2 + a_2x_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Lambda_a x,$$

where $\Lambda_a x$ is the multiplication of matrices Λ_a and x . Hence $\Lambda_a \in \Lambda(\mathbb{H}^*)$ and the mapping $\Lambda : \mathbb{H}^* \rightarrow \Lambda(\mathbb{H}^*)$, where $\Lambda(a) = \Lambda_a$, is a linear representation of the group $\mathbb{H}^*$ in $\mathbb{R}^2$. $\Lambda(\mathbb{H}^*)$ is a group with respect to the multiplication of matrices.

Put $S(\mathbb{H}^*) = \{x \in \mathbb{H} \mid Q(x) = 1\}$. It is a subgroup of the group $\mathbb{H}^*$. $\Lambda(S(\mathbb{H}^*))$ is a subgroup of the group $\Lambda(\mathbb{H}^*)$ and the mapping $\Lambda : S(\mathbb{H}^*) \rightarrow \Lambda(\mathbb{H}^*)$, where $\Lambda(a) = \Lambda_a$, is a linear representation of the group $S(\mathbb{H}^*)$ in $\mathbb{R}^2$.

3. FUNDAMENTAL GROUPS OF TRANSFORMATIONS OF THE 2-DIMENSIONAL MINKOWSKI SPACE

Let $\mathcal{M}$ be two-dimensional Minkowski space with the Lorentz scalar product $\langle a, b \rangle = a_1b_1 - a_2b_2$, where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$.

Definition 3.1. *A mapping $F : \mathcal{M} \rightarrow \mathcal{M}$ is called pseudo-orthogonal if $\langle Fx, Fy \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{M}$.*

Denote the set of all pseudo-orthogonal transformations of $\mathcal{M}$ by $O(1, 1)$.

The following propositions are well known.

Proposition 3.2. ([8, Proposition 3.4]) *Every pseudo-orthogonal transformation of $\mathcal{M}$ is linear.*

Proposition 3.3. *$O(1, 1)$ is a group with respect to the multiplication operation of matrices.*

Let $a = a_1 + ja_2, b = b_1 + jb_2 \in \mathbb{H}$. Denote the matrix of the bilinear form $\langle a, b \rangle = a_1b_1 - a_2b_2$ by $\Delta = \|\delta_{ij}\|_{i,j=1,2}$, where $\delta_{11} = 1, \delta_{22} = -1, \delta_{12} = \delta_{21} = 0$. By Proposition 5, we can consider an element of $O(1, 1)$ as a 2×2 -matrix. Let $C \in O(1, 1)$, where $C = \|c_{ij}\|_{i,j=1,2}$. Let C^T be the transpose matrix of C . It is known that the equality $\langle Cx, Cy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{H}$ is equivalent to the equality

$$(2) \quad C^T \Delta C = \Delta.$$

This equality implies the following

Proposition 3.4. *Let $C \in O(1, 1)$. Then $\det(C) = 1$ or $\det(C) = -1$.*

We denote by $O^+(1, 1)$ the set $\{C \in O(1, 1) : \det(C) = 1\}$. $O^+(1, 1)$ is a subgroup of $O(1, 1)$. $O(1, 1) = O^+(1, 1) \cup \{C\Delta \mid C \in O^+(1, 1)\}$, where $C\Delta$ is the multiplication of matrices C and Δ .

Theorem 3.5. *The equality $O^+(1, 1) = \Lambda(S(\mathbb{H}^*))$ holds.*

Proof. A proof is given in [8, Theorem 3.7] □

Denote by $MO(1, 1)$ the group of all transformations of $\mathcal{M}$ generated by the group $O(1, 1)$ and all translations of $\mathcal{M}$. Elements of the group $MO(1, 1)$ has the following form $F : \mathcal{M} \rightarrow \mathcal{M}$, where $F(x) = g(x) + a, g \in O(1, 1), a \in \mathcal{M}$. Denote by $MO^+(1, 1)$ the group of all transformations of $\mathcal{M}$ generated by the group $O^+(1, 1)$ and all translations of $\mathcal{M}$. Elements of the group $MO^+(1, 1)$ has the following form $F : \mathcal{M} \rightarrow \mathcal{M}$, where $F(x) = g(x) + a, g \in O^+(1, 1), a \in \mathcal{M}$.

4. COMPLETE SYSTEMS OF G -INVARIANTS OF A T -OBJECT IN $\mathcal{M}$ FOR THE GROUP $O^+(1, 1)$

Let T be a set such that it has at least two elements. Denote by $\Omega(T, \mathcal{M})$ the set of all mappings of the set T to $\mathcal{M}$. An element of $\Omega(T, \mathcal{M})$ will be called a T -object in the space $\mathcal{M}$.

Let G be a subgroup of the group $MO(1, 1)$.

Definition 4.1. Two T -objects $\alpha(t)$ and $\beta(t)$ in $\mathcal{M}$ is called G -equivalent if there exists $g \in G$ such that $\alpha(t) = g\beta(t), \forall t \in T$. In this case, we also write as follows: $\alpha(t) \stackrel{G}{\sim} \beta(t)$.

Definition 4.2. A subset $A \subseteq \Omega(T, \mathcal{M})$ is called G -invariant if $g(\alpha(t)) \in A, \forall \alpha(t) \in A, \forall g \in G$.

Definition 4.3. Let A be a G -invariant subset of $\Omega(T, \mathcal{M})$ and K be a set such that it has at least two elements. A function $f : A \rightarrow K$ is called G -invariant on A if $\alpha \in A, \beta \in A$ and $\alpha(t) \stackrel{G}{\sim} \beta(t)$ implies $f(\alpha) = f(\beta)$.

The set of all G -invariant functions $f : A \rightarrow K$ on the set A denote by $\text{Map}(A, K)^G$.

Example 4.4. By the definitions of the groups $G = O(1, 1), O^+(1, 1)$ we obtain that the quadratic form $Q(x) = \langle x, x \rangle$ and the bilinear form $\langle x, y \rangle$ are G -invariant functions on the set $\mathcal{M}$.

Example 4.5. Denote by $[xy]$ the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{M}$. Using the equality $\det(g) = 1, \forall g \in O^+(1, 1)$, we obtain $[(gx)(gy)] = \det(g)[xy] = [xy], \forall g \in O^+(1, 1)$. This means that $[xy]$ is an $O^+(1, 1)$ -invariant function on the set $\mathcal{M}^2$.

Example 4.6. By definitions of the groups $G = MO(1, 1), MO^+(1, 1)$ we obtain that function $f(x, y) = \langle x - y, x - y \rangle$ is an G -invariant function on the set $\mathcal{M}^2$.

Definition 4.7. (see [38, 1.1]) Let A be a G -invariant subset of $\Omega(T, \mathcal{M})$. A system $\{f_j | j \in J\}$, where $f_j \in \text{Map}(A, K)^G, \forall j \in J$, will be called a complete system of G -invariant functions on A if $\alpha(t), \beta(t) \in A$ and equalities $f_j(\alpha(t)) = f_j(\beta(t)), \forall j \in J$, imply $\alpha(t) \stackrel{G}{\sim} \beta(t)$.

Denote by θ the vector $(0, 0) \in \mathcal{M}$. Let $\alpha(t)$ be a T -object. Denote by $Z[Q(\alpha(t))]$ the set $\{t \in T | Q(\alpha(t)) = \langle \alpha(t), \alpha(t) \rangle = 0\}$. Denote by $\theta_T(t)$ the T -object such that $\theta_T(t) = \theta, \forall t \in T$. Denote by $P(T)$ the set of all subsets of the set T .

Proposition 4.8. Let G be a subgroup of $O(1, 1)$. Assume that $\alpha(t), \beta(t) \in \Omega(T, \mathcal{M})$ such that $\alpha(t) \stackrel{G}{\sim} \beta(t)$. Then $Z[Q(\alpha(t))] = Z[Q(\beta(t))]$. That is the function $Z : \Omega(T, \mathcal{M}) \rightarrow P(T)$ is a G -invariant function on $\Omega(T, \mathcal{M})$.

