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GEOMETRIC STRUCTURES ON 3-DIMENSIONAL HOM-LIE ALGEBRAS

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Abstract. Some classes of almost contact structures on 5-dimensional nilpotent Lie algebras were investigated in the literature. On the other hand, there are several generalizations of Lie algebras and their characterizations. In particular, an effective characterization of all Hom-Lie algebras of $sl(2)$ type and construction of all the 3-dimensional Hom-Lie algebras for a class of twisting homomorphisms has been presented. A list of twenty 3-dimensional Hom-Lie algebras has been given. In this paper, we consider some classes of almost contact metric structure on all those twenty 3-dimensional Hom-Lie algebras separately. We study semi-cosymplectic, almost cosymplectic and cosymlectic structures. Moreover, we study K-contact and Sasakian structures on 3-dimensional Hom-Lie algebras.

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1. Introduction

Contact manifolds are the odd-dimensional counterpart of symplectic manifolds and there isn't any doubt about the great importance of symplectic and contact geometry. An almost contact metric structure on an odd-dimensional differentiable manifold M^{2n+1} is a reduction of the structure group of its tangent bundle to $U(n) \times 1$. In [7], a classification of almost contact metric manifolds was obtained via the study of the covariant derivative of the fundamental two-form. For example, if we have $\nabla\Phi = 0$, we say that the almost contact metric structure is a cosymplectic structure.

On the other hand, the set of vector felds on a Lie group that are invariant under all left translations forms a fnite-dimensional vector space, which carries a natural bilinear product making it into an algebraic structure known as a Lie algebra. Many of the properties of a Lie group are reflected in the algebraic structure of its Lie algebra [14]. In fact, Lie algebras are related to Lie groups, and both concepts have important applications to geometry and physics. Also, Hom-algebra structures are given on linear spaces by multiplications twisted by linear maps. For example, in [3], [10], [9], [13], [15], [16], [20], [22], [23], Hom-structures including Hom-Lie algebras, Hom-algebras, Hom-coalgebras, Hom-modules, Hom-Hopf modules, biHom-associative algebras, biHom-Lie algebras and biHom-bialgebras were studied.

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [11] as part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity.

In [17], Abdenacer Makhlouf and Sergei D. Silverstrov, presented an effective characterization of all Hom-Lie algebras of $sl(2)$ type and construct all the 3-dimensional Hom-Lie algebras for a class of twisting homomorphisms. They provided a list of twenty 3-dimensional Hom-Lie algebras which was very helpful for us.

Recently, the relationship between Lie group, Lie algebra and their generalizations and contact gometry has been considered by many geometeres, for example, in [1], [2], [5], [6], [19], [12]. Particularly, in [18], Nulifer Ozdemir, Mehmet Solgun and Sirin Aktay, by using the classification of five-dimensional nilpotent Lie algebras given in [8], could determine almost contact metric structures on them ingeniously. All conclusions in their paper have been gained by very complicated calculations. In fact, we believe that, their paper is in the category of high quality papers and it motivated us to determine almost contact metric structures on all the 3-dimensional Hom-Lie algebras.

We consider all Hom-Lie algebras associated to the homomorphism given with respect to the basis e_1, e_2, e_3 of the three-dimensional V and we study some classes of almost contact metric structure on all those twenty 3-dimensional Hom-Lie algebras in this research. We consider semi-cosymplectic, almost cosymplectic and cosymlectic structures. In addition, we investigate K-contact and Sasakian structures on them.

In section 2, we present some definitions and theorems about some classes of almost contact metric structure, Lie algebra and Hom-Lie algebra. Also, all 3-dimensional Hom-Lie algebras are listed. Section 3 is devoted to gain three systems of equations for each 3-dimensional Hom-Lie algebra which characterize semi-cosymplectic, almost cosymplectic and cosymplectic structures on them. After each one, we gain a system of equations for each 3-dimensional Hom-Lie algebra which characterizes a Killing vector field, a K-contact and a Sasakian structure on them.

2. Preliminaries

M^{2n+1} has an almost contact structure or (φ, ξ, η)-structure if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Suppose that M^{2n+1} has an almost contact structure (φ, ξ, η). Then, φ has rank $2n$ and $\varphi\xi = 0$, $\eta \circ \varphi = 0$.

If M^{2n+1} with a (φ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

we say that M^{2n+1} has an almost contact metric structure and g is called a compatible metric. The fundamental 2-form Φ of an almost contact metric manifold is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

for all $X, Y \in TM$. Since φ has rank $2n$, we have $\eta \wedge \Phi^n \neq 0$. This means that every almost contact metric manifold is orientable. An almost contact metric structure with $\Phi = d\eta$ is a contact metric structure [22], [23].

In theorem 6.8 in [4], it is proved that an almost contact metric structure (φ, ξ, η, g) is cosymplectic if and only if φ is parallel. It is easy to prove that

$$(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z).$$

Therefore, an almost contact metric structure (φ, ξ, η, g) is cosymplectic if and only if Φ is parallel.

According to the classifications in [7], an almost contact metric structure
(φ, ξ, η, g) in M^{2n+1} is said to be

i) Almost cosymplectic if $d\Phi = 0$ and $d\eta = 0$, where d denotes the exterior derivative of a differential form.

ii) Cosymplectie if it is almost cosymplectic and normal.

iii) Semi cosymplectie if $\delta\Phi = 0$ and $\delta\eta = 0$, where δ denotes the coderivative of a differential form.

