

GEOMETRICAL ASPECTS OF MOTION OF CHARGED PARTICLES IN MAGNETIC AND KILLING MAGNETIC FIELDS AND THEIR CORRESPONDING TRAJECTORIES IN ANTI-DE SITTER 3-SPACE

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ABSTRACT. We inquire the motion of charged particles varying under the effect of Lorentz force produced by magnetic and Killing magnetic fields in anti-de Sitter 3-space $\mathbb{H}_1^3$. Primarily, we characterize magnetic trajectories in $\mathbb{H}_1^3$ in terms of their Frenet apparatus. Thereafter, utilizing a geometrical model of split-quaternions for $\mathbb{H}_1^3$ (where $\mathbb{H}_1^3$ corresponds to a subspace of the Lie group $\mathbf{H}$ of split-quaternions) we find 6 independent unit Killing vector fields on $\mathbb{H}_1^3$ which constitutes a basis for the corresponding 6D Lie algebra $i(\mathbb{H}_1^3)$. We conclude with characterizations of Killing magnetic trajectories in $\mathbb{H}_1^3$ with respect to quasi-slope, curvature and torsion or pseudo-torsion (depending on the situation).

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1. INTRODUCTION

Let us consider the study involving the motion of varying charged particles in magnetic fields and their trajectories in semi-Riemannian manifolds. Such study may be treated as a significant overlapping domain of *Differential Geometry* and *Physics*. In *Physics*, the branch *Magnetostatics* is concerned with the study of *magnetic fields* and associated *Lorentz forces* (or *magnetic forces*). In a magnetostatic phenomenon, a charged particle varies in a magnetic field with constant speed under the effect of Lorentz force that is orthogonal to the velocity of the particle as well as the magnetic field [13, 25, 24]. In language of *Differential Geometry*, the motion of such charged particles may be expounded as follows [4, 23]:

Let $(\mathcal{M}, g)$ denote a semi-Riemannian manifold (of dimension at least 2) and $\mathcal{D}$ denote its Levi-Civita connection. A closed 2-form B on $\mathcal{M}$ is referred to as a *magnetic field* and an endomorphism field $\mathcal{F}$ on $\mathcal{M}$ satisfying

$$g(\mathcal{F}X, Y) = B(X, Y)$$

for all $X, Y \in \mathfrak{X}(\mathcal{M})$ is the associated *Lorentz force*. Such a magnetic field B gives rise to a *magnetic flow* composed of trajectories in $\mathcal{M}$ admitting the *Lorentz equation*

$$\mathcal{D}_{\alpha'}\alpha' = \mathcal{F}\alpha'.$$

With this outset in mind, *magnetic trajectories* in a magnetic field B are considered as smooth curves α in $\mathcal{M}$ pleasing the associated Lorentz equation. The skew-symmetry nature of $\mathcal{F}$ yields

$$g(\mathcal{D}_{\alpha'}\alpha', \alpha') = 0$$

which shows that $g(\alpha', \alpha')$ is constant on α . As a result, magnetic trajectories possess constant speed. In particular, unit-speed magnetic trajectories are called *normal magnetic trajectories*. On the other hand, any *geodesic* α of $\mathcal{M}$ satisfies the geodesic equation

$$\mathcal{D}_{\alpha'}\alpha' = 0$$

which is nothing but the Lorentz equation when B is trivial (i.e., when $B \equiv 0$). Thus, geodesics are regarded as trivial magnetic trajectories.

Now, if $\mathcal{M}$ is three-dimensional, there is a bijection between *2-forms* and *smooth vector fields* on $\mathcal{M}$ by means of *volume form* Ω_g and *Hodge star operator* $\star$ on

$\mathcal{M}$. Thus, 2-forms and smooth vector fields on $\mathcal{M}$ are identified objects. For any $X \in \mathfrak{X}(\mathcal{M})$, one finds

$$\mathcal{L}_X \Omega_g = d(\iota_X \Omega_g) = (\operatorname{div} X) \Omega_g,$$

where $\mathcal{L}_X$ and ι_X denote respectively the Lie derivative with respect to X and the interior contraction with X . The 2-form $\star X^\flat = \iota_X \Omega_{\hat{g}}$ becomes closed iff $\operatorname{div} X = 0$. Consequently, magnetic fields may be viewed as vector fields with vanishing divergence (cf. [1] or [2]). The Lorentz force $\mathcal{F}_X$ of a magnetic field B_X may be expressed as

$$\mathcal{F}_X Y = X \wedge Y$$

for all $Y \in \mathfrak{X}(\mathcal{M})$, where $\wedge$ stands for the cross product on $\mathcal{M}$. Then the associated Lorentz equation reduces to

$$\mathcal{D}_{\alpha'} \alpha' = X \wedge \alpha'.$$

As a matter of fact, any Killing vector field on $\mathcal{M}$ has vanishing divergence and therefore describes a particular kind of magnetic field, known as *Killing magnetic field*, that produces a magnetic flow made up of trajectories, referred to as *Killing magnetic trajectories* (cf. [1] or [2]). Such trajectories are intensively studied in assorted ambient spaces, e.g., *Euclidean 3-space* [10], *2D and 3D sphere* [5], *Minkowski 3-space* [9], *3D warped product* [18], *de Sitter 3-space* [17], etc.

In [6], the authors studied *Hopf magnetic curves* in *anti-de Sitter 3-space* $\mathbb{H}_1^3$ using the technique of hyperbolic Hopf fibration. Predominantly, they investigated magnetic curves of the hyperbolic Hopf vector field which is appeared as a Killing vector field on $\mathbb{H}_1^3$. In this article, we consider more general features of the motion of charged particles in magnetic and Killing magnetic fields and their trajectories in $\mathbb{H}_1^3$. The article is structured into the following main sections: In section 2, we discourse some basic framework on $\mathbb{H}_1^3$. Section 3 is dedicated to recall some fundamental notions of curves in $\mathbb{H}_1^3$ including their unit-speed (arc length or pseudo-arc length) parametrizations, Frenet apparatus and the Frenet formulae for regular curves in $\mathbb{H}_1^3$ in different causality situations. In section 4, we first describe the motion in magnetic fields and then characterize normal magnetic trajectories of non-trivial magnetic fields on $\mathbb{H}_1^3$ in respect of their Frenet apparatus. Section 5 is devoted to obtain Killing vector fields on $\mathbb{H}_1^3$ and characterize the same on regular curves. In section 6, we describe the motion in Killing magnetic fields on $\mathbb{H}_1^3$. Here,

our main focus is to distinguish Killing magnetic trajectories in $\mathbb{H}_1^3$ with respect to quasi-slope, curvature and torsion or pseudo-torsion.

2. BASIC FRAMEWORK

Let $\mathbb{E}_2^4$ denote a four-dimensional *semi-Euclidean space* with index 2, i.e., a 4-dimensional smooth manifold $\mathbb{R}^4$ endowed with a semi-Euclidean metric with signature $(-, -, +, +)$ or $(+, +, -, -)$. Throughout this article, we prefer to deal with $\mathbb{E}_2^4$ possessing the first signature. In this regard, the semi-Euclidean metric g on $\mathbb{R}^4$ with signature $(-, -, +, +)$ is defined by

$$g \equiv - \sum_{i=1}^2 dx^i \otimes dx^i + \sum_{j=3}^4 dx^j \otimes dx^j,$$

where x^1, x^2, x^3 and x^4 denote the natural coordinates in $\mathbb{R}^4$. Since the metric g is indefinite everywhere on $\mathbb{E}_2^4$, a tangent vector x to $\mathbb{E}_2^4$ may have one among three *causal characters* [21]: x is *spacelike* iff $g(x, x) > 0$ or $x = 0$; x is *null* iff $g(x, x) = 0$ and $x \neq 0$; x is *timelike* iff $g(x, x) < 0$. This terminology is adopted from the *Theory of Relativity*. For a tangent vector x to $\mathbb{E}_2^4$, its *norm* is denoted and defined by

$$\|x\| := \sqrt{\text{sign}(x) g(x, x)},$$

where $\text{sign}(x)$ stands for the *signature* of x defined by

$$\text{sign}(x) := \begin{cases} 1 & \text{if } x \text{ is spacelike,} \\ 0 & \text{if } x \text{ is null,} \\ -1 & \text{if } x \text{ is timelike.} \end{cases}$$

It is trivial to mention that x is a *unit vector* iff $\|x\| = 1$, i.e., iff $g(x, x) = \text{sign}(x)$. As usual, tangent vectors x and y are *orthogonal* iff $g(x, y) = 0$. Causality of tangent vectors to $\mathbb{E}_2^4$ can be generalized to the same of vector fields on $\mathbb{E}_2^4$. An *orthonormal frame field* on $\mathbb{E}_2^4$ is an ordered set consisting of four mutually orthogonal unit vector fields on $\mathbb{E}_2^4$ [21]. For an orthonormal frame field on $\mathbb{E}_2^4$, it necessarily includes two timelike together with two spacelike unit vector fields on $\mathbb{E}_2^4$. However, there are other combinations of vector fields if someone wishes to deal with non-orthonormal frame field on $\mathbb{E}_2^4$. For example, it is possible to construct a frame field consisting some or entirely of null vector fields referred to as a *pseudo-orthonormal* or *null*

frame field on $\mathbb{E}_2^4$. The *standard orthonormal frame field* on $\mathbb{E}_2^4$ is the orthonormal frame field $\{\partial_{x^1}, \partial_{x^2}, \partial_{x^3}, \partial_{x^4}\}$ consisting of coordinate vector fields given by

$$\partial_{x^i} = \frac{\partial}{\partial x^i}, \quad i \in \{1, 2, 3, 4\},$$

such that

$$\begin{cases} g(\partial_{x^1}, \partial_{x^1}) = -1, & g(\partial_{x^2}, \partial_{x^2}) = -1, \\ g(\partial_{x^3}, \partial_{x^3}) = 1 & g(\partial_{x^4}, \partial_{x^4}) = 1. \end{cases}$$

The *anti-de Sitter 3-space*, denoted by $\mathbb{H}_1^3$, is a semi-Riemannian hypersurface [21] of $\mathbb{E}_2^4$ defined by

$$\mathbb{H}_1^3 := \{x \in \mathbb{E}_2^4 : g(x, x) = -1\}$$

which is the unit *pseudo-hyperbolic space* with centre at the origin of $\mathbb{E}_2^4$. It is the standard model for a 3D *non-flat Lorentzian space form* with negative constant sectional curvature, where $\mathbb{H}_1^3$ is equipped with the *Lorentzian metric* $\bar{g}$ originated from the semi-Euclidean metric g on $\mathbb{E}_2^4$, that is, $\bar{g} = g|_{\mathbb{H}_1^3}$. Causality in $\mathbb{E}_2^4$ generates the same in $\mathbb{H}_1^3$ as: Given $p \in \mathbb{H}_1^3$, a tangent vector $v \in T_p\mathbb{H}_1^3 \subset \mathbb{E}_2^4$ is *spacelike* iff $\bar{g}(v, v) > 0$ or $v = 0$; *null* (popularly known as *lightlike* for Lorentzian case) iff $\bar{g}(v, v) = 0$ and $v \neq 0$; *timelike* iff $\bar{g}(v, v) < 0$. The same may be extended to vector fields on $\mathbb{H}_1^3$.

2.1. Gauss formula for $\mathbb{H}_1^3$. Let $\mathcal{D}$ and $\bar{\mathcal{D}}$ denote respectively the Levi-Civita connections on $\mathbb{E}_2^4$ and $\mathbb{H}_1^3$. Then they are related by the *Gauss formula*:

$$(1) \quad \mathcal{D}_X Y = \bar{\mathcal{D}}_X Y - h(X, Y) \xi$$

for all vector fields $X, Y \in \mathfrak{X}(\mathbb{H}_1^3)$. Here h is the *scalar-valued second fundamental form* of $\mathbb{H}_1^3$ in $\mathbb{E}_2^4$ and ξ is the *unit normal* to $\mathbb{H}_1^3$ (cf. [21]). As a matter of fact, $\xi = P$, where $P : \mathbb{H}_1^3 \rightarrow \mathbb{E}_2^4$ is the *position vector field* of $\mathbb{H}_1^3$ in $\mathbb{E}_2^4$ given by $P = \sum_{i=1}^4 x^i \partial_{x^i}$, $-\sum_{i=1}^2 (x^i)^2 + \sum_{j=3}^4 (x^j)^2 = -1$ [21]. Let S_P denote the shape operator on $\mathbb{H}_1^3$ derived from P . Then one can obtain the *Weingarten formula* as follows:

$$(2) \quad \mathcal{D}_X P = -S_P(X)$$

for all $X \in \mathfrak{X}(\mathbb{H}_1^3)$ (cf. [21]). On the other hand, we have

$$h(X, Y) = \bar{g}(S_\xi(X), Y)$$

for all $X, Y \in \mathfrak{X}(\mathbb{H}_1^3)$. Moreover, totally umbilic nature of $\mathbb{H}_1^3$ gives $S_P \equiv -I_{\mathfrak{X}(\mathbb{H}_1^3)}$, where $I_{\mathfrak{X}(\mathbb{H}_1^3)}$ stands for the identity transformation. Hence, the Gauss formula (1) becomes

$$(3) \quad \mathcal{D}_X Y = \bar{\mathcal{D}}_X Y + \bar{g}(X, Y)P$$

for all $X, Y \in \mathfrak{X}(\mathbb{H}_1^3)$. Because of this formula one may exhibit the geometry of $\mathbb{H}_1^3$ in connection with the same of $\mathbb{E}_2^4$.