Proof. It is obvious. □

Proposition 4.9. *Let $\mathbb{H}$ be the algebra of hyperbolic numbers and $x, y \in \mathbb{H}$ such that $Q(x) \neq 0$. Then*

- (1) *The element yx^{-1} exists, the equality $yx^{-1} = \frac{\langle x, y \rangle}{Q(x)} + j \frac{[x, y]}{Q(x)}$ and the following equality hold*

$$(3) \quad \Lambda_{yx^{-1}} = \begin{pmatrix} \frac{\langle x, y \rangle}{Q(x)} & \frac{[x, y]}{Q(x)} \\ \frac{[x, y]}{Q(x)} & \frac{\langle x, y \rangle}{Q(x)} \end{pmatrix}.$$

- (2) *$\det(\Lambda_{yx^{-1}}) \neq 0$ if and only if $Q(y) \neq 0$.*

Proof. It is given Proposition 4.3 in [8]. □

Now we consider the G -equivalence problem of T -objects for the group $O^+(1, 1)$.

Let $\alpha(t)$ and $\beta(t)$ be T -objects such that $\alpha(t) = \beta(t) = \theta = (0, 0), \forall t \in T$. that is $Z[Q(\alpha(t))] = Z[Q(\beta(t))] = T$. Then $\alpha(t) = \beta(t) = \theta_T(t)$. In this case, it is obvious that $\alpha(t) \stackrel{\mathcal{G}}{\sim} \beta(t)$.

Theorem 4.10. *Let $\alpha(t)$ be a T -object in $\mathcal{M}$ such that $Z[Q(\alpha(t))] \neq T$ and $t_0 \in T \setminus Z[Q(\alpha(t))]$.*

- (i) *Suppose that a T -object $\beta(t)$ in $\mathcal{M}$ such that $\alpha(t) \stackrel{O^+(1,1)}{\sim} \beta(t), \forall t \in T$. Then the following equalities hold:*

$$(4) \quad \begin{cases} Z[Q(\alpha(t))] = Z[Q(\beta(t))] \\ \langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle \\ [\alpha(t_0)\alpha(t)] = [\beta(t_0)\beta(t)], \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t))]$.

- (ii) *Conversely, assume that a T -object $\beta(t)$ in $\mathcal{M}$ such that the equalities (4) hold. Then there exists a single matrix $C \in O^+(1, 1)$ such that $\beta(t) = C\alpha(t), \forall t \in T$. In this case, C has the following form*

$$(5) \quad C = \begin{pmatrix} \frac{\langle \alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{[\alpha(t_0)\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} \\ \frac{[\alpha(t_0)\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{\langle \alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} \end{pmatrix},$$

where $\det(C) = 1$.

Proof. Assume that $\alpha(t) \stackrel{O^+(1,1)}{\sim} \beta(t), \forall t \in T$. Then, by Proposition 4.8, $Z[Q(\alpha(t))] = Z[Q(\beta(t))]$. This equality and the inequality $Z[Q(\alpha(t))] \neq T$ imply inequality

$Z[Q(\beta(t))] \neq T$. Since functions $\langle \alpha(t_0), \alpha(t) \rangle$ and $[\alpha(t_0)\alpha(t)]$ are $O^+(1, 1)$ -invariant, the $O^+(1, 1)$ -equivalence $\alpha(t) \stackrel{O^+(1, 1)}{\sim} \beta(t), \forall t \in T$ implies equalities (4).

Conversely, assume that a T -object $\beta(t)$ in $\mathcal{M}$ such that equalities (4) hold. Put $x = \alpha(t_0)$ and $y = \alpha(t)$ in Proposition 4.9. Since $t_0 \in T \setminus Z[Q(\alpha(t))]$, we have $\langle \alpha(t_0), \alpha(t_0) \rangle \neq 0$. Then $(\alpha(t_0))^{-1}$ exists as an element of $\mathbb{H}$. By Proposition 4.9, we obtain following equality $(\alpha(t_0))^{-1}\alpha(t) = \frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} + j \frac{[\alpha(t_0)\alpha(t)]}{Q(\alpha(t_0))}, \forall t \in T$.

Using the above equality $Z[Q(\alpha(t))] = Z[Q(\beta(t))]$ and $t_0 \in T \setminus Z[Q(\alpha(t))]$, we obtain that $t_0 \in T \setminus Z[Q(\beta(t))]$. Put $x = \beta(t_0)$ and $y = \beta(t)$ in Proposition 4.9. Since $t_0 \in T \setminus Z[Q(\beta(t))]$, we have $\langle \beta(t_0), \beta(t_0) \rangle \neq 0$. Then $(\beta(t_0))^{-1}$ exists as an element of $\mathbb{H}$. By Proposition 4.9, we obtain following equality $(\beta(t_0))^{-1}\beta(t) = \frac{\langle \beta(t_0), \beta(t) \rangle}{Q(\beta(t_0))} + j \frac{[\beta(t_0)\beta(t)]}{Q(\beta(t_0))}, \forall t \in T$.

Using equalities (4), equalities $(\alpha(t_0))^{-1}\alpha(t) = \frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} + j \frac{[\alpha(t_0)\alpha(t)]}{Q(\alpha(t_0))}, \forall t \in T$, and equalities $(\beta(t_0))^{-1}\beta(t) = \frac{\langle \beta(t_0), \beta(t) \rangle}{Q(\beta(t_0))} + j \frac{[\beta(t_0)\beta(t)]}{Q(\beta(t_0))}, \forall t \in T$, we obtain following equalities

$$(6) \quad (\alpha(t_0))^{-1}\alpha(t) = (\beta(t_0))^{-1}\beta(t), \forall t \in T.$$

Consider element $g = \beta(t_0)(\alpha(t_0))^{-1} \in \mathbb{H}^*$. Since $\beta(t_0) = \beta(t_0)((\alpha(t_0))^{-1}\alpha(t_0)) =$

$(\beta(t_0)(\alpha(t_0))^{-1})\alpha(t_0)$, we have $\beta(t_0) = g\alpha(t_0)$. Using this equality and (1), we obtain that $\beta(t_0) = \Lambda_g\alpha(t_0)$. Using the equality $\langle \alpha(t_0), \alpha(t_0) \rangle = \langle \beta(t_0), \beta(t_0) \rangle$ in (4), equalities $Q(\alpha(t_0)) = \langle \alpha(t_0), \alpha(t_0) \rangle = \langle \beta(t_0), \beta(t_0) \rangle = Q(\beta(t_0))$ and Propositions 2.2, 2.4, we obtain $\det(\Lambda_g) = Q(g) = Q(\beta(t_0)(\alpha(t_0))^{-1}) = Q(\beta(t_0))Q((\alpha(t_0))^{-1}) = Q(\beta(t_0))Q(\alpha(t_0))^{-1} = 1$. Hence $g \in S(\mathbb{H}^*)$. By Theorem 3.5, $\Lambda_g \in O^+(1, 1)$. By $g = \beta(t_0)(\alpha(t_0))^{-1}$ and Proposition 5.1, Λ_g has the form (5) By $\det(\Lambda_g) = 1$ and Proposition 4.9, we obtain the equality $\det(\Lambda_g) = (\frac{\langle \alpha(t_0), \beta(t_0) \rangle}{Q(\alpha(t_0))})^2 - (\frac{[\alpha(t_0)\beta(t_0)]}{Q(\alpha(t_0))})^2 = 1$.

Consider the element $g = \beta(t_0)(\alpha(t_0))^{-1} \in \mathbb{H}^*$.

Since $\beta(t_0) = \beta(t_0)((\alpha(t_0))^{-1}\alpha(t_0)) = (\beta(t_0)(\alpha(t_0))^{-1})\alpha(t_0)$, we have $\beta(t_0) = g\alpha(t_0)$.

Using equalities (6), we obtain following equalities:

$\beta(t_0)((\alpha(t_0))^{-1}\alpha(t)) = \beta(t_0)((\beta(t_0))^{-1}\beta(t)), \forall t \in T$. These equalities and the above equality $g = \beta(t_0)(\alpha(t_0))^{-1}$ imply $\beta(t) = (\beta(t_0)(\beta(t_0))^{-1})\beta(t) = \beta(t_0)((\beta(t_0))^{-1}\beta(t)) = \beta(t_0)(\alpha(t_0))^{-1}\alpha(t) = (\beta(t_0)(\alpha(t_0))^{-1})\alpha(t) = g\alpha(t), \forall t \in T$. Thus we have $\beta(t) = g\alpha(t), \forall t \in T$, where $g = \beta(t_0)(\alpha(t_0))^{-1} \in \mathbb{H}^*$. Then by (1), we obtain that $\beta(t) = \Lambda_g\alpha(t), \forall t \in T$, where $\Lambda_g \in \Lambda(\mathbb{H}^*)$. As above we obtain that $\det(\Lambda_g) = 1$.