A vector field X on M is called a Killing vector field if and only if $L_X g = 0$. A contact metric structure (φ, ξ, η, g) is said to be K-contact if ξ is a Killing vector field. Moreover, let M be a contact metric manifold. Then, M is a K-contact manifold if and only if

(2) $$\nabla_X \xi = -\varphi X.$$

The almost contact structure (φ, ξ, η) is normal if and only if

$$[\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0.$$

A Sasakian manifold is a normal contact metric manifold. Also, an almost contact metric structure (φ, ξ, η, g) is Sasakian if and only if

(3) $$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

A Sasakian manifold is K-contact and in dimension three the converse is true. For more details we refer to [1], [22].

On the other hand, let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and V be a vector space on $\mathbb{K}$. For a homomorphism α of V, a Hom-Lie algebra is a triple $(V, [.,.], \alpha)$ consisting of a vector space V, bilinear map $[.,.] : V \times V \to V$ and a linear space homomorphism $\alpha : V \to V$ satisfying

$$\begin{aligned}
&[X, Y] = -[Y, X], \\
&[\alpha(X), [Y, Z]] + [\alpha(Z), [X, Y]] + [\alpha(Y), [Z, X]] = 0,
\end{aligned}$$

for all X, Y, Z in V.

There is a list of all Hom-Lie algebras associated to the homomorphism given with respect to the basis e_1, e_2, e_3 of the 3-dimensional V by the martrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

The list is obtained by solving the system of equations for the structure constants equivalent to axioms of Hom-Lie algebras. This list may be considered as a first partial step towards establishing a classification of

3-dimensional Hom-Lie algebras. We will show these Hom-Lie algebras by HLg_i and $i = 1, ..., 20$. In the list, C_{ij}^k for $i.j, k = 1, 2, 3$ are parameters in $\mathbb{K}$. [17]

$$\mathbf{HLg_1}: \quad [e_1, e_2]_1 = C_{12}^1 e_1 + C_{12}^3 e_3, \quad [e_1, e_3]_1 = C_{13}^2 e_2, \quad [e_2, e_3]_1 = C_{23}^1 e_1 + \frac{bC_{12}^1}{a} e_3.$$

$$\mathbf{HLg_2}: \quad [e_1, e_2]_2 = \frac{aC_{23}^3}{b} e_1 - C_{13}^3 e_2 + C_{12}^3 e_3, \quad [e_1, e_3]_2 = -\frac{aC_{23}^2}{b} e_1 + C_{13}^2 e_2 + C_{13}^3 e_3,$$
$$[e_2, e_3]_2 = C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3.$$

$$\mathbf{HLg_3}: \quad [e_1, e_2]_3 = 0, \quad [e_1, e_3]_3 = C_{13}^1 e_1 + C_{13}^2 e_2, \quad [e_2, e_3]_3 = C_{23}^1 e_1.$$

$$\mathbf{HLg_4}: \quad [e_1, e_2]_4 = C_{12}^3 e_3, \quad [e_1, e_3]_4 = C_{13}^2 e_2 + C_{13}^3 e_3, \quad [e_2, e_3]_4 = 0.$$

$$\mathbf{HLg_5}: \quad [e_1, e_2]_5 = C_{12}^3 e_3, \quad [e_1, e_3]_5 = C_{13}^2 e_2, \quad [e_2, e_3]_5 = C_{23}^1 e_1.$$

$$\mathbf{HLg_6}: \quad [e_1, e_2]_6 = C_{12}^1 e_1 + C_{12}^3 e_3, \quad [e_1, e_3]_6 = 0, \quad [e_2, e_3]_6 = C_{23}^1 e_1.$$

$$\mathbf{HLg_7}: \quad [e_1, e_2]_7 = C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, \quad [e_1, e_3]_7 = 0, \quad [e_2, e_3]_7 = C_{23}^1 e_1 + C_{23}^3 e_3.$$

$$\mathbf{HLg_8}: \quad [e_1, e_2]_8 = -C_{13}^3 e_2 + C_{12}^3 e_3, \quad [e_1, e_3]_8 = C_{13}^2 e_2 + C_{13}^3 e_3, \quad [e_2, e_3]_8 = C_{23}^1 e_1.$$

$$\mathbf{HLg_9}: \quad [e_1, e_2]_9 = -C_{13}^3 e_2 + C_{12}^3 e_3, \quad [e_1, e_3]_9 = -\frac{aC_{23}^2}{b} e_1 + C_{13}^2 e_2 + C_{13}^3 e_3,$$
$$[e_2, e_3]_9 = C_{23}^1 e_1 + C_{23}^2 e_2.$$

$$\mathbf{HLg_{10}}: \quad [e_1, e_2]_{10} = 0, \quad [e_1, e_3]_{10} = C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, \quad [e_2, e_3]_{10} = C_{23}^1 e_1 + C_{23}^2 e_2.$$

$$\mathbf{HLg_{11}}: \quad [e_1, e_2]_{11} = C_{12}^2 e_2 + C_{12}^3 e_3, \quad [e_1, e_3]_{11} = C_{13}^2 e_2 + C_{13}^3 e_3, \quad [e_2, e_3]_{11} = 0.$$

$$\mathbf{HLg_{12}}: \quad [e_1, e_2]_{12} = -\frac{C_{13}^1 C_{23}^3}{C_{23}^2} e_1 + C_{12}^2 e_2 - \frac{C_{13}^3 C_{23}^3}{C_{23}^2} e_3,$$
$$[e_1, e_3]_{12} = C_{13}^1 e_1 - \frac{C_{12}^2 C_{23}^2}{C_{23}^3} e_2 + C_{13}^3 e_3, \quad [e_2, e_3]_{12} = C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3.$$