2.2. Cross product on $\mathbb{H}_1^3$. The *cross product* on $\mathbb{E}_2^4$, $\times : \mathfrak{X}(\mathbb{E}_2^4) \times \mathfrak{X}(\mathbb{E}_2^4) \times \mathfrak{X}(\mathbb{E}_2^4) \rightarrow \mathfrak{X}(\mathbb{E}_2^4)$, is defined as follows: For $X, Y, Z \in \mathfrak{X}(\mathbb{E}_2^4)$, their cross product is the unique vector field $X \times Y \times Z \in \mathfrak{X}(\mathbb{E}_2^4)$ determined by

$$g(X \times Y \times Z, V) = \Omega_g(X, Y, Z, V)$$

for any $V \in \mathfrak{X}(\mathbb{E}_2^4)$, where Ω_g denotes the *volume form* on $\mathbb{E}_2^4$. Relative to the orthonormal frame field $\{\partial_{x^1}, \partial_{x^2}, \partial_{x^3}, \partial_{x^4}\}$ on $\mathbb{E}_2^4$, one can define the following:

$$\begin{cases} \partial_{x^1} \times \partial_{x^2} \times \partial_{x^3} = \partial_{x^4}, & \partial_{x^2} \times \partial_{x^3} \times \partial_{x^4} = -\partial_{x^1}, \\ \partial_{x^3} \times \partial_{x^4} \times \partial_{x^1} = -\partial_{x^2}, & \partial_{x^4} \times \partial_{x^1} \times \partial_{x^2} = \partial_{x^3}. \end{cases}$$

Now, the cross product $\times$ on $\mathbb{E}_2^4$ induces another cross product on $\mathbb{H}_1^3$, which we denote by $\wedge : \mathfrak{X}(\mathbb{H}_1^3) \times \mathfrak{X}(\mathbb{H}_1^3) \rightarrow \mathfrak{X}(\mathbb{H}_1^3)$, as follows: Given $X, Y \in \mathfrak{X}(\mathbb{H}_1^3) \subset \mathfrak{X}(\mathbb{E}_2^4)$, the *cross product* $X \wedge Y \in \mathfrak{X}(\mathbb{H}_1^3)$ satisfies

$$g(X \wedge Y, Z) = \Omega_{\bar{g}}(X, Y, Z) = \Omega_g(P, X, Y, Z),$$

for all $Z \in \mathfrak{X}(\mathbb{H}_1^3)$, where $\Omega_{\bar{g}}$ stands for the *volume form* on $\mathbb{H}_1^3$. Therefore, we have

$$(4) \quad X \wedge Y = P \times X \times Y.$$

Thus, the standard orientation on $\mathbb{E}_2^4$ induces an orientation on $\mathbb{H}_1^3$ in such a way that a frame field $\{U, V, W\}$ on $\mathbb{H}_1^3$ is positively oriented iff $\Omega_g(P, U, V, W) > 0$, i.e., $\{P, U, V, W\}$ is a positively oriented frame of $\mathbb{E}_2^4$.

3. FUNDAMENTALS OF CURVES IN $\mathbb{H}_1^3$

Let I and J be two non-trivial intervals in $\mathbb{R}$ (i.e., connected subsets of $\mathbb{R}$ consisting of more than one point). Let $\alpha : J \rightarrow \mathbb{H}_1^3$ be an arbitrary curve in $\mathbb{H}_1^3$ and α' denote its *velocity vector field*. Then α may globally or locally have one among three *causalities* which is determined by the same of its velocities $\alpha'(t)$ [21]: for all

possible t , α is *spacelike* iff $\alpha'(t)$ are spacelike; α is *lightlike* iff $\alpha'(t)$ are lightlike; α is *timelike* iff $\alpha'(t)$ are timelike. Thus, α is globally spacelike, lightlike or timelike in $\mathbb{H}_1^3$ iff its velocity field α' is respectively spacelike, null or timelike on α . If α' vanishes nowhere, then α is a regular curve in $\mathbb{H}_1^3$ and the Gauss formula on α is obtained from (3) by

$$(5) \quad \mathcal{D}_{\alpha'} X_\alpha = \bar{\mathcal{D}}_{\alpha'} X_\alpha + \bar{g}(\alpha', X_\alpha) \alpha$$

for all vector field $X_\alpha \in \mathfrak{X}(\alpha(J)) \subset \mathfrak{X}(\mathbb{H}_1^3)$. Moreover, α reduces to a geodesic of $\mathbb{H}_1^3$ iff $\alpha' \in \mathfrak{X}(\alpha(J))$ is parallel on α , i.e., $\bar{\mathcal{D}}_{\alpha'} \alpha' = 0$, i.e., $\mathcal{D}_{\alpha'} \alpha' = \bar{g}(\alpha', \alpha') \alpha$.

If α is *non-lightlike* (*spacelike* or *timelike*) and we change parameter t by *arc length function* $s : J \rightarrow I$ based at some point $t_0 \in I$ given by

$$s(t) = \int_{t_0}^t \sqrt{\text{sign}(\alpha'(u)) \bar{g}(\alpha'(u), \alpha'(u))} du,$$

then we have $g(\alpha'(s), \alpha'(s)) = \delta$ for all $s \in I$, where δ can take two values: $\delta = 1$ if α is spacelike and $\delta = -1$ if α is timelike. In this case, $\alpha : I \rightarrow \mathbb{H}_1^3$ is said to be *parametrized by arc length s* or *unit-speed* in $\mathbb{H}_1^3$. On the other hand, if α is *lightlike*, then its velocity vector field α' is also lightlike and hence cannot be normalized in the usual way as arc length function vanishes everywhere. To overcome this, one requires to introduce another parameter, referred to as *pseudo-arc length function*, that normalizes the acceleration vector field $\bar{\mathcal{D}}_{\alpha'} \alpha'$ (which is spacelike) unless it vanishes nowhere. If we change an arbitrary parameter t by *pseudo-arc length function* $s : J \rightarrow I$ based at some point $t_0 \in I$ given by

$$s(t) = \int_{t_0}^t \sqrt[4]{\bar{g}(\bar{\mathcal{D}}_{\alpha'} \alpha'(u), \bar{\mathcal{D}}_{\alpha'} \alpha'(u))} du,$$

then we have $g(\bar{\mathcal{D}}_{\alpha'} \alpha'(s), \bar{\mathcal{D}}_{\alpha'} \alpha'(s)) = 1$ for all $s \in I$. In this case, $\alpha : I \rightarrow \mathbb{H}_1^3$ is said to be *pseudo-arc length parametrized* or *unit-speed* in $\mathbb{H}_1^3$.

3.1. Frenet apparatus on non-lightlike curves possessing non-lightlike acceleration. Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed non-lightlike curves possessing non-lightlike acceleration (i.e., α be either spacelike possessing non-lightlike acceleration or timelike). If α is not a geodesic of $\mathbb{H}_1^3$, we consider the *Frenet apparatus* $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ on α defined as follows: $T_\alpha = \alpha'$ is the *unit tangent field* of α ; N_α is the *unit principal normal field* of α which is determined by normalizing the acceleration field $\bar{\mathcal{D}}_{\alpha'} T_\alpha$; B_α is the *unit binormal field* of α which is orthogonal to

T_α as well as N_α so that $\{T_\alpha, N_\alpha, B_\alpha\}$ forms the *dynamic Frenet frame* (preserving the same orientation as that of $\mathbb{H}_1^3$) on α ; $\kappa_\alpha > 0$ is the *curvature* and τ_α is the *torsion* of α . *Frenet formulae* (cf. [3, 15]) for α are

$$(6) \quad \begin{cases} \bar{\mathcal{D}}_{\alpha'} T_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \delta_1 \alpha = \delta_2 \kappa_\alpha N_\alpha, \\ \bar{\mathcal{D}}_{\alpha'} N_\alpha = \mathcal{D}_{\alpha'} N_\alpha = -\delta_1 \kappa_\alpha T_\alpha + \delta_3 \tau_\alpha B_\alpha, \\ \bar{\mathcal{D}}_{\alpha'} B_\alpha = \mathcal{D}_{\alpha'} B_\alpha = -\delta_2 \tau_\alpha N_\alpha, \end{cases}$$

where $\delta_1 = \text{sign}(T_\alpha)$, $\delta_2 = \text{sign}(N_\alpha)$ and $\delta_3 = \text{sign}(B_\alpha)$ such that $\delta_1 \delta_2 \delta_3 = -1$. It is to be noted that for positive orientation of the Frenet frame $\{T_\alpha, N_\alpha, B_\alpha\}$ on α , we have $\Omega_{\bar{g}}(T_\alpha, N_\alpha, B_\alpha) = \Omega_g(\alpha, T_\alpha, N_\alpha, B_\alpha) = 1$ for which we must have

$$T_\alpha \wedge N_\alpha = \delta_3 B_\alpha, \quad N_\alpha \wedge B_\alpha = \delta_1 T_\alpha, \quad B_\alpha \wedge T_\alpha = \delta_2 N_\alpha.$$

3.2. Frenet apparatus on spacelike curves possessing lightlike acceleration. Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed spacelike curve possessing lightlike acceleration. If α is not a geodesic of $\mathbb{H}_1^3$, we take the *Frenet apparatus* $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ for α described as follows: $T_\alpha = \alpha'$ is the *unit tangent field* of α ; N_α is the *principal normal field* of α obtained by $N_\alpha = \bar{\mathcal{D}}_{\alpha'} T_\alpha$ and it is lightlike on α ; B_α is the *binormal field* of α determined by $\bar{g}(N_\alpha, B_\alpha) = 1$ and it is orthogonal to T_α such that $\{T_\alpha, N_\alpha, B_\alpha\}$ structures the *moving pseudo-orthonormal frame* (also known as the *Cartan frame*) on α ; $\kappa_\alpha \equiv 1$ is the *curvature* and τ_α is the *pseudo-torsion* of α . *Frenet formulae* for α are given by

$$(7) \quad \begin{cases} \bar{\mathcal{D}}_{\alpha'} T_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \alpha = N_\alpha, \\ \bar{\mathcal{D}}_{\alpha'} N_\alpha = \mathcal{D}_{\alpha'} N_\alpha = \tau_\alpha B_\alpha, \\ \bar{\mathcal{D}}_{\alpha'} B_\alpha = \mathcal{D}_{\alpha'} B_\alpha = -T_\alpha - \tau_\alpha B_\alpha. \end{cases}$$

In this case, we must have

$$T_\alpha \wedge N_\alpha = N_\alpha, \quad N_\alpha \wedge B_\alpha = T_\alpha, \quad B_\alpha \wedge T_\alpha = B_\alpha.$$

Moreover, if $\Omega_{\bar{g}}(T_\alpha, N_\alpha, B_\alpha) = \Omega_g(\alpha, T_\alpha, N_\alpha, B_\alpha) = 1$, then $\{T_\alpha, N_\alpha, B_\alpha\}$ is positively oriented.

3.3. Frenet apparatus on lightlike curves. Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed lightlike curve. If α is not a lightlike geodesic of $\mathbb{H}_1^3$, we consider the *Frenet apparatus* $\{T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha\}$ on α which is defined as follows: $T_\alpha = \alpha'$ is the *tangent field* of α which is a lightlike vector field on α ; N_α is the *unit principal normal field* of

α which is a unit spacelike vector field on α such that $N_\alpha = \bar{\mathcal{D}}_{\alpha'} T_\alpha$; B_α is the *binormal field* of α which is the unique lightlike vector field on α determined by $\bar{g}(T_\alpha, B_\alpha) = -1$ and orthogonal to N_α so that $\{T_\alpha, N_\alpha, B_\alpha\}$ is the *dynamic pseudo-orthonormal frame*, referred to as the *Cartan frame*, on α ; $\kappa_\alpha \equiv 1$ is the *curvature* and τ_α is the *pseudo-torsion* of α . *Frenet formulae* (cf. [11, 12]) are given by

$$(8) \quad \begin{cases} \bar{\mathcal{D}}_{\alpha'} T_\alpha = \mathcal{D}_{\alpha'} T_\alpha = N_\alpha, \\ \bar{\mathcal{D}}_{\alpha'} N_\alpha = \mathcal{D}_{\alpha'} N_\alpha = -\tau_\alpha T_\alpha + B_\alpha, \\ \bar{\mathcal{D}}_{\alpha'} B_\alpha = \mathcal{D}_{\alpha'} B_\alpha - \alpha = -\tau_\alpha N_\alpha. \end{cases}$$

In this case, we must have

$$T_\alpha \wedge N_\alpha = -T_\alpha, \quad N_\alpha \wedge B_\alpha = -B_\alpha, \quad B_\alpha \wedge T_\alpha = N_\alpha.$$

Moreover, if $\Omega_{\bar{g}}(T_\alpha, N_\alpha, B_\alpha) = 1 = \Omega_g(\alpha, T_\alpha, N_\alpha, B_\alpha)$, then $\{T_\alpha, N_\alpha, B_\alpha\}$ is positively oriented.