Since $\det(\Lambda_g) = 1$, by Theorem 3.5, $\Lambda_g \in O^+(1, 1)$. Hence we obtain $\alpha(t) \stackrel{O^+(1,1)}{\sim} \beta(t), \forall t \in T$.

Prove the uniqueness of $C \in O^+(1, 1)$ satisfying the conditions $\beta(t) = C\alpha(t), \forall t \in T$. Assume that $D \in O^+(1, 1)$ such that $\beta(t) = D\alpha(t), \forall t \in T$. Then by (1) and Theorem 3.5, there exists a single $b \in S(\mathbb{H}^*)$ such that $D = \Lambda_b$. Hence we have $\beta(t) = \Lambda_b\alpha(t), \forall t \in T$. By (1), we obtain $\beta(t) = b\alpha(t), \forall t \in T$. Since $Q(\alpha(t_0)) \neq 0$, the equality $\beta(t_0) = b\alpha(t_0)$ implies that $b = \beta(t_0)(\alpha(t_0))^{-1} = g \in S(\mathbb{H}^*)$. The uniqueness of C is proved.

Let us obtain the evident form of Λ_g . By Proposition 4.9, the element $g = \beta(t_0)(\alpha(t_0))^{-1}$ is equal to $\frac{\langle \alpha(t_0), \beta(t_0) \rangle}{Q(\alpha(t_0))} + j \frac{[\alpha(t_0) \beta(t_0)]}{Q(\alpha(t_0))}$. Hence the matrix Λ_g has the form (5). Since $g \in S(\mathbb{H}^*)$, we have $\det(\Lambda_g) = 1$ by Theorem 3.5. $\square$

Remark 4.11. Assume that T be a set such that it has at least two elements. By Theorem 2, the system

$$(7) \quad \{Z[Q(\alpha(t))], \langle \alpha(t_0), \alpha(t) \rangle, [\alpha(t_0) \alpha(t)]\}$$

is a complete system of $O^+(1, 1)$ -invariant functions on the set of all T -objects $\alpha(t)$ such that $Z[Q(\alpha(t))] \neq T$, and $t_0 \in T \setminus Z[Q(\alpha(t))]$.

Now let us find a complete system of relations between elements of this complete system.

Theorem 4.12. Let (7) be the complete system of $O^+(1, 1)$ -invariants of a T -object $\alpha(t)$ in $\mathcal{M}$. Here a T -object $\alpha(t)$ has following properties:

$$(1.1) \quad Z[Q(\alpha(t))] \text{ be a subset of } T \text{ such that } Z[Q(\alpha(t))] \neq T,$$

$$(1.2) \quad t_0 \in T \setminus Z[Q(\alpha(t))],$$

$$(1.3) \quad \langle \alpha(t_0), \alpha(t_0) \rangle \neq 0 \text{ and } [\alpha(t_0) \alpha(t_0)] = 0.$$

Assume that:

$$(2.1) \quad U \text{ is a subset of } T \text{ such that } U \neq T,$$

$$(2.2) \quad t_0 \in T \setminus U,$$

$$(2.3) \quad r \text{ be a real number such that } r \neq 0,$$

$$(2.4) \quad a(t) = (a_1(t), a_2(t)) \text{ be a mapping } a : T \rightarrow \mathcal{M} \text{ such that following two properties hold:}$$

$$(2.4.1) \quad a_1^2(t) = a_2^2(t) \quad \forall t \in U$$

$$(2.4.2) \quad a_1(t_0) = r \text{ and } a_2(t_0) = 0 \text{ for } t_0 \in T \setminus U.$$

Then there exists a T -object $\alpha(t)$ in $\mathcal{M}$ such that following equalities hold:

$$(3.1) \quad Z[Q(\alpha(t))] = U,$$

$$(3.2) \quad \langle \alpha(t_0), \alpha(t) \rangle = a_1(t), \forall t \in T,$$

$$(3.3) \quad [\alpha(t_0) \alpha(t)] = a_2(t), \forall t \in T.$$

Proof. (1.1)-(1.3). Assume that $\alpha(t)$ is a T -object in $\mathcal{M}$ such that $Z[Q(\alpha(t))] \neq T$ and $t_0 \in T \setminus Z[Q(\alpha(t))]$. Then properties (1.1)-(1.4) are obvious.

(3.1) – (3.3). We choose a T -object $\alpha(t)$ as follows. Put $\alpha(t_0) = (\sqrt{r}, 0)$. Then we obtain $\langle \alpha(t_0), \alpha(t_0) \rangle = r$. Hence $\langle \alpha(t_0), \alpha(t_0) \rangle = a_1(t_0) = r$. We choose $\alpha(t)$ on the set U as follows: We put $\alpha(t) = (a_1(t), a_2(t))$ such that $a_1^2(t) = a_2^2(t) \quad \forall t \in U$. This equality implies $\langle \alpha(t), \alpha(t) \rangle = a(t) = 0, \forall t \in U$.

For fixed $t \in T$, we consider $a(t)$ and $\alpha(t)$ as elements of the algebra $\mathbb{H}$ of hyperbolic numbers: $a(t) = a_1(t) + ja_2(t)$, $\alpha(t) = \alpha_1(t) + j\alpha_2(t)$. We put $\alpha(t) = \frac{a(t)\alpha(t_0)}{r^2}, \forall t \in T \setminus (U \cup \{t_0\})$. For $\alpha(t_0) = (r, 0) \neq 0$, $(\alpha(t_0))^{-1}$ exists. Then the equalities $\alpha(t) = \frac{a(t)\alpha(t_0)}{r^2}, \forall t \in T \setminus (U \cup \{t_0\})$ imply equalities $(\alpha(t_0))^{-1}\alpha(t) = \frac{a(t)}{r^2}, \forall t \in T \setminus (U \cup \{t_0\})$. By Proposition 9, $(\alpha(t_0))^{-1}\alpha(t) = \frac{\langle \alpha(t_0), \alpha(t) \rangle}{Q(\alpha(t_0))} + i \frac{[\alpha(t_0) \alpha(t)]}{Q(\alpha(t_0))}, \forall t \in T$. The equality $Q(\alpha(t_0)) = \langle \alpha(t_0), \alpha(t_0) \rangle = r^2$, the last two systems of equalities and equalities $\langle \alpha(t), \alpha(t) \rangle = 0, \forall t \in U$ imply equalities $\frac{\langle \alpha(t_0), \alpha(t) \rangle}{r} + i \frac{[\alpha(t_0) \alpha(t)]}{r} = \frac{a(t)}{r}, \forall t \in T$. These equalities imply $Z[Q(\alpha(t))] = U$, $\langle \alpha(t_0), \alpha(t) \rangle = a_1(t), \forall t \in T$ and $[\alpha(t_0) \alpha(t)] = a_2(t), \forall t \in T$. The statements (3.1)-(3.3) are proved. $\square$

5. COMPLETE SYSTEMS OF G -INVARIANTS OF A T -OBJECT IN $\mathcal{M}$ FOR THE GROUP $G = O(1, 1)$

By Proposition 3.4 the following equality holds:

$O(1, 1) = O^+(1, 1) \cup \{C\Delta \mid C \in O^+(1, 1)\}$, where $C\Delta$ is the multiplication of matrices C and Δ . For shortness, denote the set

$\{C\Delta \mid C \in O^+(1, 1)\}$ by $O^+(1, 1) \cdot \Delta$. We note that $O^+(1, 1) \cap O^+(1, 1) \cdot \Delta = \emptyset$.

Let $\alpha(t)$ and $\beta(t)$ be T -objects in $\mathcal{M}$. Assume that $\alpha(t) \stackrel{O(1,1)}{\sim} \beta(t)$. Then there exists $F \in O(1, 1)$ such that $\beta(t) = F\alpha(t), \forall t \in T$. Denote by $Eq(\alpha(t), \beta(t))$ the set of all $F \in O(1, 1)$ such that $\beta(t) = F\alpha(t), \forall t \in T$.