$$\mathbf{HLg_{13}}: \quad [e_1, e_2]_{13} = C_{12}^3 e_3, \quad [e_1, e_3]_{13} = -\frac{aC_{23}^2}{b} e_1 + C_{13}^2 e_2, \quad [e_2, e_3]_{13} = C_{23}^1 e_1 + C_{23}^2 e_2.$$

$$\mathbf{HLg_{14}}: \quad [e_1,e_2]_{14} = C_{12}^1 e_1 - C_{13}^3 e_2 + \frac{C_{12}^1 C_{13}^3}{C_{13}^1} e_3,$$
$$[e_1,e_3]_{14} = C_{13}^1 e_1 - \frac{C_{13}^1 C_{13}^3}{C_{12}^1} e_2 + C_{13}^3 e_3, \quad [e_2,e_3]_{14} = C_{23}^1 e_1.$$

$$\mathbf{HLg_{15}}: \quad [e_1,e_2]_{15} = \frac{aC_{23}^3}{b} e_1 + C_{12}^2 e_2 + \frac{C_{12}^2 C_{23}^3}{C_{23}^2} e_3,$$
$$[e_1,e_3]_{15} = -\frac{((a-b)C_{12}^2 - bC_{13}^3)C_{23}^2}{bC_{12}^2} e_1 + C_{13}^2 e_2 + C_{13}^3 e_3,$$
$$[e_2,e_3]_{15} = \frac{aC_{23}^2 C_{23}^3}{bC_{12}^2} e_1 + C_{23}^2 e_2 + C_{23}^3 e_3.$$

$$\mathbf{HLg_{16}}: \quad [e_1,e_2]_{16} = \frac{aC_{23}^3}{b} e_1 + \frac{C_{13}^3 C_{23}^3}{C_{13}^1} e_3, \quad [e_1,e_3]_{16} = C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3,$$
$$[e_2,e_3]_{16} = \frac{aC_{13}^1 C_{23}^3}{bC_{13}^3} e_1 + C_{23}^3 e_3.$$

$$\mathbf{HLg_{17}}: \quad [e_1,e_2]_{17} = -\frac{C_{13}^1 C_{23}^3}{C_{23}^2} e_1 - C_{13}^3 e_2 - \frac{C_{13}^3 C_{23}^3}{C_{23}^2} e_3,$$
$$[e_1,e_3]_{17} = C_{13}^1 e_1 + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} e_2 + C_{13}^3 e_3,$$
$$[e_2,e_3]_{17} = C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3.$$

$$\mathbf{HLg_{18}}: \quad [e_1,e_2]_{18} = C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, \quad [e_1,e_3]_{18} = -\frac{aC_{23}^2}{b} e_1 - \frac{C_{12}^2 C_{23}^2}{C_{12}^1} e_2,$$
$$[e_2,e_3]_{18} = \frac{aC_{12}^1 C_{23}^2}{bC_{12}^2} e_1 + C_{23}^2 e_2.$$

$$\mathbf{HLg_{19}}: \quad [e_1,e_2]_{19} = -\frac{(bC_{12}^2 + (b-a)C_{13}^3)C_{23}^3}{bC_{13}^3} e_1 + C_{12}^2 e_2 + C_{12}^3 e_3,$$
$$[e_1,e_3]_{19} = -\frac{aC_{23}^2}{b} e_1 + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} e_2 + C_{13}^3 e_3,$$
$$[e_2,e_3]_{19} = -\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} e_1 + C_{23}^2 e_2 + C_{23}^3 e_3.$$

$$\mathbf{HLg_{20}}: \quad [e_1,e_2]_{20} = C_{12}^1 e_1 + C_{12}^2 e_2 + \frac{-aC_{13}^3 C_{23}^3 + b(C_{12}^1 C_{13}^3 + (C_{12}^2 + C_{13}^3)C_{23}^3)}{bC_{13}^1 + aC_{23}^2} e_3,$$
$$[e_1,e_3]_{20} = C_{13}^1 e_1 + \frac{bC_{12}^2 C_{13}^1 + aC_{12}^2 C_{23}^2 - bC_{12}^2 C_{23}^2 - bC_{13}^3 C_{23}^2}{bC_{12}^1 - aC_{23}^3} e_2 + C_{13}^3 e_3,$$
$$[e_2,e_3]_{20} = \frac{a(C_{12}^1 C_{23}^2 + C_{13}^1 C_{23}^3)}{b(C_{12}^2 + C_{13}^3)} e_1 + C_{23}^2 e_2 + C_{23}^3 e_3.$$

3. Structures on 3-dimensional Hom-Lie algebras

In this section we consider all 3-dimensional Hom-Lie algebras in twenty subsections and we obtain three systems of equations for each Hom-Lie algebra which characterize semi-cosymplectic, almost cosymplectic and cosymplectic structures on them. Moreover, the conditions of a K-contact and a Sasakian structure on them are investigated.