4. MOTION IN MAGNETIC FIELDS: MAGNETIC TRAJECTORIES IN $\mathbb{H}_1^3$

According to the general theory, a closed 2-form B on $\mathbb{H}_1^3$ is a *magnetic field* and a skew-symmetric tensor field $\mathcal{F}$ of type $(1, 1)$ which satisfies $\bar{g}(\mathcal{F}X, Y) = B(X, Y)$ for all $X, Y \in \mathfrak{X}(\mathbb{H}_1^3)$ is its *Lorentz force*. The *magnetic trajectory* in a magnetic field B with Lorentz force $\mathcal{F}$ is a smooth curve α in $\mathbb{H}_1^3$ admitting the *Lorentz equation*

$$(9) \quad \bar{\mathcal{D}}_{\alpha'} \alpha' = \mathcal{F} \alpha'.$$

Now, the skew-symmetry nature of $\mathcal{F}$ gives

$$\frac{d}{dt} \bar{g}(\alpha', \alpha') = 2\bar{g}(\bar{\mathcal{D}}_{\alpha'} \alpha', \alpha') = 2\bar{g}(\mathcal{F} \alpha', \alpha') = 0.$$

This indicates $\bar{g}(\alpha', \alpha') = \text{a constant}$. Therefore, magnetic trajectories in $\mathbb{H}_1^3$ are smooth curves of constant speed. Specifically, unit-speed magnetic trajectories (i.e., parametrized by arc length or pseudo-arc length) are known as *normal magnetic trajectories*. Using Gauss formula (5), (9) can be written in connection with the Levi-Civita connection $\mathcal{D}$ on $\mathbb{E}_2^4$ as

$$(10) \quad \mathcal{D}_{\alpha'} \alpha' - \text{sign}(\alpha') \|\alpha'\|^2 \alpha = \mathcal{F} \alpha'.$$

Now, when α is lightlike, (10) reduces to

$$\mathcal{D}_{\alpha'} \alpha' = \mathcal{F} \alpha'$$

which yields magnetic trajectories in $\mathbb{E}_2^4$. Therefore, the following result holds:

Theorem 4.1. *Lightlike magnetic trajectories in a magnetic field B in $\mathbb{H}_1^3$ are lightlike magnetic trajectories in the magnetic extension $\tilde{F}$ (i.e., push forward $\tilde{B} = P_*B$) of B in $\mathbb{E}_2^4$.*

As $\mathbb{H}_1^3$ is a three-dimensional, there is a bijection between 2-forms and smooth vector fields on $\mathbb{H}_1^3$ via volume form $\Omega_{\bar{g}}$ and Hodge star operator on $\mathbb{H}_1^3$ as: If one selects $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with its $\bar{g}$ -equivalent 1-form $X^\flat$, then it accords with a 2-form on $\mathbb{H}_1^3$ as

$$\star X^\flat = \iota_X \Omega_{\bar{g}},$$

where ι_X denotes the contraction with X . On the converse, if one picks a 2-form B on $\mathbb{H}_1^3$, then it accords with a 1-form $\star B$ and therefore $\bar{g}$ -equivalent vector field $(\star F)^\sharp \in \mathfrak{X}(\mathbb{H}_1^3)$. In this aspect, 2-forms and smooth vector fields on $\mathbb{H}_1^3$ are identified objects. Again, we have

$$\mathcal{L}_X \Omega_{\bar{g}} = d(\iota_X \Omega_{\bar{g}}) = \text{div}(X) \Omega_{\bar{g}}$$

for all $X \in \mathfrak{X}(\mathbb{H}_1^3)$. The 2-form $\star X^\flat = \iota_X \Omega_{\bar{g}}$ on $\mathbb{H}_1^3$ becomes closed iff $\text{div}(X) = 0$. In this regard, vector fields on $\mathbb{H}_1^3$ with vanishing divergence are considered as magnetic fields on $\mathbb{H}_1^3$.

Let $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with $\text{div}(X) = 0$. Then it corresponds to a magnetic field $B_X = \iota_X \Omega_{\bar{g}}$ with Lorentz force $\mathcal{F}_X$ such that

$$\bar{g}(\mathcal{F}_X Y, Z) = B_X(Y, Z) = (\iota_X \Omega_{\bar{g}})(Y, Z) = \Omega_{\bar{g}}(X, Y, Z) = \bar{g}(X \wedge Y, Z).$$

for all $Y, Z \in \mathfrak{X}(\mathbb{H}_1^3)$. Therefore, we have

$$\mathcal{F}_X Y = X \wedge Y = P \times X \times Y$$

for all $Y \in \mathfrak{X}(\mathbb{H}_1^3)$, where $\wedge$ and $\times$ are cross products respectively on $\mathbb{H}_1^3$ and $\mathbb{E}_2^4$ and $P : \mathbb{H}_1^3 \rightarrow \mathbb{E}_2^4$ stands for the position field. Hence, (10) becomes

$$(11) \quad \mathcal{D}_{\alpha'} \alpha' - \text{sign}(\alpha') \|\alpha'\|^2 \alpha = P \times X \times \alpha'.$$

4.1. Trivial magnetic trajectories in $\mathbb{H}_1^3$: Geodesics of $\mathbb{H}_1^3$. If a magnetic field B is *trivial* on $\mathbb{H}_1^3$ (i.e., $F \equiv 0$), then its Lorentz force $\mathcal{F}$ vanishes and the associated Lorentz equation becomes $\bar{\mathcal{D}}_{\alpha'}\alpha' = 0$. In this case, magnetic trajectories are nothing but the *geodesics* of $\mathbb{S}_1^3$. A regular curve α in $\mathbb{S}_1^3$ is a geodesic iff it pleases

$$\mathcal{D}_{\alpha'}\alpha' - \text{sign}(\alpha')\|\alpha'\|^2\alpha = 0.$$

The geodesics of $\mathbb{H}_1^3$ can be found in [21].

4.2. Non-trivial magnetic trajectories in $\mathbb{H}_1^3$. We describe the motion of charged particles in non-trivial magnetic fields on $\mathbb{H}_1^3$ and characterize geometrically their associated magnetic trajectories in $\mathbb{H}_1^3$.

4.2.1. Non-lightlike magnetic trajectories possessing non-lightlike acceleration. Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed (arc length parametrized) non-lightlike curve possessing non-lightlike acceleration and B_X be a non-trivial magnetic field associated to $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with $\text{div}(X) = 0$. If α is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then (11) is expressed in connection with its tangent field T_α as follows:

$$(12) \quad \mathcal{D}_{\alpha'}T_\alpha - \delta_1\alpha = \mathcal{F}_XT_\alpha = P \times X \times T_\alpha.$$

The next result exhibits the action of $\mathcal{F}_X$ on the Frenet frame $\{T_\alpha, N_\alpha, B_\alpha\}$.

Proposition 4.2. *If $\alpha : I \rightarrow \mathbb{H}_1^3$ is a unit-speed non-lightlike curve possessing non-lightlike acceleration, then $\mathcal{F}_X$ satisfies*

$$(13) \quad \begin{cases} \mathcal{F}_XT_\alpha = \delta_2\kappa_\alpha N_\alpha, \\ \mathcal{F}_XN_\alpha = -\delta_1\kappa_\alpha T_\alpha + \delta_3\mu_\alpha B_\alpha, \\ \mathcal{F}_XB_\alpha = -\delta_2\mu_\alpha N_\alpha, \end{cases}$$

where $\mu_\alpha : I \rightarrow \mathbb{R}$ is smooth given by $\mu_\alpha = \bar{g}(\mathcal{F}_XN_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_XB_\alpha)$.

Proof. Applying (12) and (6), we get

$$(14) \quad \mathcal{F}_XT_\alpha = \mathcal{D}_{\alpha'}T_\alpha - \delta_1\alpha = \delta_2\kappa_\alpha N_\alpha$$

which gives the first one of (13). Now, there exist functions $f_1, f_2, f_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(15) \quad \mathcal{F}_XN_\alpha = f_1T_\alpha + f_2N_\alpha + f_3B_\alpha.$$

Then using (15), skew-symmetry of $\mathcal{F}_X$ and (14), we find

$$f_1 = \delta_1 \bar{g}(\mathcal{F}_X N_\alpha, T_\alpha) = -\delta_1 \bar{g}(N_\alpha, \mathcal{F}_X T_\alpha) = -\delta_1 \delta_2 \kappa_\alpha \bar{g}(N_\alpha, N_\alpha) = -\delta_1 \kappa_\alpha.$$

Again, skew-symmetry nature of $\mathcal{F}_X$ yields $\bar{g}(\mathcal{F}_X N_\alpha, N_\alpha) = 0$ and hence (15) gives

$$f_2 = \delta_2 \bar{g}(\mathcal{F}_X N_\alpha, N_\alpha) = 0.$$

We define $\mu_\alpha : I \rightarrow \mathbb{R}$ by $\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha)$. Evidently, μ_α is smooth and using (15), we get

$$f_3 = \delta_3 \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\delta_3 \bar{g}(N_\alpha, \mathcal{F}_X B_\alpha) = -\delta_3 \mu_\alpha.$$

Substitution of f_1 , f_2 and f_3 in (15) gives the second one of (13). On the other hand, there exist functions $h_1, h_2, h_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(16) \quad \mathcal{F}_X B_\alpha = h_1 T_\alpha + h_2 N_\alpha + h_3 B_\alpha.$$

Then using (16), skew-symmetry of $\mathcal{F}_X$ and (14), we find

$$h_1 = \delta_1 \bar{g}(\mathcal{F}_X B_\alpha, T_\alpha) = -\delta_1 \bar{g}(B_\alpha, \mathcal{F}_X T_\alpha) = -\delta_1 \delta_2 \kappa_\alpha \bar{g}(B_\alpha, N_\alpha) = 0.$$

Again, using (16), we get $h_2 = \delta_2 \bar{g}(\mathcal{F}_X B_\alpha, N_\alpha) = -\delta_2 \mu_\alpha$. Also, skew-symmetry of $\mathcal{F}_X$ implies that $\bar{g}(\mathcal{F}_X B_\alpha, B_\alpha) = 0$, and hence we obtain $h_3 = \delta_3 \bar{g}(\mathcal{F}_X B_\alpha, B_\alpha) = 0$. Substituting the values of h_1 , h_2 and h_3 in (16), we find the last equation of (13). $\square$

Now, we obtain the next result characterizing spacelike magnetic trajectories possessing non-lightlike acceleration or timelike magnetic trajectories in magnetic background $(\mathbb{H}_1^3, \bar{g}, B_X)$ in respect of their Frenet apparatus.

Theorem 4.3. *A unit-speed non-lightlike curve possessing non-lightlike acceleration $\alpha : I \rightarrow \mathbb{H}_1^3$ is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$ iff the restriction of X to α , $X_\alpha \in \mathfrak{X}(\alpha(I))$, takes the form*

$$(17) \quad X_\alpha = \delta_2 \mu_\alpha T_\alpha + \delta_2 \kappa_\alpha B_\alpha.$$

Proof. Let α be a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$. Then for the restriction of X to α , i.e., $X_\alpha \in \mathfrak{X}(\alpha(I))$, there exist three functions $m_1, m_2, m_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(18) \quad X_\alpha = m_1 T_\alpha + m_2 N_\alpha + m_3 B_\alpha.$$

Now, Proposition 4.2 gives the action of $\mathcal{F}_X$ on the Frenet frame $\{T_\alpha, N_\alpha, B_\alpha\}$. On the other hand, we have $\mathcal{F}_X X_\alpha = P \times X \times X_\alpha = 0$. Therefore, using (13) and (18), we obtain

$$0 = -\delta_1 m_2 \kappa_\alpha T_\alpha + \delta_2 (m_1 \kappa_\alpha - m_3 \mu_\alpha) N_\alpha + \delta_3 m_2 \mu_\alpha B_\alpha$$

which implies $m_2 = 0$ and $m_1 \kappa_\alpha - m_3 \mu_\alpha = 0$. Therefore, we find $m_1 = \delta_2 \mu_\alpha$ and $m_3 = \delta_2 \kappa_\alpha$. Substituting the values of m_1 , m_2 and m_3 in (18), we find (17).

Conversely, let $X_\alpha \in \mathfrak{X}(\alpha(I))$ be given by (17). Using (17), we find

$$\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha = \delta_2 \kappa_\alpha N_\alpha.$$

Then applying (6), we obtain

$$\mathcal{F}_X T_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \delta_1 \alpha$$

which is the respective Lorentz equation. Consequently, α is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$. $\square$

Remark 4.4. *For a non-lightlike magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$ possessing non-lightlike acceleration, $X_\alpha \in \mathfrak{X}(\alpha(I))$ always resides in rectifying planes of α .*

Remark 4.5. *The function μ_α obtained by $\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha)$ is called the “quasi-slope” [2] of α with respect to the non-trivial magnetic field B_X .*

4.2.2. Spacelike magnetic trajectories possessing lightlike acceleration. Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed (arc length parametrized) spacelike curve possessing lightlike acceleration and B_X be a non-trivial magnetic field corresponding to $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with $\text{div}(X) = 0$. If α is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then the associated Lorentz equation (11) may be written as

$$(19) \quad \mathcal{D}_{\alpha'} T_\alpha - \alpha = \mathcal{F}_X T_\alpha = P \times X \times T_\alpha.$$

We obtain the following result that expresses how $\mathcal{F}_X$ acts on the pseudo-orthonormal frame $\{T_\alpha, N_\alpha, B_\alpha\}$ on α :

Proposition 4.6. *If $\alpha : I \rightarrow \mathbb{H}_1^3$ is a unit-speed spacelike curve possessing lightlike acceleration, then $\mathcal{F}_X$ satisfies*

$$(20) \quad \begin{cases} \mathcal{F}_X T_\alpha = N_\alpha, \\ \mathcal{F}_X N_\alpha = \mu_\alpha N_\alpha, \\ \mathcal{F}_X B_\alpha = -T_\alpha - \mu_\alpha B_\alpha, \end{cases}$$

where $\mu_\alpha : I \rightarrow \mathbb{R}$ is smooth given by $\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha)$, $B_\alpha = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha)$.