Proposition 5.1. *Let $\alpha(t)$ and $\beta(t)$ be T -objects in $\mathcal{M}$ such that $\alpha(t) \stackrel{O(1,1)}{\sim} \beta(t)$. Then there exist only following three possibilities for the set $Eq(\alpha(t), \beta(t))$:*

- (I) *$Eq(\alpha(t), \beta(t))$ has only one element. In the case, $Eq(\alpha(t), \beta(t)) = \{F\}$, where $F \in O^+(1, 1)$.*
- (II) *$Eq(\alpha(t), \beta(t))$ has only one element. In the case, $Eq(\alpha(t), \beta(t)) = \{F\}$, where $F \in O^+(1, 1) \cdot \Delta$.*
- (III) *$Eq(\alpha(t), \beta(t))$ has only two elements. In the case, $Eq(\alpha(t), \beta(t)) = \{F_1, F_2\}$, where $F_1 \in O^+(1, 1), F_2 \in O^+(1, 1) \cdot \Delta$.*

Proof. Assume that $\alpha(t) \stackrel{O(1,1)}{\sim} \beta(t)$. Then there exists $F \in O(1, 1)$ such that $F \in Eq(\alpha(t), \beta(t))$. Since $F \in O(1, 1)$ and $F \in O(1, 1)$, then $F \in O^+(1, 1)$ or $F \in O^+(1, 1) \cdot \Delta$.

- (I) Let $F \in Eq(\alpha(t), \beta(t))$, where $F \in O^+(1, 1)$. By Theorem 4.10, in this case there exists a single $F \in O^+(1, 1)$ such that following equalities $\beta(t) = F\alpha(t), \forall t \in T$, hold. Hence, the set $Eq(\alpha(t), \beta(t))$ has a single element of $O^+(1, 1)$. Assume that the set $Eq(\alpha(t), \beta(t))$ has not elements of $O^+(1, 1) \cdot \Delta$. Then, the set $Eq(\alpha(t), \beta(t))$ has only a single element $F \in O(1, 1)$ such that $F \in O^+(1, 1)$.
- (II) Let $F \in Eq(\alpha(t), \beta(t))$, where $F \in O^+(1, 1) \cdot \Delta$. Then following equality $\beta(t) = F\alpha(t), \forall t \in T$ holds. Since $F \in \{C\Delta \mid C \in O^+(1, 1)\}$, there exists $C \in O^+(1, 1)$ such that $F = C\Delta$. Then we have following equality $\beta(t) = C\Delta\alpha(t), \forall t \in T$. By Theorem 4.10, there exists a single $C \in O^+(1, 1)$

such that following equalities $\beta(t) = C\Delta\alpha(t), \forall t \in T$, hold. Hence, the set $Eq(\alpha(t), \beta(t))$ has a single element of $O^+(1, 1) \cdot \Delta$. Assume that the set $Eq(\alpha(t), \beta(t))$ has not elements of $O^+(1, 1)$. Then, the set $Eq(\alpha(t), \beta(t))$ has a single element of $O^+(1, 1) \cdot \Delta$ such that $Eq(\alpha(t), \beta(t)) = \{F\}$, where $F \in O^+(1, 1) \cdot \Delta$.

- (III) Let $Eq(\alpha(t), \beta(t))$ be such that for $F_1 \in O^+(1, 1)$ and $F_2 \in O^+(1, 1) \cdot \Delta$ following equalities hold: $\beta(t) = F_1\alpha(t), \forall t \in T$, and $\beta(t) = F_2\alpha(t) = C\Delta\alpha(t), \forall t \in T$, where $C \in O^+(1, 1)$. By Theorem 4.10, there exists a single $F_1 \in O^+(1, 1)$ such that following equalities $\beta(t) = F_1\alpha(t), \forall t \in T$, hold. Hence, the set $Eq(\alpha(t), \beta(t))$ has a single element of $O^+(1, 1)$. By Theorem 4.10, there exists a single element $F_2 \in O^+(1, 1) \cdot \Delta$ such that following equalities $\beta(t) = F_2\alpha(t) = C\Delta\alpha(t), \forall t \in T$ hold, where $C \in O^+(1, 1)$. Then, the set $Eq(\alpha(t), \beta(t))$ have only two elements, where these are a single element of $O^+(1, 1)$ and a single element of $O^+(1, 1) \cdot \Delta$.

□

Theorem 5.2. *Let $\alpha(t)$ be a T -object in $\mathcal{M}$ such that $Z[Q(\alpha(t))] \neq T$ and $t_0 \in T \setminus Z[Q(\alpha(t))]$.*

- (i) *Suppose that a T -object $\beta(t)$ in $\mathcal{M}$ such that the following equalities $\beta(t) = C\Delta\alpha(t), \forall t \in T$, hold for some $C \in O^+(1, 1)$. Then following equalities hold:*

$$(8) \quad \begin{cases} Z[Q(\alpha(t))] = Z[Q(\beta(t))] \\ \langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle \\ -[\alpha(t_0)\alpha(t)] = [\beta(t_0)\beta(t)], \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t))]$.

- (ii) *Conversely, assume that a T -object $\beta(t)$ in $\mathcal{M}$ such that the equalities (8) hold. Then there exists a single matrix $U \in O^+(1, 1)$ such that $\beta(t) = U\Delta\alpha(t), \forall t \in T$. In this case, U has the following form*

$$(9) \quad U = \begin{pmatrix} \frac{\langle \Delta\alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{[\Delta\alpha(t_0)\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} \\ \frac{[\Delta\alpha(t_0)\beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{\langle \Delta\alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} \end{pmatrix},$$

where $\det(U) = 1$.

Proof. Suppose that a T -object $\beta(t)$ in $\mathcal{M}$ such that the following equalities $\beta(t) = C\Delta\alpha(t), \forall t \in T$ hold for some $C \in O^+(1, 1)$. This means T -objects $\Delta\alpha(t)$ and $\beta(t)$ are $O^+(1, 1)$ -equivalent. Then, by Theorem 4.10, we obtain following equalities:

$$(10) \quad \begin{cases} Z[Q(\Delta\alpha(t))] = Z[Q(\beta(t))] \\ \langle \Delta\alpha(t_0), \Delta\alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle \\ [\Delta\alpha(t_0) \ \Delta\alpha(t)] = [\beta(t_0) \ \beta(t)], \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t))]$. These equalities and equalities $Z[Q(\Delta\alpha(t))] = Z[Q(\alpha(t))]$, $\langle \Delta\alpha(t_0), \Delta\alpha(t) \rangle = \langle \alpha(t_0), \alpha(t) \rangle$, $[\Delta\alpha(t_0) \ \Delta\alpha(t)] = -[\alpha(t_0) \ \alpha(t)]$ imply equalities (8).

Conversely, assume that a T -object $\beta(t)$ in $\mathcal{M}$ such that the equalities (8) hold. Then equalities (8) and equalities $Z[Q(\Delta\alpha(t))] = Z[Q(\alpha(t))]$, $\langle \Delta\alpha(t_0), \Delta\alpha(t) \rangle = \langle \alpha(t_0), \alpha(t) \rangle$, $[\Delta\alpha(t_0) \ \Delta\alpha(t)] = -[\alpha(t_0) \ \alpha(t)]$ imply equalities (10). By Theorem 4.10, equalities (10) imply an existence $U \in O^+(1, 1)$ such that following equalities $\beta(t) = U\Delta\alpha(t), \forall t \in T$, hold. By Theorem 4.10, the matrix U has the form (9). $\square$

Remark 5.3. Assume that T be a set such that it has at least two elements. By Theorem 5.2, the system $\{Z[Q(\alpha(t))], \langle \alpha(t_0), \alpha(t) \rangle, [\Delta\alpha(t_0) \ \Delta\alpha(t)]\}$ is a complete system of $O^+(1, 1)$ -invariant functions on the set of all T -objects $\Delta\alpha(t)$ such that $Z[Q(\alpha(t))] \neq T$, and $t_0 \in T \setminus Z[Q(\alpha(t))]$. Complete systems of relations between elements of this system as in Theorem 4.12.

Theorem 5.4. Let $\alpha(t)$ and $\beta(t)$ be T -objects in $\mathcal{M}$. Assume that $Z[Q(\alpha(t))] \neq T$ and $t_0 \in T \setminus Z[Q(\alpha(t))]$.

- (i) Suppose that matrices $F_1, F_2 \in O^+(1, 1)$ exist such that $\beta(t) = F_1\alpha(t), \forall t \in T$, and $\beta(t) = F_2\Delta\alpha(t), \forall t \in T$. Then following equalities hold:

$$(11) \quad \begin{cases} Z[Q(\alpha(t))] = Z[Q(\beta(t))] \\ \langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle \\ \text{rank}(\alpha(t)) = \text{rank}(\beta(t)) = 1. \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t))]$.