3.1. **The Hom-Lie algebra HLg_1.** Let $\{e_1, e_2, e_3\}$ be a basis on HLg_1. By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned} \nabla_{e_1}e_2 &= C_{12}^1 e_1 + \frac{1}{2}(-C_{23}^1 - C_{13}^2 + C_{12}^3)e_3, \quad \nabla_{e_1}e_3 = \frac{1}{2}(C_{23}^1 - C_{12}^3 + C_{13}^2)e_2, \\ \nabla_{e_2}e_1 &= -\frac{1}{2}(C_{23}^1 + C_{12}^3 + C_{13}^2)e_3, \quad \nabla_{e_3}e_2 = \frac{1}{2}(C_{12}^3 + C_{13}^2 - C_{23}^1)e_1 - \frac{b}{a}C_{12}^1 e_3, \\ \nabla_{e_3}e_1 &= \frac{1}{2}(C_{23}^1 - C_{12}^3 - C_{13}^2)e_2, \quad \nabla_{e_2}e_3 = \frac{1}{2}(C_{23}^1 + C_{12}^3 + C_{13}^2)e_1, \\ \nabla_{e_1}e_1 &= -C_{12}^1 e_2, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_3}e_3 = \frac{b}{a}C_{12}^1 e_2. \end{aligned} \tag{4}$$

Let (φ, ξ, η, g) be an almost contact metric structure on HLg_1 and Φ is a fundamental 2-form on it. We notice that it must be $\Phi \wedge \eta \neq 0$. Also, suppose that $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$. Then,

$$de^1 = -C_{12}^1 e^{12} - C_{23}^1 e^{23}, \quad de^2 = -C_{13}^2 e^{13}, \quad de^3 = -C_{12}^3 e^{12} - \frac{b}{a}C_{12}^1 e^{23}.$$

Therefore,

$$d\eta = -(b_1 C_{12}^1 + b_3 C_{12}^3)e^{12} - b_2 C_{13}^2 e^{13} - (b_1 C_{23}^1 + b_3 \frac{b}{a} C_{12}^1)e^{23},$$

and $d\eta = 0$ if and only if

$$b_1 C_{12}^1 + b_3 C_{12}^3 = 0, \quad b_2 C_{13}^2 = 0, \quad b_1 C_{23}^1 + b_3 \frac{b}{a} C_{12}^1 = 0.$$

On the other hand,

$$d\Phi = b_{12} de^{12} + b_{13} de^{13} + b_{23} de^{23}, \quad de^{13} = (-C_{12}^1 + \frac{b}{a}C_{12}^1)e^{123}, \quad de^{12} = de^{23} = 0,$$

So,

$$d\Phi = b_{13}(-C_{12}^1 + \frac{b}{a}C_{12}^1)e^{123}.$$

We deduce that we have an almost cosymplectic structure on HLg_1 if and only if

$$b_{13}(-C_{12}^1 + \frac{b}{a}C_{12}^1) = 0, \quad b_1 C_{12}^1 + b_3 C_{12}^3 = 0, \quad b_1 C_{23}^1 + b_3 \frac{b}{a} C_{12}^1 = 0, \quad b_2 C_{13}^2 = 0.$$

Now let $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ be a vector field on HLg_1. Then, from $g(\nabla_{e_i} X, e_j) = -g(\nabla_{e_j} X, e_i)$ it will be a Killing vector field if and only if

$$\begin{aligned} &a_2 C_{12}^1 = 0, \quad -a_1(C_{13}^2 + C_{12}^3) + a_3 \frac{b}{a} C_{12}^1 = 0, a_3(C_{13}^2 + C_{23}^1) - a_1 C_{12}^1 = 0, \\ &a_2(C_{12}^3 - C_{23}^1) = 0, \quad a_2 \frac{b}{a} C_{12}^1 = 0. \end{aligned} \tag{5}$$

Let (φ, ξ, η, g) is an almost contact metric structure on HLg_1 and $\xi = a_1 e_1 + a_2 e_2 + a_3 e_3$ be a Killing vector field. So, by (2) and (5) we have

$$\begin{aligned} \varphi e_1 &= \frac{1}{2}a_3 C_{12}^3 e_2 + \frac{1}{2}a_2 C_{13}^2 e_3, \\ \varphi e_2 &= -\frac{1}{2}a_3(C_{23}^1 + C_{13}^2 + C_{12}^3)e_1 + \frac{1}{2}a_1(C_{23}^1 + C_{13}^2 + C_{12}^3)e_3, \\ \varphi e_3 &= -\frac{1}{2}a_2 C_{13}^2 e_1 - \frac{1}{2}a_1 C_{23}^1 e_2. \end{aligned}$$

Therefore, by using (1) for e_1, we have

$$\begin{aligned}
-\frac{1}{4}(a_3^2C_{12}^3(C_{23}^1+C_{13}^2+C_{12}^3)+a_2^2(C_{13}^2)^2) &= -(a_2^2+a_3^2),\\
\frac{1}{4}C_{12}^3(C_{23}^1+C_{13}^2+C_{12}^3)a_1a_3 &= a_1a_3,\\
-\frac{1}{4}a_1a_2C_{13}^2C_{23}^1 &= a_1a_2.
\end{aligned}$$

Also, by applying (1) for e_2, e_3, we obtain the following results:

$$C_{23}^1=\pm 2,\quad C_{13}^2=\pm 2,\quad C_{12}^3=\pm 2,\quad C_{23}^1+C_{13}^2=0,\quad C_{13}^2+C_{12}^3=0.$$

Moreover, it is easy to show that the condition $\Phi = d\eta$ is satisfied. That is we have

$$g(e_i,\varphi e_j)=d\eta(e_i,e_j),\quad i,j=1,2,3.$$

Then, the almost contact metric structure (φ,ξ,η,g) is a contact metric structure. Thus, it is a K-contact structure and because of dimension three, the structure must be a Sasakian structure.