Proof. Applying (19) and (7), we find

$$(21) \quad \mathcal{F}_X T_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \alpha = N_\alpha.$$

which is the first equation of (20). Now, there exist functions $f_1, f_2, f_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(22) \quad \mathcal{F}_X N_\alpha = f_1 T_\alpha + f_2 N_\alpha + f_3 B_\alpha.$$

Then using (22), skew-symmetry of $\mathcal{F}_X$ and (21), we find

$$f_1 = \bar{g}(\mathcal{F}_X N_\alpha, T_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X T_\alpha) = -\bar{g}(N_\alpha, N_\alpha) = 0.$$

We define a function $\mu_\alpha : I \rightarrow \mathbb{R}$ by

$$\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha).$$

Then μ_α is smooth and since $\bar{g}(N_\alpha, B_\alpha) = 1$, using (22), we get

$$f_2 = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = \mu_\alpha.$$

Again, skew-symmetry of $\mathcal{F}_X$ implies $\bar{g}(\mathcal{F}_X N_\alpha, N_\alpha) = 0$, and since $\bar{g}(B_\alpha, N_\alpha) = 1$, using (22), we obtain

$$f_3 = \bar{g}(\mathcal{F}_X N_\alpha, N_\alpha) = 0.$$

Substituting the values of f_1, f_2 and f_3 in (22), we find the second equation of (20). Moreover, there exist functions $h_1, h_2, h_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(23) \quad \mathcal{F}_X B_\alpha = h_1 T_\alpha + h_2 N_\alpha + h_3 B_\alpha.$$

Since $\bar{g}(B_\alpha, N_\alpha) = 1$, using (23), skew-symmetry of $\mathcal{F}_X$ and (21), we find

$$h_1 = \bar{g}(\mathcal{F}_X B_\alpha, T_\alpha) = -\bar{g}(B_\alpha, \mathcal{F}_X T_\alpha) = -\bar{g}(B_\alpha, N_\alpha) = -1.$$

Again, skew-symmetry of $\mathcal{F}_X$ implies $\bar{g}(\mathcal{F}_X B_\alpha, B_\alpha) = 0$ and since $\bar{g}(N_\alpha, B_\alpha) = 1$, using (23) we obtain

$$h_2 = \bar{g}(\mathcal{F}_X B_\alpha, B_\alpha) = 0.$$

Since $\bar{g}(B_\alpha, N_\alpha) = 1$, using (23) we get

$$h_3 = \bar{g}(\mathcal{F}_X B_\alpha, N_\alpha) = -\mu_\alpha.$$

Substituting the values of h_1, h_2 and h_3 in (23), we find the last equation of (20). $\square$

The next result characterizes spacelike normal trajectories in $(\mathbb{H}_1^3, \bar{g}, B_X)$ possessing lightlike acceleration in connection with their Frenet apparatus.

Theorem 4.7. *A unit-speed spacelike curve possessing lightlike acceleration $\alpha : I \rightarrow \mathbb{H}_1^3$ is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$ iff the restriction of X to α , $X_\alpha \in \mathfrak{X}(\alpha(I))$, has expression*

$$(24) \quad X_\alpha = \mu_\alpha T_\alpha - N_\alpha.$$

Proof. Let α be a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$. Then for the restriction of X to α , i.e., $X_\alpha \in \mathfrak{X}(\alpha(I))$, there exist three functions $m_1, m_2, m_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(25) \quad X_\alpha = m_1 T_\alpha + m_2 N_\alpha + m_3 B_\alpha.$$

Proposition 4.6 gives the action of $\mathcal{F}_X$ on the pseudo-orthonormal frame $\{T_\alpha, N_\alpha, B_\alpha\}$. Again, we have $\mathcal{F}_X X_\alpha = P \times X \times X_\alpha = 0$. Therefore, using (20) and (25), we obtain

$$0 = -m_3 T_\alpha + (m_1 + m_2 \mu_\alpha) N_\alpha - m_3 \mu_\alpha B_\alpha$$

which implies $m_1 + m_2 \mu_\alpha = 0$ and $m_3 = 0$. Again, since $\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha$, $\alpha \times T_\alpha \times T_\alpha = 0$, $\alpha \times N_\alpha \times T_\alpha = -N_\alpha$ and $\alpha \times B_\alpha \times T_\alpha = B_\alpha$, using (20) and (25), we find $m_2 = -1$ and hence $m_1 = \mu_\alpha$. Substituting the values of m_1, m_2 and m_3 in (25), we find (24).

Conversely, let $X_\alpha \in \mathfrak{X}(\alpha(I))$ be given by (24). Since $\alpha \times T_\alpha \times T_\alpha = 0$ and $\alpha \times N_\alpha \times T_\alpha = -N_\alpha$, using (24), we find

$$\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha = N_\alpha.$$

Then applying (7), we obtain

$$\mathcal{F}_X T_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \alpha$$

which is the associated Lorentz equation. Hence, α is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_U)$. $\square$

Remark 4.8. For a spacelike magnetic trajectory α in $(\mathbb{H}_1^3, \bar{g}, B_X)$ possessing lightlike acceleration, $X_\alpha \in \mathfrak{X}(\alpha(I))$ always resides in osculating planes of α .

4.2.3. *Lightlike magnetic trajectories.* Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed (pseudo-arc length parametrized) lightlike curve and B_X be a non-trivial magnetic field corresponding to $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with $\text{div}(X) = 0$. If α is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then the associated Lorentz equation (11) can be written in terms of its tangent vector field T_α as follows:

$$(26) \quad \mathcal{D}_\alpha' T_\alpha = \mathcal{F}_X T_\alpha = P \times X \times T_\alpha,$$

where $\mathcal{F}_X$ is the Lorentz force of B_X . We determine the following result that manifests how $\mathcal{F}_X$ acts on the vector fields in the dynamic pseudo-orthonormal frame $\{T_\alpha, N_\alpha, B_\alpha\}$ on α :

Proposition 4.9. If $\alpha : I \rightarrow \mathbb{H}_1^3$ is a unit-speed lightlike curve, then $\mathcal{F}_X$ satisfies

$$(27) \quad \begin{cases} \mathcal{F}_X T_\alpha = N_\alpha, \\ \mathcal{F}_X N_\alpha = -\mu_\alpha T_\alpha + B_\alpha, \\ \mathcal{F}_X B_\alpha = -\mu_\alpha N_\alpha, \end{cases}$$

where $\mu_\alpha : I \rightarrow \mathbb{R}$ is smooth such that $\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha)$.

Proof. Using (26) and (8), we find

$$(28) \quad \mathcal{F}_X T_\alpha = \mathcal{D}_\alpha' T_\alpha = N_\alpha$$

which is the first equation of (27). Now, there exist functions $f_1, f_2, f_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(29) \quad \mathcal{F}_X N_\alpha = f_1 T_\alpha + f_2 N_\alpha + f_3 B_\alpha.$$

We define a function $\mu_\alpha : I \rightarrow \mathbb{R}$ by

$$\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha).$$

Then μ_α is smooth and since $\bar{g}(T_\alpha, B_\alpha) = -1$, using (29) we obtain

$$f_1 = -\bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\mu_\alpha.$$

Again, skew-symmetry of $\mathcal{F}_X$ implies $\bar{g}(\mathcal{F}_X N_\alpha, N_\alpha) = 0$, and therefore using (15) we get

$$f_2 = \bar{g}(\mathcal{F}_X N_\alpha, N_\alpha) = 0.$$

Since $\bar{g}(B_\alpha, T_\alpha) = -1$, using (29), skew-symmetry of $\mathcal{F}_X$ and (28), we find

$$f_3 = -\bar{g}(\mathcal{F}_X N_\alpha, T_\alpha) = \bar{g}(N_\alpha, \mathcal{F}_X T_\alpha) = \bar{g}(N_\alpha, N_\alpha) = 1.$$

Substituting the values of f_1, f_2, f_3 in (29), we find the second equation of (27). On the other hand, there exist functions $h_1, h_2, h_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(30) \quad \mathcal{F}_X B_\alpha = h_1 T_\alpha + h_2 N_\alpha + h_3 B_\alpha.$$

Then skew-symmetry of $\mathcal{F}_X$ implies $\bar{g}(\mathcal{F}_X B_\alpha, B_\alpha) = 0$ and since $\bar{g}(T_\alpha, B_\alpha) = -1$, using (30) we obtain

$$h_1 = -\bar{g}(\mathcal{F}_X B_\alpha, B_\alpha) = 0.$$

Again, using (30), we get

$$h_2 = \bar{g}(\mathcal{F}_X B_\alpha, N_\alpha) = -\mu_\alpha.$$

Since $\bar{g}(B_\alpha, T_\alpha) = -1$, using (30), skew-symmetry of $\mathcal{F}_X$ and (28), we find

$$h_3 = -\bar{g}(\mathcal{F}_X B_\alpha, T_\alpha) = \bar{g}(B_\alpha, \mathcal{F}_X T_\alpha) = \bar{g}(B_\alpha, N_\alpha) = 0.$$

Substituting the values of h_1, h_2 and h_3 in (30), we find the last equation of (27). $\square$

Next, we obtain the following result characterizing normal lightlike trajectories possessing lightlike acceleration in magnetic background $(\mathbb{H}_1^3, \bar{g}, B_X)$ in respect of their Frenet apparatus:

Theorem 4.10. *A unit-speed lightlike curve $\alpha : I \rightarrow \mathbb{H}_1^3$ a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$ iff the restriction of X to α , $X_\alpha \in \mathfrak{X}(\alpha(I))$, takes the form*

$$(31) \quad X_\alpha = \mu_\alpha T_\alpha + B_\alpha.$$

Proof. Let α be a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_X)$. Then for the restriction of X to α , i.e., $X_\alpha \in \mathfrak{X}(\alpha(I))$, there exist three functions $m_1, m_2, m_3 : I \rightarrow \mathbb{R}$, not all vanishing simultaneously, such that we can express

$$(32) \quad X_\alpha = m_1 T_\alpha + m_2 N_\alpha + m_3 B_\alpha.$$

Proposition 4.9 informs the action of the associated Lorentz force $\mathcal{F}_X$ in the dynamic pseudo-orthonormal frame $\{T_\alpha, N_\alpha, B_\alpha\}$ on α . On the other hand, we have $\mathcal{F}_X X_\alpha = P \times X \times X_\alpha = 0$. Hence, applying (27) and (32), we obtain

$$0 = -m_2\mu_\alpha T_\alpha + (m_1 - m_3\mu_\alpha) N_\alpha + m_2 B_\alpha$$

which implies $m_1 - m_3\mu_\alpha = 0$ and $m_2 = 0$. Again, since $\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha$, $\alpha \times T_\alpha \times T_\alpha = 0$, $\alpha \times N_\alpha \times T_\alpha = T_\alpha$ and $\alpha \times B_\alpha \times T_\alpha = N_\alpha$, using (27) and (32), we find $m_3 = 1$ and hence $m_1 = \mu_\alpha$. Substituting these values of m_1 , m_2 and m_3 in (32), we find (31).

Conversely, let $X_\alpha \in \mathfrak{X}(\alpha(I))$ be given by (31). Since $\alpha \times T_\alpha \times T_\alpha = 0$ and $\alpha \times B_\alpha \times T_\alpha = N_\alpha$, using (31), we find

$$\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha = N_\alpha.$$

Then applying (8), we obtain

$$\mathcal{F}_X T_\alpha = \mathcal{D}_{\alpha'} T_\alpha$$

which is the associated Lorentz equation. Hence, α is a normal magnetic trajectory in $(\mathbb{H}_1^3, \bar{g}, B_U)$. $\square$

Remark 4.11. For a lightlike magnetic trajectory α in $(\mathbb{H}_1^3, \bar{g}, B_X)$, $X_\alpha \in \mathfrak{X}(\alpha(I))$ always resides in rectifying planes of α .