- (ii) Conversely, assume that the equalities (11) hold. Then only two matrices $C_1, C_2 \in O^+(1, 1)$ exist such that following equalities $\beta(t) = C_1\alpha(t), \forall t \in T$, $\beta(t) = C_2\Delta\alpha(t), \forall t \in T$, hold. Here the matrices C_1 and C_2 have the

following forms:

$$(12) \quad C_1 = \begin{pmatrix} \frac{\langle \alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{[\alpha(t_0) \beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} \\ \frac{[\alpha(t_0) \beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{\langle \alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} \end{pmatrix}$$

and

$$(13) \quad C_2 = \begin{pmatrix} \frac{\langle \Delta \alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{[\Delta \alpha(t_0) \beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} \\ \frac{[\Delta \alpha(t_0) \beta(t_0)]}{\langle \alpha(t_0), \alpha(t_0) \rangle} & \frac{\langle \Delta \alpha(t_0), \beta(t_0) \rangle}{\langle \alpha(t_0), \alpha(t_0) \rangle} \end{pmatrix},$$

where $\det(C_1) = \det(C_2) = 1$, respectively.

Proof. (i) Suppose that there exist $C_1 \in O^+(1, 1)$ such that $\beta(t) = C_1 \alpha(t), \forall t \in T$. Then, by Theorem 4.10, the following equalities hold:

$$(14) \quad \begin{cases} Z[Q(\alpha(t))] = Z[Q(\beta(t))] \\ \langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle \\ [\alpha(t_0) \alpha(t)] = [\beta(t_0) \beta(t)], \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t))]$. Suppose that there exist $C_2 \in O^+(1, 1)$ such that $\beta(t) = C_2 \Delta \alpha(t), \forall t \in T$. Then, by Theorem 5.2, the following equalities hold:

$$(15) \quad \begin{cases} Z[Q(\alpha(t))] = Z[Q(\beta(t))] \\ \langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle \\ [\alpha(t_0) \alpha(t)] = -[\beta(t_0) \beta(t)], \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t))]$. (14) and (15) imply the following equalities:

$$(16) \quad \begin{cases} Z[Q(\alpha(t))] = Z[Q(\beta(t))] \\ \langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle, \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t))]$. (14) implies the following equalities:

$$(17) \quad [\alpha(t_0) \alpha(t)] = [\beta(t_0) \beta(t)], \forall t \in T \setminus Z[Q(\alpha(t))].$$

(15) implies the following equalities:

$$(18) \quad [\alpha(t_0) \alpha(t)] = -[\beta(t_0) \beta(t)], \forall t \in T \setminus Z[Q(\alpha(t))].$$

Equalities (17) and (18) imply following equalities:

$$(19) \quad [\beta(t_0) \beta(t)] = -[\beta(t_0) \beta(t)], \forall t \in T \setminus Z[Q(\alpha(t))].$$

These equalities imply following equalities:

$$(20) \quad [\beta(t_0) \beta(t)] = 0, \forall t \in T \setminus Z[Q(\alpha(t))].$$

These equalities and the equalities (17) imply following equalities

$$(21) \quad [\alpha(t_0) \alpha(t)] = 0, \forall t \in T \setminus Z[Q(\alpha(t))].$$

The equalities (21) imply that there exists a real function $a(t)$ on T such that $a(t) = 0$ for all $t \in Z[Q(\alpha(t))]$ and $a(t) \neq 0$ for all $t \in T \setminus Z[Q(\alpha(t))]$. In this case, $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$ hold.

Similarly, equalities (20) imply that there exists a real function $b(t)$ on T such that $b(t) = 0$ for all $t \in Z[Q(\alpha(t))]$ and $b(t) \neq 0$ for all $t \in T \setminus Z[Q(\alpha(t))]$. In this case, $\beta(t) = b(t)\beta(t_0), \forall t \in T$ hold. Then, the above equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$ and $\beta(t) = b(t)\beta(t_0), \forall t \in T$ imply the equality $rank(\alpha(t)) = rank(\beta(t)) = 1$ in the equalities (11). This equality and the equalities (14) imply the equalities (11).

Conversely, assume that the equalities (11) hold. Then the equality $rank(\alpha(t)) = 1$ in (11) implies an existence of a real function $a(t)$ on T such that $a(t) = 0$ for all $t \in Z[Q(\alpha(t))]$ and $a(t) \neq 0$ for all $t \in T \setminus Z[Q(\alpha(t))]$. In this case, $\alpha(t) = a(t)\alpha(t_0)$ for all $t \in T$.

Similarly, the equality $rank(\beta(t)) = 1$ in (11) implies an existence of a real function $b(t)$ on T such that $b(t) = 0$ for all $t \in Z[Q(\alpha(t))]$ and $b(t) \neq 0$ for all $t \in T \setminus Z[Q(\alpha(t))]$. In this case, $\beta(t) = b(t)\beta(t_0), \forall t \in T$.

The equalities $Z[Q(\alpha(t))] = Z[Q(\beta(t))]$ and $\langle \alpha(t_0), \alpha(t) \rangle = \langle \beta(t_0), \beta(t) \rangle, \forall t \in T \setminus Z[Q(\alpha(t))]$ imply following equality $a(t) = b(t), \forall t \in T$. Hence we obtain following equalities $\alpha(t) = a(t)\alpha(t_0), \forall t \in T$ and $\beta(t) = a(t)\beta(t_0), \forall t \in T$.

Since $t_0 \in T \setminus Z[Q(\alpha(t))]$, we have $a(t_0) \neq 0$. By the equality $Z[Q(\alpha(t))] = Z[Q(\beta(t))]$, we obtain $\beta(t_0) \neq 0$. By Theorem 5.2 in [8], only two matrices $C_1 \in O^+(1, 1)$ and $C_2 \in O^+(1, 1)$ exist such that $\beta(t_0) = C_1\alpha(t_0)$ and $\beta(t_0) = C_2\Delta\alpha(t_0)$. By Theorem 5.1 in [8], C_1 has the form (12) and C_2 has the form (13).

The above equalities $\beta(t) = a(t)\beta(t_0), \forall t \in T$, $\beta(t_0) = C_1\alpha(t_0)$, $\beta(t_0) = C_2\Delta\alpha(t_0)$ imply following equalities: $\beta(t) = C_1\alpha(t), \forall t \in T$ and $\beta(t) = C_2\Delta\alpha(t), \forall t \in T$.

□

Remark 5.5. Assume that T be a set such that it has at least two elements. By Theorem 5.4, the system $\{Z[Q(\alpha(t))], \langle \alpha(t_0), \alpha(t) \rangle, \text{rank}(\alpha(t))\}$ is a complete system of $O^+(1, 1)$ -invariant functions on the set of all T -objects $\alpha(t)$ such that $Z[Q(\alpha(t))] \neq T$, $\text{rank}(\alpha(t)) = 1$ and $t_0 \in T \setminus Z[Q(\alpha(t))]$. Complete systems of relations between elements of this system follows easy from Theorem 4.12.

Corollary 5.5.1. Let $\alpha(t)$ and β be a T -objects in $\mathcal{M}$ such that $Z[Q(\alpha(t))] \neq T, Z[Q(\beta(t))] \neq T$. Assume that $F_1 \in O(1, 1)$ and $F_2 \in O(1, 1)$ such that:

$$(1) F_1 \neq F_2,$$

$$(2) \beta(t) = F_1 \alpha(t), \forall t \in T,$$

$$(3) \beta(t) = F_2 \alpha(t), \forall t \in T.$$

Then $\text{rank}(\alpha(t)) = \text{rank}(\beta(t)) = 1$.

Conversely, assume that $\alpha(t) \overset{O(1,1)}{\sim} \beta(t), \forall t \in T$ and $\text{rank}(\alpha(t)) = \text{rank}(\beta(t)) = 1$. Then there exist only two matrices $F_1 \in O(1, 1)$ and $F_2 \in O(1, 1)$ satisfying above the conditions (1), (2) and (3).

Proof. It follows from Theorem 5.4. □

Corollary 5.5.2. Let $\alpha(t)$ and β be a T -objects in $\mathcal{M}$ such that $Z[Q(\alpha(t))] \neq T, Z[Q(\beta(t))] \neq T$. Assume that there exists a single matrix $F \in O(1, 1)$ such that $\beta(t) = F\alpha(t), \forall t \in T$. Then $\text{rank}(\alpha(t)) = \text{rank}(\beta(t)) = 2$.

Conversely, assume that $\alpha(t) \overset{O(1,1)}{\sim} \beta(t), \forall t \in T$, and $\text{rank}(\alpha(t)) = \text{rank}(\beta(t)) = 2$. Then there exists a single matrix $F \in O(1, 1)$ such that $\beta(t) = F\alpha(t), \forall t \in T$.