Example 3.1. *The almost contact metric structure* (φ,ξ,η,g) *where*

$$\xi=e_1,\quad \eta=e^1,\quad \varphi e_2=-e_3,\quad \varphi e_3=e_2,$$

under condition $C_{23}^1=C_{12}^1=0$ *is an almost cosymplectic structure. We notice that in this case* $\Phi=e^{23}$ *and then* $\Phi\wedge\eta\neq 0$.

For any vector $X=\Sigma x_ie_i$ on HLg_1, by using (4), we have

$$\begin{aligned}
\delta\eta &= -\Sigma(\nabla_{e_i}\eta)e_i = b_2(\frac{b}{a}C_{12}^1-C_{12}^1),\\
\delta\Phi(X) &= -\Sigma(\nabla_{e_i}\Phi)(e_i,X)\\
&= x_1(C_{23}^1b_{32}+\frac{b}{a}C_{12}^1b_{21})+x_2C_{13}^2b_{31}+x_3(C_{12}^3b_{21}-C_{12}^1b_{23}),
\end{aligned}$$

So, there is a semi cosymplectic structure on HLg_1 if and only if

$$b_2(\frac{b}{a}C_{12}^1-C_{12}^1)=0,\quad C_{23}^1b_{32}+\frac{b}{a}C_{12}^1b_{21}=0,\quad C_{12}^3b_{21}-C_{12}^1b_{23}=0,\quad C_{13}^2b_{31}=0.$$

Example 3.2. *The almost contact metric structure* (φ,ξ,η,g) *where*

$$\xi=e_2,\quad \eta=e^2,\quad \varphi e_1=-e_3,\quad \varphi e_3=e_1,\quad \Phi=e^{13},$$

under condition $C_{13}^2=C_{12}^1=0$ *is a semi cosymplectic structure.*

By calculating $\nabla\Phi=0$, we obtain

$$\begin{aligned}
&(C_{12}^3-C_{13}^2-C_{23}^1)b_{13}=0,\quad (C_{12}^3+C_{13}^2-C_{23}^1)b_{13}=0,\\
&(C_{12}^3+C_{13}^2+C_{23}^1)b_{21}=0,\quad (C_{12}^3+C_{13}^2+C_{23}^1)b_{32}=0,\\
&C_{12}^1b_{13}=0,\quad C_{12}^1b_{23}-\frac{1}{2}(C_{13}^2-C_{12}^3+C_{23}^1)b_{12}=0,\\
&\frac{b}{a}C_{12}^1b_{13}=0,\quad \frac{b}{a}C_{12}^1b_{12}+\frac{1}{2}(-C_{13}^2-C_{12}^3+C_{23}^1)b_{23}=0. \qquad (6)
\end{aligned}$$

Thus, an almost contact metric structure is a cosymplectic structure if and only if it satisfies in (6).

Example 3.3. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2, \quad \Phi = e^{23},$$

under conditions $C^1_{12} = C^1_{23} = 0$ *and* $C^2_{13} = -C^3_{12}$ *is a cosymplectic structure.*

Example 3.4. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad C^1_{23} = 2, \quad C^2_{13} = -2, \quad C^3_{12} = 2, \tag{7}$$

is a contact metric structure. Also, in this example ξ *is a Killing vector field. Then, the almost contact metric structure is a* K*-contact structure and it must be a Sasakian structure.*

3.2. **The Hom-Lie algebra** HLg_2**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1} e_2 &= \frac{a}{b} C^3_{23} e_1 + \frac{1}{2}(-C^1_{23} - C^2_{13} + C^3_{12}) e_3,\\
\nabla_{e_1} e_3 &= -\frac{a}{b} C^2_{23} e_1 + \frac{1}{2}(C^1_{23} + C^2_{13} - C^3_{12}) e_2,\\
\nabla_{e_3} e_2 &= \frac{1}{2}(C^3_{12} + C^2_{13} - C^1_{23}) e_1 - C^3_{23} e_3,\\
\nabla_{e_2} e_1 &= C^3_{13} e_2 - \frac{1}{2}(C^1_{23} + C^2_{13} + C^3_{12}) e_3,\\
\nabla_{e_3} e_1 &= \frac{1}{2}(C^1_{23} - C^2_{13} - C^3_{12}) e_2 - C^3_{13} e_3,\\
\nabla_{e_2} e_3 &= \frac{1}{2}(C^3_{12} + C^2_{13} + C^1_{23}) e_1 + C^2_{23} e_2,\\
\nabla_{e_1} e_1 &= -\frac{a}{b} C^3_{23} e_2 + \frac{a}{b} C^2_{23} e_3,\\
\nabla_{e_2} e_2 &= -C^3_{13} e_1 - C^2_{23} e_3, \quad \nabla_{e_3} e_3 = C^3_{13} e_1 + C^3_{23} e_2.
\end{aligned} \tag{8}$$

let (φ, ξ, η, g) be an almost contact metric structure on HLg_2. Also,
$\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$. Then,

$$\begin{aligned}
d\eta &= (-b_1 \frac{a}{b} C^3_{23} + b_2 C^3_{13} - b_3 C^3_{12}) e^{12} + (b_1 \frac{a}{b} C^2_{23} - b_2 C^2_{13} - b_3 C^3_{13}) e^{13}\\
&\quad -(b_1 C^1_{23} + b_2 C^2_{23} + b_3 C^3_{23}) e^{23},\\
d\Phi &= \big(b_{12}(C^2_{23} - \frac{a}{b} C^2_{23}) + b_{13}(C^3_{23} - \frac{a}{b} C^3_{23})\big) e^{123}
\end{aligned}$$