5. KILLING VECTOR FIELDS ON $\mathbb{H}_1^3$

As it is, $X \in \mathfrak{X}(\mathbb{H}_1^3)$ is *Killing* iff the Lie derivative of $\bar{g}$ with respect to X vanishes identically, i.e., $\mathcal{L}_X \bar{g} \equiv 0$. In connection with the Levi-Civita connection $\bar{\mathcal{D}}$ on $\mathbb{H}_1^3$, a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ satisfies the following equation (referred to as the *Killing equation*):

$$(33) \quad \bar{g}(\bar{\mathcal{D}}_Y X, Z) + \bar{g}(Y, \bar{\mathcal{D}}_Z X) = 0$$

for all $Y, Z \in \mathfrak{X}(\mathbb{H}_1^3)$. We know that linear combinations of Killing vector fields on $\mathbb{H}_1^3$ are again Killing. Also, if $X_1, X_2 \in \mathfrak{X}(\mathbb{H}_1^3)$ are Killing, then their Lie bracket $[X_1, X_2] \in \mathfrak{X}(\mathbb{H}_1^3)$ is also Killing. Hence, the set $i(\mathbb{H}_1^3)$ of all Killing vector fields on $\mathbb{H}_1^3$ constitutes a Lie algebra of dimension $\leq \frac{3(3+1)}{2} = 6$ (cf. [21]). Now, maximally symmetry nature of $\mathbb{H}_1^3$ asserts that $i(\mathbb{H}_1^3)$ is six dimensional. Moreover, $i(\mathbb{H}_1^3)$ is isomorphic to $I(\mathbb{H}_1^3)$, group of isometries of $\mathbb{H}_1^3$, which coincides with the Lorentz

group $O(1, 3)$. We compute a basis for this Lie algebra $i(\mathbb{H}_1^3)$ with the assistance of a geometrical model of split-quaternions for $\mathbb{H}_1^3$ as follows:

5.1. A geometrical model of split-quaternions for $\mathbb{H}_1^3$. Semi-Euclidean space $\mathbb{E}_2^4$ may be regarded as the collection of *split-quaternions* (introduced by *James Cockle* in 1849 under the name of “*co-quaternions*” [7]; also referred to as *anti-quaternions* or *pseudo-quaternions* [22]) as follows: A split-quaternion is an expression of the form

$$\mathbf{q} = x^1 + x^2 \mathbf{e}_1 + x^3 \mathbf{e}_2 + x^4 \mathbf{e}_3,$$

where $x^1, x^2, x^3, x^4 \in \mathbb{R}$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are split-quaternionic units satisfying the following non-commutative multiplication rules (cf. [8]):

$$\left\{ \begin{array}{l} \mathbf{e}_1^2 = -1, \\ \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1, \\ \mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2 \mathbf{e}_1, \\ \mathbf{e}_2 \mathbf{e}_3 = -\mathbf{e}_1 = -\mathbf{e}_3 \mathbf{e}_2, \\ \mathbf{e}_3 \mathbf{e}_1 = \mathbf{e}_2 = -\mathbf{e}_1 \mathbf{e}_3. \end{array} \right.$$

A split-quaternion $\mathbf{q}$ is composed of a real part x^1 and an imaginary part $x^2 \mathbf{e}_1 + x^3 \mathbf{e}_2 + x^4 \mathbf{e}_3$. The collection of split-quaternions, denoted by $\mathbf{H}$, constitutes a 4D real linear space with $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as a basis. If $\bar{\mathbf{q}} = x^1 - x^2 \mathbf{e}_1 - x^3 \mathbf{e}_2 - x^4 \mathbf{e}_3$ is the conjugate of $\mathbf{q} = x^1 + x^2 \mathbf{e}_1 + x^3 \mathbf{e}_2 + x^4 \mathbf{e}_3$, then the product

$$\mathbf{q} \bar{\mathbf{q}} = (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2$$

gives the quadratic form on $\mathbb{E}_2^4$ according to the semi-Euclidean metric with signature $(+, +, -, -)$. Consequently, $-\mathbf{q} \bar{\mathbf{q}}$ becomes the quadratic form on $\mathbb{E}_2^4$ according to the semi-Euclidean metric g with signature $(-, -, +, +)$. Again, the map $\Sigma : \mathbf{H} \rightarrow \mathbb{E}_2^4$ such that

$$\Sigma(1) = \partial_{x^1}, \quad \Sigma(\mathbf{e}_1) = \partial_{x^2}, \quad \Sigma(\mathbf{e}_2) = \partial_{x^3}, \quad \Sigma(\mathbf{e}_3) = \partial_{x^4}$$

is a vector space isomorphism. Hence, $\mathbb{E}_2^4$ may be regarded as the Lie group $\mathbf{H}$ equipped with the semi-Riemannian metric $\Sigma^* g$ which we simply denote by g (cf. [20]). In this regard, anti-de Sitter 3-space $\mathbf{H}_1^3$ is identified with a subspace of $\mathbf{H}$ as follows:

$$\mathbf{H}_1^3 = \{\mathbf{q} \in \mathbf{H} : \mathbf{q} \bar{\mathbf{q}} = -g(\mathbf{q}, \mathbf{q}) = 1\}.$$

Rotation of $\mathbf{q} = x^1 + x^2\mathbf{e}_1 + x^3\mathbf{e}_2 + x^4\mathbf{e}_3 \in \mathbb{H}_1^3$, where

$$-\sum_{i=1}^2 (x^i(s))^2 + \sum_{j=3}^4 (x^j(s))^2 = -1,$$

yields the following set of six vectors:

$$\left\{ \begin{array}{l} \mathbf{e}_1\mathbf{q} = -x^2 + x^1\mathbf{e}_1 - x^4\mathbf{e}_2 + x^3\mathbf{e}_3, \\ \mathbf{e}_2\mathbf{q} = x^3 - x^4\mathbf{e}_1 + x^1\mathbf{e}_2 - x^2\mathbf{e}_3, \\ \mathbf{e}_3\mathbf{q} = x^4 + x^3\mathbf{e}_1 + x^2\mathbf{e}_2 + x^1\mathbf{e}_3, \\ \mathbf{q}\mathbf{e}_1 = -x^2 + x^1\mathbf{e}_1 + x^4\mathbf{e}_2 - x^3\mathbf{e}_3, \\ \mathbf{q}\mathbf{e}_2 = x^3 + x^4\mathbf{e}_1 + x^1\mathbf{e}_2 + x^2\mathbf{e}_3, \\ \mathbf{q}\mathbf{e}_3 = x^4 - x^3\mathbf{e}_1 - x^2\mathbf{e}_2 + x^1\mathbf{e}_3. \end{array} \right.$$

Under the map Σ , the above set accords with the following set of six vector fields on $\mathbb{E}_2^4$:

$$(34) \quad \left\{ \begin{array}{l} X_1 = -x^2\partial_{x^1} + x^1\partial_{x^2} - x^4\partial_{x^3} + x^3\partial_{x^4}, \\ X_2 = x^3\partial_{x^1} - x^4\partial_{x^2} + x^1\partial_{x^3} - x^2\partial_{x^4}, \\ X_3 = x^4\partial_{x^1} + x^3\partial_{x^2} + x^2\partial_{x^3} + x^1\partial_{x^4}, \\ X_4 = -x^2\partial_{x^1} + x^1\partial_{x^2} + x^4\partial_{x^3} - x^3\partial_{x^4}, \\ X_5 = x^3\partial_{x^1} + x^4\partial_{x^2} + x^1\partial_{x^3} + x^2\partial_{x^4}, \\ X_6 = x^4\partial_{x^1} - x^3\partial_{x^2} - x^2\partial_{x^3} + x^1\partial_{x^4}. \end{array} \right.$$

All of $X_1, X_2, X_3, X_4, X_5, X_6$ are unit smooth vector fields on $\mathbb{E}_2^4$ orthogonal to the position vector field $P = \sum_1^4 x^i\partial_{x^i}$ of $\mathbb{H}_1^3$ in $\mathbb{E}_2^4$, where $-\sum_{i=1}^2 (x^i(s))^2 + \sum_{j=3}^4 (x^j(s))^2 = -1$. Therefore, it is evident that $X_1, X_2, X_3, X_4, X_5, X_6 \in \mathfrak{X}(\mathbb{H}_1^3)$. Moreover, we observe that X_1 and X_4 are timelike whereas X_2, X_3, X_5 and X_6 are spacelike. Consequently, $\{X_1, X_2, X_3\}$ forms a global *right-invariant* orthonormal frame field on $\mathbb{H}_1^3$ whereas $\{X_4, X_5, X_6\}$ forms a global *left-invariant* orthonormal frame field on $\mathbb{H}_1^3$. Now, on direct computations, for each $\alpha \in \{1, \dots, 6\}$ and for any $\partial_{x^i}, \partial_{x^j}; i, j \in \{1, 2, 3, 4\}$, we find

$$g(\mathcal{D}_{\partial_{x^i}}X_\alpha, \partial_{x^j}) + g(\partial_{x^i}, \mathcal{D}_{\partial_{x^j}}X_\alpha) = 0.$$

Therefore, for any two vector fields $X = \sum_1^4 X^i \partial_{x^i}, Y = \sum_1^4 Y^i \partial_{x^i} \in \mathfrak{X}(\mathbb{H}_1^3)$, we obtain

$$g(\mathcal{D}_X X_\alpha, Y) + g(X, \mathcal{D}_Y X_\alpha) = 0$$

and using the Gauss formula (3), we get

$$\bar{g}(\bar{\mathcal{D}}_X X_\alpha, Y) + \bar{g}(X, \bar{\mathcal{D}}_Y X_\alpha) = 0.$$

Thus, each $X_\alpha \in \mathfrak{X}(\mathbb{H}_1^3)$ satisfies the Killing equation (33). Hence, we obtain the following result:

Proposition 5.1. *The unit vector fields $X_1, X_2, X_3, X_4, X_5, X_6 \in \mathfrak{X}(\mathbb{H}_1^3)$ obtained by (34) are Killing vector fields on $\mathbb{H}_1^3$.*

Remark 5.2. *Here, we obtain six linearly independent Killing vector fields $X_1, X_2, X_3, X_4, X_5, X_6$ on $\mathbb{H}_1^3$ among which X_1, X_2, X_3 are right-invariant whereas X_4, X_5, X_6 are left-invariant. It can be easily checked that any linear combination of the right-invariant Killing vector fields X_1, X_2, X_3 (resp. left-invariant Killing vector fields X_4, X_5, X_6) is again a right-invariant (resp. left-invariant) Killing vector field on $\mathbb{H}_1^3$. Thus, these Killing vector fields spans the Lie algebra $i(\mathbb{H}_1^3)$. Consequently, $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ forms a basis for $i(\mathbb{H}_1^3)$.*

Now, the following result which can be proved easily provides us a account for all spacelike, timelike or lightlike right-invariant or left-invariant Killing vector fields of constant length on $\mathbb{H}_1^3$.

Proposition 5.3. *Any right-invariant (resp. left-invariant) Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ of constant length is characterized by one of the following categories:*

- (1) *X is spacelike iff X is a scalar multiple of X_2 or X_3 (resp. X_5 or X_6).*
- (2) *X is timelike iff X is a scalar multiple of X_1 (resp. X_4).*
- (3) *X is lightlike iff X is a scalar multiple of $X_1 \pm X_2$ or $X_1 \pm X_3$ (resp. $X_4 \pm X_5$ or $X_4 \pm X_6$).*

5.2. Killing vector fields on regular curves in $\mathbb{H}_1^3$. Let us consider a regular curve $\alpha : I \rightarrow \mathbb{H}_1^3$ possessing speed $v_\alpha(t) = \|\alpha'(t)\|$ and scalar acceleration $a_\alpha(t) = \|\bar{\mathcal{D}}_{\alpha'} \alpha'(t)\|$. Let $\epsilon > 0$ and $\beta : (-\epsilon, \epsilon) \times I \rightarrow \mathbb{H}_1^3$ be a variation of α such that $\beta(0, t) = \alpha(t)$ and causality of t -curves α_p obtained by $\alpha_p(t) = \beta(p, t)$ be the same

as that of α . Then there exist $X_\beta, Y_\beta \in \mathfrak{X}(\beta)$ defined by

$$X_\beta(p, t) = \frac{\partial \alpha}{\partial p}(p, t), \quad Y_\beta(p, t) = \frac{\partial \alpha}{\partial t}(p, t).$$

Here X_β stands for the velocity field of p -curves α_t and Y_β the velocity field of t -curves α_p . As a matter of fact, $X_\alpha \in \mathfrak{X}(\alpha(I))$ given by $X_\alpha(t) = X_\beta(0, t)$ is the variation vector field and $Y_\alpha \in \mathfrak{X}(\alpha(I))$ given by $Y_\alpha(t) = Y_\beta(0, t)$ is the velocity field of α . Again, t -curves α_p have speed $v = v(p, t) = \|Y_\beta(p, t)\|$ and scalar acceleration $a = a(p, t) = \|\bar{D}_{Y_\beta} Y_\beta(p, t)\|$ such that

$$v(0, t) = v_\alpha(t) = \|\alpha'(t)\|, \quad a(0, t) = a_\alpha(t) = \|\bar{D}_{\alpha'} \alpha'(t)\|.$$

5.2.1. Killing vector fields on non-lightlike curves with non-lightlike acceleration.

Let α be unit-speed and spacelike with non-lightlike acceleration or timelike. Then s -curves α_p of the variation β have speed $v = v(p, s)$, curvature $\kappa = \kappa(p, s)$ and torsion $\tau = \tau(p, s)$ satisfying

$$v(0, s) = v_\alpha(s) = 1, \quad \kappa(0, s) = \kappa_\alpha(s), \quad \tau(0, s) = \tau_\alpha(s).$$