Proof. It follows from Corollary 5.5.1. □

6. COMPLETE SYSTEMS OF INVARIANTS OF A T -OBJECT IN $\mathcal{M}$ FOR THE GROUP

$$MO^+(1, 1)$$

Let $G = O(1, 1)$ or $G = O^+(1, 1)$. Denote by $G \ltimes Tr(2)$ the group of all transformations of $\mathcal{M}$ generated by elements of G and all translations of $\mathcal{M}$. In particularly, $MO(1, 1) = O(1, 1) \ltimes Tr(2)$ and $MO^+(1, 1) = O^+(1, 1) \ltimes Tr(2)$.

Assume that the set T has only one element. Let $\alpha(t)$ and $\beta(t)$ be T -objects. Then they are $Tr(2)$ -equivalent. Hence they are $G \ltimes Tr(2)$ -equivalent. Below we assume that T has at last two elements.

Proposition 6.1. *Let $G = O(1, 1)$ or $G = O^+(1, 1)$ and T be a set such that it has at last two elements.*

- (1) *Assume that $\alpha(t) \stackrel{G \ltimes Tr(2)}{\sim} \beta(t), \forall t \in T$ and t_0 is an arbitrary fixed element of T . Then $(\alpha(t) - \alpha(t_0)) \stackrel{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$.*
- (2) *Assume that $(\alpha(t) - \alpha(t_0)) \stackrel{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$, for some element $t_0 \in T$. Then $\alpha(t) \stackrel{G \ltimes Tr(2)}{\sim} \beta(t), \forall t \in T$.*

Proof. $\Rightarrow$ Assume that $\alpha(t) \stackrel{G \ltimes Tr(2)}{\sim} \beta(t), \forall t \in T$. Then there exists $F \in G$ and $a \in \mathcal{M}$ such that $\beta(t) = F\alpha(t) + a, \forall t \in T$. In particularly, for $t = t_0$, we have $\beta(t_0) = F\alpha(t_0) + a$. This equality implies $a = \beta(t_0) - F\alpha(t_0)$. This equality and equalities $\beta(t) = F\alpha(t) + a, \forall t \in T$ imply equalities $\beta(t) = F\alpha(t) + \beta(t_0) - F\alpha(t_0), \forall t \in T$. These equalities imply equalities $\beta(t) - \beta(t_0) = F(\alpha(t) - \alpha(t_0)), \forall t \in T$, that is $(\alpha(t) - \alpha(t_0)) \stackrel{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$.

$\Leftarrow$ Assume that $(\alpha(t) - \alpha(t_0)) \stackrel{G}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$. Then there exists $F \in G$ such that $\beta(t) - \beta(t_0) = F(\alpha(t) - \alpha(t_0)), \forall t \in T$. Put $a = \beta(t_0) - F\alpha(t_0)$. This equality implies $\beta(t_0) = F\alpha(t_0) + a$. The equality $a = \beta(t_0) - F\alpha(t_0), \beta(t) - \beta(t_0) = F(\alpha(t) - \alpha(t_0)), \forall t \in T$ and $\beta(t_0) = F\alpha(t_0) + a$ imply equalities $\beta(t) = F\alpha(t) + a, \forall t \in T$. Hence $\alpha(t) \stackrel{G \ltimes Tr(2)}{\sim} \beta(t), \forall t \in T$. $\square$

Proposition 6.2. *Let $G = O^+(1, 1)$ or $G = O(1, 1)$. Assume that $\alpha(t)$ and $\beta(t)$ are T -objects such that $\alpha(t) \stackrel{G \ltimes Tr(2)}{\sim} \beta(t)$ and $t_0 \in T$. Then $Z[Q(\alpha(t) - \alpha(t_0))] = Z[Q(\beta(t) - \beta(t_0))]$.*

Proof. This statement follows from Propositions 4.8 and 6.1 $\square$

Theorem 6.3. *Let $\alpha(t)$ be a T -object in $\mathcal{M}$ and $t_0 \in T$ such that $Z[Q(\alpha(t) - \alpha(t_0))] \neq T$ and $t_1 \in T \setminus Z[Q(\alpha(t) - \alpha(t_0))]$ be fixed.*

(i) Suppose that a T -object $\beta(t)$ in $\mathcal{M}$ such that $\alpha(t) \stackrel{O^+(1,1) \times T^{r(2)}}{\sim} \beta(t), \forall t \in T$.

Then following equalities hold:

$$(22) \quad \left\{ \begin{array}{l} Z[Q(\alpha(t) - \alpha(t_0))] = Z[Q(\beta(t) - \beta(t_0))] \\ \langle \alpha(t_1) - \alpha(t_0), \alpha(t) - \alpha(t_0) \rangle = \langle \beta(t_1) - \beta(t_0), \beta(t) - \beta(t_0) \rangle \\ [(\alpha(t_1) - \alpha(t_0)) (\alpha(t) - \alpha(t_0))] = [(\beta(t_1) - \beta(t_0)) (\beta(t) - \beta(t_0))], \end{array} \right.$$

for all $t \in T \setminus Z(\alpha(t) - \alpha(t_0))$.

(ii) Conversely, assume that a T -object $\beta(t)$ in $\mathcal{M}$ such that the equalities (22) hold. Then there exists only element $F \in MO^+(1, 1)$ such that $\beta(t) = F\alpha(t), \forall t \in T$. The evident form of F as follows: $F\alpha(t) = C\alpha(t) + a, \forall t \in T$, where $C \in O^+(1, 1)$, $a \in \mathcal{M}$. Here evident form of C as follows

$$(23) \quad C = \left(\begin{array}{cc} \frac{\langle \alpha(t_1) - \alpha(t_0), \beta(t_1) - \beta(t_0) \rangle}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} & \frac{[(\alpha(t_1) - \alpha(t_0)) (\beta(t_1) - \beta(t_0))]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} \\ \frac{[(\alpha(t_1) - \alpha(t_0)) (\beta(t_1) - \beta(t_0))]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} & \frac{\langle \alpha(t_1) - \alpha(t_0), \beta(t_1) - \beta(t_0) \rangle}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} \end{array} \right),$$

where $\det(C) = 1$.

The element a has the form such that $a = \beta(t_0) - C\alpha(t_0)$.

Proof. It follows from Theorem 4.10 and Proposition 6.1. □

Remark 6.4. Let $t_0 \in T$. By Theorem 6.3, the system

$\{Z[Q(\alpha(t) - \alpha(t_0))], \langle \alpha(t_1) - \alpha(t_0), \alpha(t) - \alpha(t_0) \rangle, [(\alpha(t_1) - \alpha(t_0)) (\alpha(t) - \alpha(t_0))]\}$ is a complete system of $MO^+(1, 1)$ -invariant functions on the set of all T -objects $\alpha(t)$ in $\mathcal{M}$ such that $Z[Q(\alpha(t) - \alpha(t_0))] \neq T$, where $t_1 \in T \setminus Z[Q(\alpha(t) - \alpha(t_0))]$ be fixed. A complete system of relations between elements of this complete system as in Theorem 4.12.

7. COMPLETE SYSTEMS OF INVARIANTS OF A T -OBJECT IN $\mathcal{M}$ FOR THE GROUP $MO(1, 1)$

Let $\alpha(t)$ and $\beta(t)$ be T -objects in $\mathcal{M}$. Assume that $\alpha(t)$ and $t_0 \in T$ such that $Z[Q(\alpha(t) - \alpha(t_0))] \neq T$. Then, by Proposition 6.1, $\alpha(t) \stackrel{MO(1,1)}{\sim} \beta(t)$ if and only if $(\alpha(t) - \alpha(t_0)) \stackrel{O(1,1)}{\sim} (\beta(t) - \beta(t_0)), \forall t \in T$. In this case, by Proposition 5.1, there exist only three following possibilities for the set $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0))$:

(I) $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0))$ has only one element. In this case, $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0)) = \{F\}$, where $F \in O^+(1, 1)$.

- (II) $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0))$ has only one element. In this case, $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0)) = \{F\}$, where $F \in O^+(1, 1) \cdot \Delta$.
- (III) $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0))$ has only two elements. In this case, $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0)) = \{F_1, F_2\}$, where $F_1 \in O^+(1, 1)$, $F_2 \in O^+(1, 1) \cdot \Delta$.

A description of the set $Eq(\alpha(t) - \alpha(t_0), \beta(t) - \beta(t_0))$ and a complete system of invariants of a T -object in $\mathcal{M}$ in the case (I) are given in Section 5.

Consider the case (II). This case is introduced in the following theorem.