We deduce that we have an almost cosymplectic structure on HLg_2 if and only if

$$\begin{aligned}
&-b_1 \frac{a}{b} C^3_{23} + b_2 C^3_{13} - b_3 C^3_{12} = 0, \quad b_1 \frac{a}{b} C^2_{23} - b_2 C^2_{13} - b_3 C^3_{13} = 0,\\
&b_1 C^1_{23} + b_2 C^2_{23} + b_3 C^3_{23} = 0, \quad b_{12}(C^2_{23} - \frac{a}{b} C^2_{23}) + b_{13}(C^3_{23} - \frac{a}{b} C^3_{23}) = 0.
\end{aligned} \tag{9}$$

Moreover, assume that $X = \Sigma x_i e_i$ is a vector on HLg_2. Then, by (8) we have

$$\delta\eta = b_2(-\frac{a}{b} C^3_{23} + C^3_{23}) + b_3(\frac{a}{b} C^2_{23} - C^2_{23}),$$

and

$$\begin{aligned}
\delta\Phi(X) &= x_1(-C^2_{23} b_{31} + C^3_{23} b_{21} + C^1_{23} b_{32}) + x_2(\frac{a}{b} C^2_{23} b_{32} + C^3_{13} b_{12} + C^2_{13} b_{31})\\
&\quad + x_3(-\frac{a}{b} C^3_{23} b_{23} - C^3_{13} b_{13} + C^3_{12} b_{21}).
\end{aligned}$$

So, there is a semi cosymplectic structure on HLg_2 if and only if

$$\begin{aligned}
&b_2(-\frac{a}{b}C^3_{23}+C^3_{23})+b_3(\frac{a}{b}C^2_{23}-C^2_{23})=0, \quad -C^2_{23}b_{31}+C^3_{23}b_{21}+C^1_{23}b_{32}=0,\\
&\frac{a}{b}C^2_{23}b_{32}+C^3_{13}b_{12}+C^2_{13}b_{31}=0, \quad -\frac{a}{b}C^3_{23}b_{23}-C^3_{13}b_{13}+C^3_{12}b_{21}=0.
\end{aligned} \tag{10}$$

Now let $X=a_1e_1+a_2e_2+a_3e_3$ be a vector field on HLg_2. Then, it will be a Killing vector field if and only if we have the following equations

$$\begin{aligned}
&a_2\frac{a}{b}C^3_{23}-a_3\frac{a}{b}C^2_{23}=0, \quad a_1C^3_{13}+a_3C^2_{23}=0, \quad a_1C^3_{13}+a_2C^3_{23}=0,\\
&-a_1\frac{a}{b}C^3_{23}-a_2C^3_{13}+a_3(C^2_{13}+C^1_{23})=0, a_1\frac{a}{b}C^2_{23}+a_3C^3_{13}+a_2(C^3_{12}-C^1_{23})=0,\\
&-a_2C^2_{23}+a_3C^3_{23}-a_1(C^3_{12}+C^2_{13})=0.
\end{aligned} \tag{11}$$

Suppose that (φ,ξ,η,g) is an almost contact metric structure on HLg_2 and $\xi=a_1e_1+a_2e_2+a_3e_3$ be a Killing vector field. So, by (2) and (11) we have

$$\begin{aligned}
\varphi e_1 &= -\big(a_2C^3_{13}-\frac{1}{2}a_3(C^1_{23}+C^2_{13}+C^3_{12})\big)e_2+\big(a_3C^3_{13}-\frac{1}{2}a_2(C^1_{23}-C^2_{13}-C^3_{12})\big)e_3,\\
\varphi e_2 &= \big(a_2C^3_{13}-\frac{1}{2}a_3(C^1_{23}+C^2_{13}+C^3_{12})\big)e_1+\big(a_2C^2_{23}+\frac{1}{2}a_1(C^1_{23}+C^2_{13}+C^3_{12})\big)e_3,\\
\varphi e_3 &= -\big(a_3C^3_{13}+\frac{1}{2}a_2(-C^1_{23}+C^2_{13}+C^3_{12})\big)e_1-\big(a_3C^3_{23}+\frac{1}{2}a_1(C^1_{23}-C^2_{13}-C^3_{12})\big)e_2.
\end{aligned}$$

Thus, if the almost contact metric structure (φ,ξ,η,g) satisfies in $g(e_i,\varphi e_j)=d\eta(e_i,e_j), i,j=1,2,3$, it will be a K-contact and a Sasakian structure.

Example 3.5. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_1, \quad \eta=e^1, \quad \varphi e_2=-e_3, \quad \varphi e_3=e_2, \quad \Phi=e^{23},$$

under condition $C^1_{23}=C^2_{23}=C^3_{23}=0$ satisffies in (9) and (10). So, it is an almost cosymplectic and a semi cosymplectic structure.

By $\nabla\Phi=0$, we conclude that any almost contact metric structure is a cosymplectic structure if and only if we have the following equations

$$\begin{aligned}
&-\frac{a}{b}C^2_{23}b_{32}-\frac{1}{2}(C^3_{12}-C^2_{13}-C^1_{23})b_{13}=0, \quad \frac{a}{b}C^3_{23}b_{23}-\frac{1}{2}(-C^3_{12}+C^2_{13}+C^1_{23})b_{12}=0,\\
&-\frac{a}{b}C^3_{23}b_{13}+\frac{a}{b}C^2_{23}b_{21}=0, \quad C^2_{23}b_{13}+\frac{1}{2}(C^3_{12}+C^2_{13}+C^1_{23})b_{32}=0,\\
&-C^3_{13}b_{23}-C^2_{23}b_{12}=0, \quad C^3_{13}b_{13}-\frac{1}{2}(C^3_{12}+C^2_{13}+C^1_{23})b_{21}=0,\\
&C^3_{13}b_{32}+C^3_{23}b_{13}=0, \quad -C^3_{23}b_{12}-\frac{1}{2}(-C^3_{12}-C^2_{13}+C^1_{23})b_{23}=0,\\
&-C^3_{13}b_{21}-\frac{1}{2}(C^3_{12}+C^2_{13}-C^1_{23})b_{13}=0.
\end{aligned}$$