A straightforward calculation yields (cf. [19] or [3])

$$\left\{ \begin{array}{l} [X_\alpha, T_\alpha] = 0, \\ \frac{\partial v}{\partial p}(0, s) = \delta_1 \bar{g}(\bar{D}_{\alpha'} X_\alpha, T_\alpha) v, \\ \frac{\partial \kappa^2}{\partial p}(0, s) = 2\delta_2 \bar{g}(\bar{D}_{\alpha'}^2 X_\alpha, \bar{D}_{\alpha'} T_\alpha) - 4\delta_1 \kappa_\alpha^2 \bar{g}(\bar{D}_{\alpha'} X_\alpha, T_\alpha) + 2\delta_2 \bar{g}(\bar{R}(X_\alpha, T_\alpha) T_\alpha, \bar{D}_{\alpha'} T_\alpha), \\ \frac{\partial \tau^2}{\partial p}(0, s) = 2\delta_2 \bar{g}\left(\frac{1}{\kappa_\alpha} \bar{D}_{\alpha'}^3 X_\alpha - \frac{\kappa'_\alpha}{\kappa_\alpha^2} \bar{D}_{\alpha'}^2 X_\alpha + \delta_1 \left(\delta_2 \kappa_\alpha - \frac{1}{\kappa_\alpha}\right) \bar{D}_{\alpha'} X_\alpha + \delta_1 \frac{\kappa'_\alpha}{\kappa_\alpha^2} X_\alpha, \tau_\alpha B_\alpha\right). \end{array} \right.$$

The vector field X_α is Killing on α iff it maintains extremely small arc length of α and α develops along X_α without altering its curvature and torsion (i.e., shape of α stays unaltered). In this regard, X_α is Killing on α iff

$$\frac{\partial v}{\partial p}(0, s) = 0, \quad \frac{\partial \kappa^2}{\partial p}(0, s) = 0, \quad \frac{\partial \tau^2}{\partial p}(0, s) = 0,$$

i.e., iff

$$(35) \quad \left\{ \begin{array}{l} \bar{g}(\bar{D}_{\alpha'} X_\alpha, T_\alpha) = 0, \quad \bar{g}(\bar{D}_{\alpha'}^2 X_\alpha - \delta_1 X_\alpha, N_\alpha) = 0, \\ \bar{g}(\kappa_\alpha \bar{D}_{\alpha'}^3 X_\alpha - \kappa'_\alpha \bar{D}_{\alpha'}^2 X_\alpha + \delta_1 \kappa_\alpha (\delta_2 \kappa_\alpha^2 - 1) \bar{D}_{\alpha'} X_\alpha + \delta_1 \kappa'_\alpha X_\alpha, \tau_\alpha B_\alpha) = 0. \end{array} \right.$$

On solving (35), one can find a 6D vector space and obtain

Proposition 5.4 ([19, 3]). *A vector field $X_\alpha \in \mathfrak{X}(\alpha(I))$ is Killing on α iff it is the restriction of a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ to α .*

5.2.2. *Killing vector fields on spacelike curves possessing lightlike acceleration.* Let α be unit-speed and spacelike possessing lightlike acceleration. Then s -curves α_p of the variation β have speed $v = v(p, s)$, scalar acceleration $a = a(p, s)$ and pseudo-torsion $\tau = \tau(p, s)$ satisfying

$$v(0, s) = v_\alpha(s) = 1, \quad a(0, s) = a_\alpha(s) = 0, \quad \tau(0, s) = \tau_\alpha(s).$$

In this setting, X_α is a Killing on α iff

$$\frac{\partial v}{\partial p}(0, s) = 0, \quad \frac{\partial a}{\partial p}(0, s) = 0, \quad \frac{\partial \tau}{\partial p}(0, s) = 0,$$

i.e., iff

$$(36) \quad \begin{cases} \bar{g}(\bar{\mathcal{D}}_{\alpha'} X_\alpha, T_\alpha) = 0, & \bar{g}(\bar{\mathcal{D}}_{\alpha'}^2 X_\alpha - X_\alpha, N_\alpha) = 0, \\ \bar{g}(\bar{\mathcal{D}}_{\alpha'}^3 X_\alpha - \tau_\alpha \bar{\mathcal{D}}_{\alpha'}^2 X_\alpha - \bar{\mathcal{D}}_{\alpha'} X_\alpha + \tau_\alpha X_\alpha, B_\alpha) - \bar{g}(X_\alpha, T_\alpha) = 0. \end{cases}$$

Solutions of (36) yields a 6D vector space. Accordingly, we find

Proposition 5.5. *A vector field $X_\alpha \in \mathfrak{X}(\alpha(I))$ is Killing on α iff it is the restriction of a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ to α .*

5.2.3. *Killing vector fields on lightlike curves.* Let α be unit-speed and lightlike. Then s -curves α_p of the variation β have speed $v = v(p, s)$, scalar acceleration $a = a(p, s)$ and pseudo-torsion $\tau = \tau(p, s)$ satisfying

$$v(0, s) = v_\alpha(s) = 0, \quad a(0, s) = a_\alpha(s), \quad \tau_\alpha(0, s) = \tau_\alpha(s).$$

In this setting, X_α is a Killing vector field on α iff

$$\frac{\partial v}{\partial p}(0, s) = 0, \quad \frac{\partial a}{\partial p}(0, s) = 0, \quad \frac{\partial \tau}{\partial p}(0, s) = 0,$$

i.e., iff (cf. [16] or [14])

$$(37) \quad \begin{cases} \bar{g}(\bar{\mathcal{D}}_{\alpha'} X_\alpha, T_\alpha) = 0, & \bar{g}(\bar{\mathcal{D}}_{\alpha'}^2 X_\alpha, N_\alpha) = 0, \\ \bar{g}(\bar{\mathcal{D}}_{\alpha'}^3 X_\alpha + \tau_\alpha \bar{\mathcal{D}}_{\alpha'} X_\alpha, B_\alpha) - 2\bar{g}(X_\alpha, N_\alpha) = 0. \end{cases}$$

By solving (37), one can constitute a 6D vector space and obtain

Proposition 5.6 ([16, 14]). *A vector field $X_\alpha \in \mathfrak{X}(\alpha(I))$ is Killing on α iff it is the restriction of a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ to α .*

6. MOTION IN KILLING MAGNETIC FIELDS: KILLING MAGNETIC TRAJECTORIES IN $\mathbb{H}_1^3$

Let us consider a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ generated by $X_1, X_2, X_3, X_4, X_5, X_6 \in \mathfrak{X}(\mathbb{H}_1^3)$ given by (34). Evidently, X has vanishing divergence and therefore corresponds to a *Killing magnetic field* $B_X = \iota_X \Omega_{\bar{g}}$ on $\mathbb{H}_1^3$. Hence, its Lorentz force $\mathcal{F}_X$ is given by

$$\mathcal{F}_X Y = X \wedge Y = P \times X \times Y$$

for all $Y \in \mathfrak{X}(\mathbb{H}_1^3)$. In this setting, the Lorentz equation is

$$\mathcal{D}_{\alpha'} \alpha' - \text{sign}(\alpha') \|\alpha'\|^2 \alpha = P \times X \times \alpha'$$

that defines a magnetic flow composed of *Killing magnetic trajectories* in Killing magnetic background $(\mathbb{H}_1^3, \bar{g}, B_X)$.

6.1. Non-lightlike Killing magnetic trajectories possessing non-lightlike acceleration. We characterize spacelike magnetic trajectories possessing non-lightlike acceleration or timelike magnetic trajectories corresponding to Killing magnetic fields in terms of their quasi-slope, curvature and torsion.

Theorem 6.1. *Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed non-lightlike curve with non-lightlike acceleration and B_X be the Killing magnetic field corresponding to a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$. If α is a normal KMT¹ in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then it possesses constant quasi-slope μ_0 and one characterization among the following:*

- (1) α resides in a totally geodesic surface in $\mathbb{H}_1^3$.
- (2) α possesses curvature κ_α and torsion τ_α which are connected as follows:

$$(38) \quad \begin{cases} \kappa_\alpha^2 (\mu_0 - 2\tau_\alpha) = c_1, \\ \delta_2 \kappa_\alpha'' + \delta_3 \kappa_\alpha \tau_\alpha (\mu_0 - \tau_\alpha) + \frac{1}{2} \delta_1 \kappa_\alpha^3 = c_2 \kappa_\alpha, \end{cases}$$

where c_1 and c_2 are undetermined constants.

Conversely, if α is a unit-speed non-lightlike curve possessing non-lightlike acceleration having one of two characterizations mentioned as above, then there is a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ so that α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ possessing constant quasi-slope μ_0 .

¹KMT stands for *Killing magnetic trajectory*.

Proof. Let α be a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$. Then Theorem 4.3 gives the following expression for $X_\alpha \in \mathfrak{X}(\alpha(I))$:

$$(39) \quad X_\alpha = \delta_2 \mu_\alpha T_\alpha + \delta_2 \kappa_\alpha B_\alpha,$$

where μ_α is the quasi-slope of α obtained by $\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha)$, $\mathcal{F}_X$ being the associated Lorentz force. Again, Proposition 5.4 asserts $X_\alpha \in \mathfrak{X}(\alpha(I))$ is Killing on α and hence (35) is satisfied. Now, (39) and (6) yields

$$\bar{\mathcal{D}}_{\alpha'} X_\alpha = \delta_2 \mu'_\alpha T_\alpha + \kappa_\alpha (\mu_\alpha - \tau_\alpha) N_\alpha + \delta_2 \kappa'_\alpha B_\alpha.$$

Then the first one of (35) gives $\mu'_\alpha = 0$ and hence μ is a constant, say μ_0 . Consequently, (39) and the last equation become

$$(40) \quad X_\alpha = \delta_2 \mu_0 T_\alpha + \delta_2 \kappa_\alpha B_\alpha,$$

$$(41) \quad \bar{\mathcal{D}}_{\alpha'} X_\alpha = \kappa_\alpha (\mu_0 - \tau_\alpha) N_\alpha + \delta_2 \kappa'_\alpha B_\alpha.$$

Using (41) and (6), we compute

$$(42) \quad \begin{aligned} \bar{\mathcal{D}}_{\alpha'}^2 X_\alpha &= [-\delta_1 \kappa_\alpha^2 (\mu_0 - \tau_\alpha)] T_\alpha + [\kappa'_\alpha (\mu_0 - 2\tau_\alpha) - \kappa_\alpha \tau'_\alpha] N_\alpha \\ &\quad + [\delta_2 \kappa''_\alpha + \delta_3 \kappa_\alpha \tau_\alpha (\mu_0 - \tau_\alpha)] B_\alpha. \end{aligned}$$

Again, applying (40) and (42) in the second one of (35), we find

$$\kappa'_\alpha (\mu_0 - 2\tau_\alpha) - \kappa_\alpha \tau'_\alpha = 0$$

which gives

$$(43) \quad \kappa_\alpha^2 (\mu_0 - 2\tau_\alpha) = c_1,$$

where c_1 is an undetermined constant. Also, (42) reduces to

$$(44) \quad \bar{\mathcal{D}}_{\alpha'}^2 X_\alpha = [-\delta_1 \kappa_\alpha^2 (\mu_0 - \tau_\alpha)] T_\alpha + [\delta_2 \kappa''_\alpha + \delta_3 \kappa_\alpha \tau_\alpha (\mu_0 - \tau_\alpha)] B_\alpha.$$

Moreover, using (44) and (6), we compute

$$\begin{aligned} \bar{\mathcal{D}}_{\alpha'}^3 X_\alpha &= \left[-\delta_1 (\kappa_\alpha^2 (\mu_0 - \tau_\alpha))' \right] T_\alpha + \left[-\tau_\alpha \kappa''_\alpha + \kappa_\alpha (\delta_3 \kappa_\alpha^2 + \delta_1 \tau_\alpha^2) (\mu_0 - \tau_\alpha) \right] N_\alpha \\ &\quad + \left[(\delta_2 \kappa''_\alpha + \delta_3 \kappa_\alpha \tau_\alpha (\mu_0 - \tau_\alpha))' \right] B_\alpha. \end{aligned}$$

Then applying it including (40), (41) and (44) in the third one of (35), after some steps, we obtain

$$(45) \quad \tau_\alpha \left[\frac{1}{\kappa_\alpha} (\delta_2 \kappa''_\alpha + \delta_3 \kappa_\alpha \tau_\alpha (\mu_0 - \tau_\alpha)) + \frac{1}{2} \delta_1 \kappa_\alpha^2 \right]' = 0.$$

If $\tau_\alpha \equiv 0$, then from (43) we find κ_α is a constant and hence α resides in a totally geodesic surface in $\mathbb{H}_1^3$. On the other hand, if $\tau_\alpha \not\equiv 0$, then (45) yields

$$(46) \quad \delta_2 \kappa_\alpha'' + \delta_3 \kappa_\alpha \tau_\alpha (\mu_0 - \tau_\alpha) + \frac{1}{2} \delta_1 \kappa_\alpha^3 = c_2 \kappa_\alpha,$$

c_2 being an undetermined constant, and consequently κ_α and τ_α are connected by (43) and (46).

Conversely, let α reside in a totally geodesic surface in $\mathbb{H}_1^3$. Then κ_α is a constant, say κ_0 , and $\tau_\alpha \equiv 0$. We define $X_\alpha \in \mathfrak{X}(\alpha(I))$ by

$$X_\alpha = \delta_2 \mu_0 T_\alpha + \delta_2 \kappa_0 B_\alpha,$$

where μ_0 is a constant. Using (6) and computing $\bar{\mathcal{D}}_{\alpha'} X_\alpha$, $\bar{\mathcal{D}}_{\alpha'}^2 X_\alpha$ and $\bar{\mathcal{D}}_{\alpha'}^3 X_\alpha$, it can be easily examine that the conditions of (35) is satisfied. Hence, $X_\alpha \in \mathfrak{X}(\alpha(I))$ is Killing on α and therefore according to Proposition 5.4, X_α is extended to a Killing field $X \in \mathfrak{X}(\mathbb{H}_1^3)$. Also, using (6), we find

$$\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha = \delta_2 \kappa_0 N_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \delta_1 \alpha.$$

This indicates α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$.