Theorem 7.1. *Let $\alpha(t)$ be a T -object in $\mathcal{M}$ such that $Z[Q(\alpha(t) - \alpha(t_0))] \neq T$ for some $t_0 \in T$ and $t_1 \in T \setminus Z[Q(\alpha(t) - \alpha(t_0))]$ be fixed.*

- (i) *Suppose that a T -object $\beta(t)$ such that the following equalities $\beta(t) = H\Delta\alpha(t) + a, \forall t \in T$, hold for some $C \in O^+(1, 1)$ and some $a \in \mathcal{M}$. Then following equalities hold:*

$$(24) \quad \begin{cases} Z[Q(\alpha(t) - \alpha(t_0))] = Z[Q(\beta(t) - \beta(t_0))] \\ \langle \alpha(t_1) - \alpha(t_0), \alpha(t) - \alpha(t_0) \rangle = \langle \beta(t_1) - \beta(t_0), \beta(t) - \beta(t_0) \rangle \\ -[\alpha(t_1) - \alpha(t_0) \alpha(t) - \alpha(t_0)] = [\beta(t_1) - \beta(t_0) \beta(t) - \beta(t_0)], \end{cases}$$

for all $t \in T \setminus Z[Q(\alpha(t) - \alpha(t_0))]$.

- (ii) *Conversely, assume that a T -object $\beta(t)$ in $\mathcal{M}$ such that the equalities (24) hold. Then a single matrix $U \in O^+(1, 1)$ and a single $a \in \mathcal{M}$ exist such that $\beta(t) = U\Delta\alpha(t) + a, \forall t \in T$. In this case, U has following form*

$$(25) \quad U = \begin{pmatrix} \frac{\langle \Delta(\alpha(t_1) - \alpha(t_0)), (\beta(t_1) - \beta(t_0)) \rangle}{\langle (\alpha(t_1) - \alpha(t_0)), (\alpha(t_1) - \alpha(t_0)) \rangle} & \frac{[\Delta(\alpha(t_1) - \alpha(t_0)) (\beta(t_1) - \beta(t_0))]}{\langle (\alpha(t_1) - \alpha(t_0)), (\alpha(t_1) - \alpha(t_0)) \rangle} \\ \frac{[\Delta(\alpha(t_1) - \alpha(t_0)) (\beta(t_1) - \beta(t_0))]}{\langle (\alpha(t_1) - \alpha(t_0)), (\alpha(t_1) - \alpha(t_0)) \rangle} & \frac{\langle \Delta(\alpha(t_1) - \alpha(t_0)), (\beta(t_1) - \beta(t_0)) \rangle}{\langle (\alpha(t_1) - \alpha(t_0)), (\alpha(t_1) - \alpha(t_0)) \rangle} \end{pmatrix},$$

where $\det(U) = 1$.

The element a has the form such that $a = \beta(t_0) - U\Delta\alpha(t_0)$.

Proof. It follows easy from Proposition 6.1 and Theorem 5.2. □

Corollary 7.1.1. *Let $\alpha(t)$ and $\beta(t)$ be T -objects in $\mathcal{M}$. Assume that $\alpha(t)$ and $t_0 \in T$ are such that $Z[Q(\alpha(t) - \alpha(t_0))] \neq T$. Assume that $F_1 \in O^+(1, 1)$, $a_1 \in \mathcal{M}$, $F_2 \in O^+(1, 1)$, $a_2 \in \mathcal{M}$ such that:*

- 1) $\beta(t) = F_1\Delta\alpha(t) + a_1, \forall t \in T$,
 2) $\beta(t) = F_1\Delta\alpha(t) + a_2, \forall t \in T$.

Then $F_1 = F_2$, and $a_1 = a_2$.

Proof. It follows easy from Theorem 7.1. $\square$

Consider the case (III). This case is given in the following corollary.

Corollary 7.1.2. *Let $\alpha(t)$ and $\beta(t)$ be T -objects in $\mathcal{M}$. Assume that $\alpha(t)$ and $t_0 \in T$ are such that $Z[Q(\alpha(t) - \alpha(t_0))] \neq T$. Assume that $F_1 \in O(1, 1)$, $a_1 \in \mathcal{M}$, $F_2 \in O(1, 1)$, a_2 such that:*

- 1) $F_1 \neq F_2$,
- 2) $\beta(t) = F_1\alpha(t) + a_1, \forall t \in T$,
- 3) $\beta(t) = F_2\alpha(t) + a_2, \forall t \in T$.

Then $F_1 \in O^+(1, 1)$, $F_2 \in O^+(1, 1) \cdot \Delta$ or $F_2 \in O^+(1, 1)$, $F_1 \in O^+(1, 1) \cdot \Delta$.

Proof. It follows easy from Theorems 4.10 and 5.2. $\square$

Theorem 7.2. *Let $\alpha(t)$ be a T -object in $\mathcal{M}$ such that $Z[Q(\alpha(t) - \alpha(t_0))] \neq T$ for some $t_0 \in T$ and $t_1 \in T \setminus Z[Q(\alpha(t) - \alpha(t_0))]$ be fixed.*

- (i) *Suppose that matrices $F_1, F_2 \in O^+(1, 1)$ and vectors $a_1, a_2 \in \mathcal{M}$ exist such that $\beta(t) = F_1\alpha(t) + a_1, \forall t \in T$ and $\beta(t) = F_2\Delta\alpha(t) + a_2, \forall t \in T$. Then following equalities hold:*

$$(26) \quad \begin{cases} Z[Q(\alpha(t) - \alpha(t_0))] = Z[Q(\beta(t) - \beta(t_0))] \\ \langle \alpha(t_1) - \alpha(t_0), \alpha(t) - \alpha(t_0) \rangle = \langle \beta(t_1) - \beta(t_0), \beta(t) - \beta(t_0) \rangle \\ \text{rank}(\alpha(t) - \alpha(t_0)) = \text{rank}(\beta(t) - \beta(t_0)) = 1. \end{cases}$$

for all $t \in T \setminus Z(\alpha(t) - \alpha(t_0))$.

- (ii) *Conversely, assume that the equalities (26) hold. Then only two matrices $C_1, C_2 \in O^+(1, 1)$ and only two vectors $a_1, a_2 \in \mathcal{M}$ exist such that following equalities $\beta(t) = C_1\alpha(t) + a_1, \forall t \in T$ and $\beta(t) = C_2\Delta\alpha(t) + a_2, \forall t \in T$, hold. Here the matrices C_1 and C_2 have following forms:*

$$(27) \quad C_1 = \begin{pmatrix} \frac{\langle \alpha(t_1) - \alpha(t_0), \beta(t_1) - \beta(t_0) \rangle}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} & \frac{[\alpha(t_1) - \alpha(t_0) \mid \beta(t_1) - \beta(t_0)]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} \\ \frac{[\alpha(t_1) - \alpha(t_0) \mid \beta(t_1) - \beta(t_0)]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} & \frac{[\alpha(t_1) - \alpha(t_0) \mid \alpha(t_1) - \alpha(t_0)]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} \end{pmatrix},$$

and

$$(28) \quad C_2 = \begin{pmatrix} \frac{\langle \Delta\alpha(t_1) - \Delta\alpha(t_0), \beta(t_1) - \beta(t_0) \rangle}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} & \frac{[\Delta\alpha(t_1) - \Delta\alpha(t_0) \mid \beta(t_1) - \beta(t_0)]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} \\ \frac{[\Delta\alpha(t_1) - \Delta\alpha(t_0) \mid \beta(t_1) - \beta(t_0)]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} & \frac{[\Delta\alpha(t_1) - \Delta\alpha(t_0) \mid \beta(t_1) - \beta(t_0)]}{\langle \alpha(t_1) - \alpha(t_0), \alpha(t_1) - \alpha(t_0) \rangle} \end{pmatrix},$$

where $\det(C_1) = \det(C_2) = 1.$, respectively.

Vectors a_1 and a_2 have the forms $a_1 = \beta(t_0) - C_1\alpha(t_0)$ and $a_2 = \beta(t_0) - C_2\alpha(t_0)$, respectively.

Proof. It follows easy from Proposition 6.1 and Theorem 5.4. $\square$

8. CONCLUSION

This paper propose a global approach for the complete systems of invariants of a T-object and the solutions of the equivalence problem between two T-objects in respect to the two dimensional Minkowski space. The main contributions of the approach are delivered as follows:

1. It achieves the solution of the equivalence problem between two m-point systems, two paths or two vector fields in terms of their global invariants and they are investigated as a single concept.
2. For given two objects with the global invariants, this approach finds the transformation between two T-objects, and this transformation maps one into the other in terms of their common global invariants.