Example 3.6. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_1, \quad \eta=e^1, \quad \varphi e_2=-e_3, \quad \varphi e_3=e_2, \quad \Phi=e^{23},$$

under conditions $C^2_{23}=C^3_{23}=C^1_{23}=C^3_{13}=0$ and $C^1_{23}=-C^3_{12}$ is a cosymplectic structure.

Example 3.7. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad C^1_{23} + C^2_{13} + C^3_{12} = 2, \quad C^3_{13} = C^3_{23} = C^2_{23} = 0,$$

is a K*-contact structure and so a Sasakian structure.*

3.3. **The Hom-Lie algebra** HLg_3**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1} e_2 &= -\frac{1}{2}(C^2_{13} + C^1_{23})e_3, \quad \nabla_{e_1} e_3 = C^1_{13} e_1 + \frac{1}{2}(C^2_{13} + C^1_{23})e_2, \\
\nabla_{e_3} e_2 &= \frac{1}{2}(C^2_{13} - C^1_{23})e_1, \quad \nabla_{e_2} e_1 = -\frac{1}{2}(C^2_{13} + C^1_{23})e_3, \\
\nabla_{e_3} e_1 &= \frac{1}{2}(C^1_{23} - C^2_{13})e_2, \quad \nabla_{e_2} e_3 = \frac{1}{2}(C^2_{13} + C^1_{23})e_1, \\
\nabla_{e_1} e_1 &= -C^1_{13} e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\end{aligned}$$

Let $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$. The almost contact metric structure (φ, ξ, η, g) on HLg_3 is an almost cosymplectic structure if and only if

$$b_1 C^1_{23} = 0, \quad b_1 C^1_{13} + b_2 C^2_{13} = 0, \quad b_{12} C^1_{13} = 0.$$

Also, for any vector $X = \Sigma x_i e_i$ on HLg_3, we have a semi cosymplectic structure if and only if

$$b_3 C^1_{13} = 0, \quad C^1_{23} b_{23} = 0, \quad C^1_{13} b_{23} - C^2_{13} b_{13} = 0.$$

We know $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a Killing vector field on HLg_3 if and only if we have

$$a_3 C^1_{13} = 0, \quad a_3(C^2_{13} + C^1_{23}) = 0, \quad a_1 C^1_{13} + a_2 C^1_{23} = 0, \quad a_1 C^2_{13} = 0. \tag{12}$$

Let $\xi = a_1 e_1 + a_2 e_2 + a_3 e_3$ be a Killing vector field and (φ, ξ, η, g) is an almost contact metric structure on HLg_3 . So, by applying (1), (2) and (12), we obtain

$$C^1_{23} = \pm 2, \quad C^2_{13} = 0, \quad a_3 = 0. \tag{13}$$

Then, the almost contact metric structure (φ, ξ, η, g) under conditions (13) is a contact metric structure. So, it is a K-contact and a Sasakian structure.

Example 3.8. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under condition $C^1_{13} = 0$ *is a semi and an almost cosymplectic structure.*

By calculating $\nabla \Phi = 0$, we obtain

$$\begin{aligned}
&C^1_{13} b_{21} = 0, \quad (C^1_{23} - C^2_{13}) b_{31} = 0, \quad (C^1_{23} - C^2_{13}) b_{23} = 0, \\
&(C^1_{23} + C^2_{13}) b_{23} = 0, \quad (C^1_{23} + C^2_{13}) b_{12} = 0, \quad C^1_{13} b_{32} + \frac{1}{2}(C^1_{23} + C^2_{13}) b_{13} = 0,
\end{aligned} \tag{14}$$

Thus, any almost contact metric structure which satisfies in (14) is a cosymplectic structure and vice versa.

Example 3.9. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under conditions $C^1_{13} = 0$ *and* $C^2_{13} = -C^1_{23}$ *is a cosymplectic structure.*

Example 3.10. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_1 = 0, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2, \quad C_{23}^1 = 2, \quad C_{13}^2 = C_{13}^1 = 0,$$

is a K*-contact structure and so be a Sasakian structure.*

3.4. **The Hom-Lie algebra** HLg_4**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1} e_2 &= \frac{1}{2}(C_{12}^3 - C_{13}^2)e_3, \quad \nabla_{e_1} e_3 = \frac{1}{2}(C_{13}^2 - C_{12}^3)e_2, \\
\nabla_{e_3} e_2 &= \frac{1}{2}(C_{13}^2 + C_{12}^3)e_1, \quad \nabla_{e_2} e_1 = -\frac{1}{2}(C_{12}^3 + C_{13}^2)e_3, \\
\nabla_{e_3} e_1 &= -\frac{1}{2}(C_{13}^2 + C_{12}^3)e_2 - C_{13}^3 e_3, \quad \nabla_{e_2} e_3 = \frac{1}{2}(C_{13}^2 + C_{12}^3)e_1, \\
\nabla_{e_1} e_1 &= 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = C_{13}^3 e_1.
\end{aligned}$$