On the other hand, let κ_α and torsion τ_α be connected by (38). We define $X_\alpha \in \mathfrak{X}(\alpha(I))$ by

$$X_\alpha = \delta_2 \mu_0 T_\alpha + \delta_2 \kappa_\alpha B_\alpha,$$

where μ_0 is a constant. Then using (6), we compute $\bar{\mathcal{D}}_{\alpha'} X_\alpha$, $\bar{\mathcal{D}}_{\alpha'}^2 X_\alpha$ and $\bar{\mathcal{D}}_{\alpha'}^3 X_\alpha$ and see, on a long but simple calculation, (35) holds. Hence, $X_\alpha \in \mathfrak{X}(\alpha(I))$ is Killing on α . Therefore, Proposition 5.4 ensures X_α is extended to a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$. Moreover, using (6), we get

$$\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha = \delta_2 \kappa_\alpha N_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \delta_1 \alpha.$$

This implies α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$. □

Remark 6.2. According to Theorem 6.1, if a unit-speed non-lightlike curve possessing non-lightlike acceleration $\alpha : I \rightarrow \mathbb{H}_1^3$ is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then it has constant quasi-slope μ_0 and the Killing vector field $X_\alpha \in \mathfrak{X}(\alpha(I))$ rewritten as

$$X_\alpha = \delta_2 \mu_0 T_\alpha + \delta_2 \kappa_\alpha B_\alpha$$

resides in rectifying planes of α . Now, if N_α is timelike, then rectifying planes of α are spacelike and therefore X_α is spacelike and so is X . On the other hand, if

N_α is spacelike, then rectifying planes of α are timelike (and hence Lorentzian) and therefore X_α is spacelike or timelike or lightlike and so is X .

Now, if $X \in \mathfrak{X}(\mathbb{H}_1^3)$ is a unit Killing vector field with $\text{sign}(X) = \delta$, then we have

$$\bar{g}(X_\alpha, X_\alpha) = \delta_1 \mu_0^2 + \delta_3 \kappa_\alpha^2 = \delta$$

and hence α has constant curvature κ_0 given by $\kappa_0 = \sqrt{|\delta_3 \delta + \delta_2 \mu_0^2|}$. Moreover, if $X = X_1$ (when timelike), X_2, X_3 (when spacelike), $X_1 \pm X_2$ or $X_1 \pm X_3$ (when lightlike), then X_α is right-invariant on α and hence, by applying Lemma 2 in [20], we have

$$\bar{\mathcal{D}}_{\alpha'} X_\alpha = \alpha \times X_\alpha \times T_\alpha.$$

After some computations, we find that α has either everywhere vanishing torsion or constant torsion τ_0 given by $\tau_0 = \mu_0 - \delta_2$. On the other hand, if $X = X_4$ (when timelike), X_5, X_6 (when spacelike), $X_4 \pm X_5$ or $X_4 \pm X_6$ (when lightlike), then X_α is left-invariant on α and hence, by applying Lemma 2 in [20], we have

$$\bar{\mathcal{D}}_{\alpha'} X_\alpha = \alpha \times T_\alpha \times X_\alpha.$$

After some computations, we find that α has either everywhere vanishing torsion or constant torsion τ_0 given by $\tau_0 = \mu_0 + \delta_2$. Hence, we find the following result:

Theorem 6.3. *Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed non-lightlike curve possessing non-lightlike acceleration and B_X be the Killing magnetic field corresponding to a right-invariant (resp. left-invariant) unit Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with $\text{sign}(X) = \delta$. Then α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ iff*

- (1) α possesses constant quasi-slope μ_0 ,
- (2) α possesses constant curvature $\kappa_0 = \sqrt{|\delta_3 \delta + \delta_2 \mu_0^2|}$,
- (3) α possesses either zero torsion or non-zero constant torsion $\tau_0 = \mu_0 - \delta_2$ (resp. $\tau_0 = \mu_0 + \delta_2$).

Now, a curve α in $\mathbb{H}_1^3$ is a *general helix* [3] iff there is a Killing vector field $X_\alpha \in \mathfrak{X}(\alpha(I))$ of constant length admitting $\bar{g}(\bar{\mathcal{D}}_{\alpha'} \alpha', X_\alpha) = 0$ (i.e., X_α and the acceleration field of α are orthogonal). Here X_α is called an axis of the general helix. Obviously, a curve with everywhere vanishing torsion is a non-degenerate general

helix as its unit binormal vector field is regarded as an axis which is obviously non-lightlike. Now, for a curve α possessing constant curvature and constant torsion, it is evident that the vector field X_α on α given by

$$X_\alpha = aT_\alpha + bB_\alpha$$

for some constants a and b is always Killing and it resides in rectifying planes of α . If N_α is timelike, then rectifying planes of α are spacelike and therefore all Killing vector fields on α are non-lightlike. In this case, α is a non-degenerate general helix. On the other hand, if N_α is spacelike, then rectifying planes of α are timelike (and hence Lorentzian) and therefore Killing vector fields on α may be non-lightlike or lightlike. Therefore, we cannot conclude whether α is a degenerate general helix or not. Nevertheless, we can find out a unique Killing vector field X_α on α by forcibly making it parallel on α . In this case, α is called a non-degenerate or degenerate general helix if X_α is respectively non-lightlike or lightlike (cf. [3]).

Also, it happens that a non-lightlike curve α possessing non-lightlike acceleration is a *general helix* in $\mathbb{H}_1^3$ iff it has any one among the following characterizations (cf. [3]):

- α possesses zero torsion and resides in a totally geodesic surface in $\mathbb{H}_1^3$. In this case, α is a non-degenerate general helix admitting a unique non-lightlike axis.
- α has constant curvature κ_0 as well as constant torsion τ_0 , i.e., α is a helix in $\mathbb{H}_1^3$. In this case, α is a general helix admitting a plane (rectifying plane) of axes among which one is parallel on α , and it is degenerate iff N_α is spacelike and $\tau_0 = \pm \kappa_0$, otherwise it is non-degenerate.
- α has curvature κ_α and torsion τ_α related by $\tau_\alpha = c_0 \kappa_\alpha \pm 1$ for some undetermined constant c_0 . In this case, α is a general helix admitting a unique axis that cannot be parallel on α , and it is degenerate iff N_α is spacelike and $c_0 = \pm 1$, otherwise it is non-degenerate.

Accordingly, we obtain the following consequence of Theorems 6.1 and 6.3:

Theorem 6.4. *Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed non-lightlike curve possessing non-lightlike acceleration and B_X be the Killing magnetic field corresponding to a right-invariant (resp. left-invariant) unit Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$. Then α is a*

normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ iff it has constant quasi-slope μ_0 , and it belongs to one among the following families:

- (1) If X is spacelike, then α is a non-degenerate general helix admitting either the support of X as axis or a plane (rectifying plane) of axes which includes X . In this case, α has constant curvature $\kappa_0 = \sqrt{|1 - \mu_0^2|}$ and either everywhere vanishing torsion or constant torsion $\tau_0 = \mu_0 + 1$ (resp. $\tau_0 = \mu_0 - 1$).
- (2) If X is timelike, then α is a non-degenerate general helix admitting either the support of X as axis or a plane (rectifying plane) of axes which includes X . In this case, α has constant curvature $\kappa_0 = \sqrt{|-\delta_3 + \mu_0^2|}$ and either everywhere vanishing torsion or constant torsion $\tau_0 = \mu_0 - 1$ (resp. $\tau_0 = \mu_0 + 1$).
- (3) If X is lightlike, then α is a degenerate general helix admitting either the support of X as axis or a plane (rectifying plane) of axes which includes X . In this case, μ_0 is non-zero (otherwise X becomes non-lightlike), and α has constant curvature $\kappa_0 = |\mu_0|$ and constant torsion $\tau_0 = \mu_0 - 1$ (resp. $\tau_0 = \mu_0 + 1$).

6.2. Spacelike Killing magnetic trajectories possessing lightlike acceleration. We characterize spacelike magnetic trajectories possessing lightlike acceleration corresponding to Killing magnetic fields in terms of their quasi-slope and pseudo-torsion.

Theorem 6.5. *Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed spacelike curve possessing lightlike acceleration and B_X be the Killing magnetic field corresponding to a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$. If α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then quasi-slope of α is a constant, say μ_0 , and pseudo-torsion τ_α of α agrees*

$$(47) \quad \tau'_\alpha - (\mu_0 - \tau_\alpha) \tau_\alpha + 2\mu_0 s = c,$$

for some undetermined constant c .

Conversely, if τ_α agrees (47), then there is a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ so that α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ possessing constant quasi-slope μ_0 .

Proof. Let α be a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$. Then by Theorem 4.7, the restriction of X to α , $X_\alpha \in \mathfrak{X}(\alpha(I))$, can be expressed as

$$(48) \quad X_\alpha = \mu_\alpha T_\alpha - N_\alpha,$$

where μ_α is the quasi-slope of α obtained by $\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha)$, where $\mathcal{F}_X$ is the associated Lorentz force. Again, according to Proposition 5.5, $X_\alpha \in \mathfrak{X}(\alpha(I))$ is a Killing vector field on α and hence the conditions (36) are satisfied. Now, using (48) and (7), we compute

$$\bar{\mathcal{D}}_{\alpha'} X_\alpha = \mu'_\alpha T_\alpha + (\mu_\alpha - \tau_\alpha) N_\alpha$$

and substituting it in the first condition of (36), we find $\mu'_\alpha = 0$ and therefore μ_α is a constant, say μ_0 . Hence, α has constant quasi-slope μ_0 . Then (48) and the previous equation reduce to

$$(49) \quad X_\alpha = \mu_0 T_\alpha - N_\alpha, \quad \bar{\mathcal{D}}_{\alpha'} X_\alpha = (\mu_0 - \tau_\alpha) N_\alpha.$$

Moreover, using (7) and (47), we find

$$\begin{cases} \bar{\mathcal{D}}_{\alpha'}^2 X_\alpha = [-\tau'_\alpha + (\mu_0 - \tau_\alpha) \tau_\alpha] N_\alpha, \\ \bar{\mathcal{D}}_{\alpha'}^3 X_\alpha = [-\tau''_\alpha + (\mu_0 - 3\tau_\alpha) \tau'_\alpha + (\mu_0 - \tau_\alpha) \tau_\alpha^2] N_\alpha \end{cases}$$

and hence the third one of (36) yields (47).

Conversely, let τ_α satisfies (47). We define $X_\alpha \in \mathfrak{X}(\alpha(I))$ by

$$X_\alpha = \mu_0 T_\alpha - N_\alpha,$$

where μ_0 is a constant. Then using (7), we compute $\bar{\mathcal{D}}_{\alpha'} X_\alpha$, $\bar{\mathcal{D}}_{\alpha'}^2 X_\alpha$ and $\bar{\mathcal{D}}_{\alpha'}^3 X_\alpha$ and observe (36) is satisfied. Hence, $X_\alpha \in \mathfrak{X}(\alpha(I))$ is a Killing vector field on α . Consequently, by Proposition 5.5, there exists a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ whose restriction to α is X_α . Applying (7), we obtain

$$\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha = N_\alpha = \mathcal{D}_{\alpha'} T_\alpha - \alpha$$

which is the associated Lorentz equation. Hence, α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$. $\square$

Remark 6.6. According to Theorem 6.5, if a unit-speed spacelike curve $\alpha : I \rightarrow \mathbb{H}_1^3$ possessing lightlike acceleration is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then it has constant quasi-slope μ_0 and the restriction of the Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ to α , $X_\alpha \in \mathfrak{X}(\alpha(I))$, can be expressed as $X_\alpha = \mu_0 T_\alpha - N_\alpha$ which resides in osculating

planes of α . Since B_α is lightlike, osculating planes of α are lightlike (i.e., degenerate). Now, it is well-known that a lightlike subspace contains a lightlike vector but not a timelike one, and its intersection with the lightlike cone together with the zero-vector forms a one-dimensional subspace ([21], p. 142). Hence, it follows that X_α is either spacelike or lightlike and so is X .