The results of this paper may be useful in many areas of pure and applied mathematics, physic, mechanics, image processing, computer vision theory, etc.

In classical mechanics, the global invariants of paths under similarity transformations in the two dimensional Euclidean space and their applications to theoretical Newtonian mechanics are given in [21]. In pattern recognition, for given two paths are related by means of an orthogonal transformation, we find the transformation between two paths, and the transformation transform one into the other in terms of their common global invariants of two paths. For this, in [19], we use examples in [31]. In areas such as image processing and point set registration plays an important role. In the identification problems of two m-point systems or two paths, important results are obtained(see [5, 7, 26, 28, 29, 31, 32, 34, 36] and references therein). Therefore, we consider that the results obtained in this paper can also be useful for similar application problems in two dimensional Minkowski space.

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REFERENCES

- [1] M. Berger, *Geometry I*, Springer-Verlag, Berlin, Heidelberg, 1987.
- [2] R. Höfer, *m*-point invariants of real geometries, *Beitrage Algebra Geom.*, Vol. 40, 261-266 (1999).
- [3] E. Stasiak, Scalar concomitants of a system of vectors in pseudo-Euclidean geometry of index 1, *Publ.Math. Debrecen*, Vol.57, Number 1-2,55-69 (2000).
- [4] . Ören, On invariants of *m*-vectors in Lorentzian Geometry, *International Electronic Journal of Geometry*, Vol. 9, Number 1, 38-44 (2016).
- [5] . Ören, Equivalence conditions of two Bézier Curves in the Euclidean geometry, *Iranian Journal of Science and Technology, Transactions A: Science (ISTT)*, Vol. 42, Number 3, 1563-1577 (2018).
- [6] . Ören, The equivalence problem of vectors in the two-dimensional Minkowski spacetime and its applications to Bézier curves, *J. math. Comput.Sci.*, Vol. 6, Number 1, 1-21 (2016).
- [7] . Ören, M. ncesu, Recognition of complex polynomial Bezier curves under similarity transformations, *Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.*, Vol. 69, Number 2,1377-1388 (2020).
- [8] D. Khadjiev, Y. Góksal, Applications of hyperbolic numbers to the invariant theory in two-dimensional pseudo-Euclidean space, *Adv. Appl. Clifford Algebras*, Vol.26, 645-668 (2016).
- [9] D. Khadjiev, S. Ayupov, G.Beshimov, Complete systems of invariant of *m*-tuples for fundamental groups of the two-dimensional Euclidean space, *Uzbek Mathematical Journal*, Vol. 1,57-84 (2020).
- [10] D. Khadjiev, U. Bekbaev, R.Aripov, On equivalence of vector value maps, <https://arxiv.org/abs/2005.08707>, (2020).
- [11] D. Khadjiev, Projective invariants of *m*-tuples in the one-dimensional projective space, *Uzbek Mathematical Journal*, Vol. 1, 60-72 (2019).
- [12] H. E. Bez, Generalized invariant-geometry conditions for the rational Bzier paths, *International Journal of Computer Mathematics*, Vol. 87, Number 12, 2722-2732 (2010).
- [13] H. A. oban, Complete system of invariants of vectors for isometry group in *n*-dimensional unitary space, *Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.*, Vol. 68, Number 1,362-370 (2019).
- [14] R. Aripov, D. Khadjiev, The complete system of global differential and integral invariants of a curve in Euclidean geometry, *Russian Mathematics*, Vol. 51, 1-14 (2007).
- [15] D. Khadjiev, . Ören, Ö. Peken, Generating systems of differential invariants and the theorem on existence for curves in the pseudo-Euclidean geometry, *Turkish Journal of Mathematics*, Vol. 37, 80-94 (2013).
- [16] Ö. Peken, D. Khadjiev, On invariants of curves in centro-affine geometry, *J. Math. Kyoto Univ.(JMKYAZ)* Vol.44, Number 3, 603-613(2004).

- [17] Ö. Peken, D. Khadjiev, . Ören, Invariant parametrizations and complete systems of global invariants of curves in the pseudo-euclidean geometry, Turkish Journal of Mathematics, Vol. 36, 147-160 (2012).
- [18] . Ören, D. Khadjiev, Recognition of paths and curves in the 2-dimensional Euclidean geometry, International Electronic Journal of Geometry, Vol. 13, Number 2, 116-1348 (2020).
- [19] . Ören, D. Khadjiev, Global invariants of paths and curves for the group of orthogonal transformations in the two-dimensional Euclidean space, An. St. Univ. Ovidius Constanta, Vol. 27, Number 2, 37-65 (2019).
- [20] . Ören, D. Khadjiev, Recognition of plane paths and plane curves under linear pseudo-similarity transformations, Journal of Geometry, Vol. 111, Number 3, 1-24 (2020).
- [21] . Ören, D. Khadjiev, Ö. Peken, Identifications of paths and curves under the plane similarity transformations and their applications to mechanics, Journal of Geometry and Physics, Vol. 151, 103619, 1-17 (2020).
- [22] D. Khadjiev, . Ören, Ö. Peken, Global invariants of paths and curves for the group of all linear similarities in the two-dimensional Euclidean space, International Journal of Geometric Methods in Modern Physics, Vol. 15, Number 6, 1850092 (2018).
- [23] K.K. Muminov, Equivalence of Curves with Respect to the Action of the Symplectic Group, Russian Mathematics (Iz. VUZ), Vol. 53, Number 6, 2428 (2009).
- [24] V. Chilin, K. Muminov, Differential invariants of curves in Galilean geometry, AIP Conference Proceedings, 23-65 (2021).
- [25] H. Weyl, The Classical Groups: Their Invariants and Representations, Princeton Univ. Press, Princeton, New Jersey, 1946.
- [26] C. Zhang, S. Du, J. Liu, Y. Li, J. Xue, Y. Liu, Robust iterative closest point algorithm with bounded rotation angle for 2D registration, Neurocomputing, Vol. 195, 172-180 (2016).
- [27] P.J. Besl, Neil D. McKay, A method for registration of 3-D shapes, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 14, Number 2, 239-256 (1992).
- [28] S. Du, N. Zheng, S. Ying, J. Liu, Affine iterative closest point algorithm for point set registration, Pattern Recognition Letters, Vol. 31, 791-799 (2010).
- [29] H. Chen, X. Zhang, S. Du, Z. Wu, and N. Zheng, A Correntropy-based Affine Iterative Closest Point Algorithm for Robust Point Set Registration, IEEE/CAA J. Autom. Sin. Vol. 6, Number 4, 981-991 (2019).
- [30] I. Weiss, Geometric invariants and object recognition, J. Math. Imaging Vision, Vol. 10, Number 3, 201-231 (1993).
- [31] J. G. Alcázar, Hermoso, M. Carlos, G. Muntingh, Detecting similarity of rational plane curves, J. Comput. Appl. Math., Vol. 269, 113 (2014).
- [32] J. G. Alcázar, C. Hermoso, G. Muntingh, Similarity detection of rational space curves, J. Symbolic Comput., Vol. 85, 424 (2018).
- [33] J. G. Alcázar, Gema M. Diaz-Toca, C. Hermoso On the Problem of Detecting When Two Implicit Plane Algebraic Curves Are Similar, Internat. J. Algebra Comput, Vol. 29, Number 5, 775-793 (2019).

- [34] M. Hauer, B. Jüttler, Detecting affine equivalences of planar rational curves, EuroCG 2016, Lugano, Switzerland, March 30-April 1, 2016.
- [35] J. Sánchez Reyes, Detecting symmetries in polynomial Bézier curves, J. Comput. Appl. Math. , Vol.288, 274283 (2015).
- [36] J. Mozo-Fernandez and C. Munuera, Recognition of Polynomial Plane Curves Under Affine Transformations, AAEECC 13, 121-136 (2002).
- [37] M. İncesu, O. Gürsoy ,LS(2)-Equivalence conditions of control points and application to planar Bézier curves, New Trends in Mathematical Science, Vol. 3, 70-84 (2017).
- [38] K. S. Sibirskii, Introduction to the Algebraic Invariants of Differential Equations, Manchester University Press, New York, 1988.
- [39] D. Khadjiev, On invariants of immersions of an n-dimensional manifold in an n-dimensional pseudo-euclidean space, J. Nonlinear Math. Phys. , Vol. 17, Number 1, 49-70 (2010).