Now, if $X \in \mathfrak{X}(\mathbb{H}_1^3)$ is a unit Killing vector field with $\text{sign}(X) = \delta$, where $\delta = 1$ or 0, then we have $\bar{g}(X_\alpha, X_\alpha) = \mu_0^2 = \delta$ and hence α has constant quasi-slope μ_0 given by $\mu_0 = \pm \delta$. Moreover, if $X = X_2, X_3$ (when spacelike), $X_1 \pm X_2$ or $X_1 \pm X_3$ (when lightlike), then X_α is right-invariant on α and hence, by applying Lemma 2 in [20], we have $\bar{\mathcal{D}}_\alpha X_\alpha = \alpha \times X_\alpha \times T_\alpha$. After some computations, we find that α has constant pseudo-torsion τ_0 given by $\tau_0 = \pm \delta - 1$. On the other hand, if $X = X_5, X_6$ (when spacelike), $X_4 \pm X_5$ or $X_4 \pm X_6$ (when lightlike), then X_α is left-invariant on α and hence, by applying Lemma 2 in [20], we have $\bar{\mathcal{D}}_\alpha X_\alpha = \alpha \times T_\alpha \times X_\alpha$. After some computations, we find that α has constant pseudo-torsion τ_0 given by $\tau_0 = \pm \delta + 1$. Accordingly, we find the following result:

Theorem 6.7. *Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed spacelike curve possessing lightlike acceleration and B_X be the Killing magnetic field corresponding to a right-invariant (resp. left-invariant) spacelike or lightlike unit Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with $\text{sign}(X) = \delta$ (where $\delta = 1$ or 0). Then α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ iff it has constant quasi-slope $\mu_0 = \pm \delta$, and it is a spacelike helix possessing constant pseudo-torsion $\tau_0 = \pm \delta - 1$ (resp. $\tau_0 = \pm \delta + 1$) with axis X .*

6.3. Lightlike Killing magnetic trajectories.

Theorem 6.8. *Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed lightlike curve and B_X be the Killing magnetic field corresponding to a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$. If α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then its pseudo-torsion τ_α and quasi-slope μ_α satisfy*

$$(50) \quad \begin{cases} 2\mu_\alpha - \tau_\alpha = c_1, \\ \mu_\alpha'' - \mu_\alpha^2 + \tau_\alpha^2 = c_2, \end{cases}$$

for some undetermined constants c_1 and c_2 .

Conversely, if τ_α satisfies (50) for some smooth function μ_α , then there is a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ so that α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ with μ_α as quasi-slope.

Proof. Let us assume that α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$. Then by Theorem 4.10, the restriction of X to α , $X_\alpha \in \mathfrak{X}(\alpha(I))$, can be written as

$$(51) \quad X_\alpha = \mu_\alpha T_\alpha + B_\alpha,$$

where μ_α is the quasi-slope of α obtained by $\mu_\alpha = \bar{g}(\mathcal{F}_X N_\alpha, B_\alpha) = -\bar{g}(N_\alpha, \mathcal{F}_X B_\alpha)$, where $\mathcal{F}_X$ is the associated Lorentz force. Again, according to Proposition 5.6, $X_\alpha \in \mathfrak{X}(\alpha(I))$ is a Killing vector field on α and hence the conditions (37) are satisfied. Applying (51) and (8), we find

$$(52) \quad \bar{\mathcal{D}}_{\alpha'} X_\alpha = \mu'_\alpha T_\alpha + (\mu_\alpha - \tau_\alpha) N_\alpha$$

and hence

$$\bar{\mathcal{D}}_{\alpha'}^2 X_\alpha = [\mu''_\alpha - (\mu_\alpha - \tau_\alpha) \tau_\alpha] T_\alpha + [2\mu'_\alpha - \tau'_\alpha] N_\alpha + [\mu_\alpha - \tau_\alpha] B_\alpha.$$

Substituting it in the second condition of (37), we find $2\mu'_\alpha - \tau'_\alpha = 0$ which gives first equation of (50). Therefore, the previous equation reduces to

$$(53) \quad \bar{\mathcal{D}}_{\alpha'}^2 X_\alpha = [\mu''_\alpha - (\mu_\alpha - \tau_\alpha) \tau_\alpha] T_\alpha + [\mu_\alpha - \tau_\alpha] B_\alpha.$$

Also, applying (53) and (8), we obtain

$$(54) \quad \begin{aligned} \bar{\mathcal{D}}_{\alpha'}^3 X_\alpha = & \left[\mu_\alpha^{(3)} - (\mu'_\alpha - \tau'_\alpha) \tau_\alpha - (\mu_\alpha - \tau_\alpha) \tau'_\alpha \right] T_\alpha \\ & + [\mu''_\alpha - 2(\mu_\alpha - \tau_\alpha) \tau_\alpha] N_\alpha + [\mu'_\alpha - \tau'_\alpha] B_\alpha. \end{aligned}$$

Substituting (51), (52) and (54) in the third condition of (37), some computation gives second equation of (50).

For converse, let τ_α satisfies (50). We define $X_\alpha \in \mathfrak{X}(\alpha(I))$ by

$$X_\alpha = \mu_\alpha T_\alpha + B_\alpha.$$

Then using (8) and (50), it is easy to compute $\bar{\mathcal{D}}_{\alpha'} X_\alpha$, $\bar{\mathcal{D}}_{\alpha'}^2 X_\alpha$ and $\bar{\mathcal{D}}_{\alpha'}^3 X_\alpha$, and check that the conditions of (37) are easily satisfied. Hence, $X_\alpha \in \mathfrak{X}(\alpha(I))$ is a Killing vector field on α . Consequently, by Proposition 5.6, there exists a Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ whose restriction to α is X_α . Using first one of (8), we find

$$\mathcal{F}_X T_\alpha = P \times X \times T_\alpha = \alpha \times X_\alpha \times T_\alpha = N_\alpha = \mathcal{D}_{\alpha'} T_\alpha$$

which is the associated Lorentz equation. Hence, α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ possessing quasi-slope μ_α . $\square$

Remark 6.9. According to Theorem 6.8, if a unit-speed lightlike curve $\alpha : I \rightarrow \mathbb{H}_1^3$ is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$, then $X_\alpha \in \mathfrak{X}(\alpha(I))$ can be expressed as $X_\alpha = \mu_\alpha T_\alpha + B_\alpha$ which resides in rectifying planes of α . Since N_α is spacelike, rectifying planes of α are timelike (and hence Lorentzian). Therefore, it follows that X_α may be spacelike, timelike or lightlike and so is X .

Now, if $X \in \mathfrak{X}(\mathbb{H}_1^3)$ is a unit Killing vector field with $\text{sign}(X) = \delta$, then we have $\bar{g}(X_\alpha, X_\alpha) = -2\mu_\alpha = \delta$ and hence α has constant quasi-slope μ_0 given by $\mu_0 = -\frac{\delta}{2}$. Again, if $X = X_1$ (when timelike), X_2, X_3 (when spacelike), $X_1 \pm X_2$ or $X_1 \pm X_3$ (when lightlike), then X_α is right-invariant on α and hence, by applying Lemma 2 in [20], we have $\bar{D}_{\alpha'} X_\alpha = \alpha \times X_\alpha \times T_\alpha$. On some computation, we find that α has constant pseudo-torsion τ_0 given by $\tau_0 = -\frac{\delta}{2} - 1$. On the other hand, if $X = X_4$ (when timelike), X_5, X_6 (when spacelike), $X_4 \pm X_5$ or $X_4 \pm X_6$ (when lightlike), then X_α is left-invariant on α and hence, by applying Lemma 2 in [20], we have $\bar{D}_{\alpha'} X_\alpha = \alpha \times T_\alpha \times X_\alpha$. After some computation, we find that α has constant pseudo-torsion τ_0 given by $\tau_0 = -\frac{\delta}{2} + 1$. In this regard, we find the following result:

Theorem 6.10. Let $\alpha : I \rightarrow \mathbb{H}_1^3$ be a unit-speed lightlike curve and B_X be the Killing magnetic field corresponding to a right-invariant (resp. left-invariant) unit Killing vector field $X \in \mathfrak{X}(\mathbb{H}_1^3)$ with $\text{sign}(X) = \delta$. Then α is a normal KMT in $(\mathbb{H}_1^3, \bar{g}, B_X)$ iff it has constant quasi-slope $\mu_0 = -\frac{\delta}{2}$, and it is a lightlike helix possessing constant pseudo-torsion $\tau_0 = -\frac{\delta}{2} - 1$ (resp. $\tau_0 = -\frac{\delta}{2} + 1$) with axis X .

7. LAST REMARK

If we express a curve $\alpha : I \rightarrow \mathbb{H}_1^3$ in terms of coordinate functions as

$$\alpha(s) = (x^1(s), x^2(s), x^3(s), x^4(s))$$

for all $s \in I$, then we have

$$(55) \quad -\sum_{i=1}^2 (x^i(s))^2 + \sum_{j=3}^4 (x^j(s))^2 = -1.$$

Now, if α is non-lightlike and unit-speed, then we have

$$(56) \quad -\sum_{i=1}^2 \left(\frac{dx^i}{ds} \right)^2 + \sum_{j=3}^4 \left(\frac{dx^j}{ds} \right)^2 = \delta,$$

where δ can take two values: $\delta = 1$ if α is spacelike and $\delta = -1$ if α is timelike. On the other hand, if α is lightlike and unit-speed, then we have

$$(57) \quad \sum_{i=1}^2 \left(\frac{dx^i}{ds} \right)^2 = \sum_{j=3}^4 \left(\frac{dx^j}{ds} \right)^2, \quad - \sum_{i=1}^2 \left(\frac{d^2x^i}{ds^2} \right)^2 + \sum_{j=3}^4 \left(\frac{d^2x^j}{ds^2} \right)^2 = 1.$$

Let us assume, WOLOG, that $0 \in I \subset \mathbb{R}$ and α begins from $\alpha(0) = (x_0^1, x_0^2, x_0^3, x_0^4)$ having initial velocity $\alpha'(0) = (u_0^1, u_0^2, u_0^3, u_0^4)$. That is, α has the following initial conditions:

$$(58) \quad \begin{cases} x^1(0) = x_0^1, & x^2(0) = x_0^2, & x^3(0) = x_0^3, & x^4(0) = x_0^4, \\ \frac{dx^1}{ds}(0) = u_0^1, & \frac{dx^2}{ds}(0) = u_0^2, & \frac{dx^3}{ds}(0) = u_0^3, & \frac{dx^4}{ds}(0) = u_0^4, \end{cases}$$

where

$$(59) \quad - \sum_{i=1}^2 (x_0^i)^2 + \sum_{j=3}^4 (x_0^j)^2 = -1.$$

If α is non-lightlike and unit-speed, then we have

$$(60) \quad - \sum_{i=1}^2 (u_0^i)^2 + \sum_{j=3}^4 (u_0^j)^2 = \delta,$$

where δ can take two values: $\delta = 1$ if α is spacelike and $\delta = -1$ if α is timelike. Again, if α is lightlike and unit-speed, then we have

$$(61) \quad \sum_{i=1}^2 (u_0^i)^2 = \sum_{j=3}^4 (u_0^j)^2, \quad - \sum_{i=1}^2 \left(\frac{d^2x^i}{ds^2}(0) \right)^2 + \sum_{j=3}^4 \left(\frac{d^2x^j}{ds^2}(0) \right)^2 = 1.$$

Let us pick any one of the basis Killing vector fields, say X_1 , on $\mathbb{H}_1^3$ obtained by (34) and consider its corresponding magnetic field B_{X_1} . Then the associated Lorentz equation gives rise to the following nonlinear system of ODEs:

$$(62) \quad \begin{cases} \frac{d^2x^1}{ds^2} = \delta_1 \frac{dx^1}{ds} + \left((x^3)^2 + (x^4)^2 \right) \frac{dx^2}{ds} + (x^1x^4 - x^2x^3) \frac{dx^3}{ds} - (x^1x^3 + x^2x^4) \frac{dx^4}{ds}, \\ \frac{d^2x^2}{ds^2} = \delta_1 \frac{dx^2}{ds} - \left((x^3)^2 + (x^4)^2 \right) \frac{dx^1}{ds} + (x^1x^3 + x^2x^4) \frac{dx^{13}}{ds} + (x^1x^4 - x^2x^3) \frac{dx^4}{ds}, \\ \frac{d^2x^3}{ds^2} = \delta_1 \frac{dx^3}{ds} + (x^2x^3 - x^1x^4) \frac{dx^1}{ds} - (x^1x^3 + x^2x^4) \frac{dx^2}{ds} + \left((x^1)^2 + (x^2)^2 \right) \frac{dx^4}{ds}, \\ \frac{d^2x^4}{ds^2} = \delta_1 \frac{dx^4}{ds} + (x^1x^3 + x^2x^4) \frac{dx^1}{ds} + (x^2x^3 - x^1x^4) \frac{dx^{12}}{ds} - \left((x^1)^2 + (x^2)^2 \right) \frac{dx^3}{ds}, \end{cases}$$

where $\delta_1 = 1, -1$ or 0 . Now, one can find explicit parametrization of the non-lightlike normal KMT α in $(\mathbb{H}_1^3, \bar{g}, B_{X_1})$ starting at $\alpha(0) = (x_0^1, x_0^2, x_0^3, x_0^4)$ with initial velocity $\alpha'(0) = (u_0^1, u_0^2, u_0^3, u_0^4)$ by solving the nonlinear system (62) with initial conditions (58) satisfying (59) and (60), where $\delta_1 = 1$ if α is spacelike and $\delta_1 =$

-1 if α is timelike. On the other hand, one can determine explicit parametrization of the lightlike normal KMT α in $(\mathbb{H}_1^3, \bar{g}, B_{X_1})$ starting at $\alpha(0) = (x_0^1, x_0^2, x_0^3, x_0^4)$ with initial velocity $\alpha'(0) = (u_0^1, u_0^2, u_0^3, u_0^4)$ by solving the nonlinear system (62) with initial conditions (58) satisfying (59) and (61), where $\delta_1 = 0$. Analogously, one can obtain parametrizations of Killing magnetic trajectories corresponding to any Killing magnetic field on $\mathbb{H}_1^3$.

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