Let $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$. The almost contact metric structure (φ, ξ, η, g) on HLg_4 is an almost cosymplectic structure if and only if

$$b_3 C_{12}^3 = 0, \quad b_2 C_{13}^2 + b_3 C_{13}^3 = 0, \quad b_{23} C_{13}^3 = 0.$$

Moreover, for any vector $X = \Sigma x_i e_i$ on HLg_4, there is an semi cosymplectic structure on it if and only if

$$b_1 C_{13}^3 = 0, \quad C_{12}^3 b_{21} = 0, \quad C_{13}^3 b_{12} + C_{13}^2 b_{31} = 0.$$

Now let $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ be a vector field on HLg_4. Then, it will be a Killing vector field if and only if

$$a_3 C_{13}^2 = 0, \quad a_1(C_{13}^2 + C_{12}^3) = 0, \quad a_3 C_{13}^3 + a_2 C_{12}^3 = 0, \quad a_1 C_{13}^3 = 0. \tag{15}$$

Suppose that (φ, ξ, η, g) is an almost contact metric structure on HLg_4 while $\xi = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a Killing vector field. Then, by using (1), (2) and (15), we obtain

$$C_{12}^3 = \pm 2, \quad C_{13}^2 = 0, \quad a_1 = 0. \tag{16}$$

Then, the almost contact metric structure (φ, ξ, η, g) under conditions (16) is a contact metric structure. So, it is a K-contact and a Sasakian structure.

Example 3.11. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2, \quad \Phi = e^{23},$$

under condition $C_{13}^3 = 0$ *is a semi and an almost cosymplectic structure.*

For an almost contact metric structure (φ, ξ, η, g), by calculating $\nabla \Phi = 0$, we obtain

$$\begin{aligned}
&C_{13}^3 b_{23} = 0, \quad (C_{12}^3 - C_{13}^2) b_{13} = 0, \quad (C_{13}^2 + C_{12}^3) b_{23} = 0, \\
&(C_{13}^2 + C_{12}^3) b_{12} = 0, \quad (C_{13}^2 - C_{12}^3) b_{12} = 0, \quad C_{13}^3 b_{12} + \frac{1}{2}(C_{12}^3 + C_{13}^2) b_{31} = 0,
\end{aligned} \tag{17}$$

So, (φ, ξ, η, g) is a cosymplectic structure if and only if we have (17).

Example 3.12. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2,$$

under conditions $C_{13}^3 = 0$ *and* $C_{13}^2 = -C_{12}^3$ *is a cosymplectic structure.*

Example 3.13. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad C_{12}^3 = 2, \quad C_{13}^3 = C_{13}^2 = 0,$$

is an example of a K*-contact structure and in result it is a Sasakian structure.*

3.5. **The Hom-Lie algebra** HLg_5**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1} e_2 &= \frac{1}{2}(C_{12}^3 - C_{13}^2 - C_{23}^1)e_3, \quad \nabla_{e_1} e_3 = \frac{1}{2}(C_{13}^2 - C_{12}^3 + C_{23}^1)e_2, \\
\nabla_{e_3} e_2 &= \frac{1}{2}(C_{12}^3 + C_{13}^2 - C_{23}^1)e_1, \quad \nabla_{e_2} e_1 = -\frac{1}{2}(C_{12}^3 + C_{13}^2 + C_{23}^1)e_3, \\
\nabla_{e_3} e_1 &= \frac{1}{2}(-C_{12}^3 - C_{13}^2 + C_{23}^1)e_2, \quad \nabla_{e_2} e_3 = \frac{1}{2}(C_{12}^3 + C_{13}^2 + C_{23}^1)e_1, \\
\nabla_{e_1} e_1 &= 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\end{aligned} \tag{18}$$

Let $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$. Also, (φ, ξ, η, g) be an almost contact metric structure on HLg_5. Then, $d\eta = -(b_3 C_{12}^3 e^{12} + b_2 C_{13}^2 e^{13} + b_1 C_{23}^1 e^{23})$ and $d\eta = 0$ if and only if

$$b_3 C_{12}^3 = 0, \quad b_2 C_{13}^2 = 0, \quad b_1 C_{23}^1 = 0. \tag{19}$$

On the other hand, $d\Phi = 0$. We deduce that we have an almost cosymplectic structure on HLg_5 if and only if we have (19). We know that $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a Killing vector field on HLg_5 if and only if

$$a_1(C_{12}^3 + C_{13}^2) = 0, \quad a_2(C_{12}^3 - C_{23}^1) = 0, \quad , a_3(C_{13}^2 + C_{23}^1) = 0, \tag{20}$$

Let (φ, ξ, η, g) is an almost contact metric structure on HLg_5 and $\xi = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a Killing vector field. Then, by (2) and (20) we have

$$\begin{aligned}
\varphi e_1 &= \frac{1}{2} a_3 C_{12}^3 e_2 + \frac{1}{2} a_2 C_{13}^2 e_3, \\
\varphi e_2 &= -\frac{1}{2} a_3 C_{12}^3 e_1 + \frac{1}{2} a_1 C_{23}^1 e_3, \\
\varphi e_3 &= -\frac{1}{2} a_2 C_{13}^2 e_1 - \frac{1}{2} a_1 C_{23}^1 e_2,
\end{aligned}$$

and by using (1) we obtain

$$C_{12}^3 = \pm 2, \quad C_{13}^2 = \pm 2, \quad C_{23}^1 = \pm 2, \quad C_{13}^2 C_{23}^1 = C_{12}^3 C_{13}^2 = -4, \quad C_{12}^3 C_{23}^1 = 4. \tag{21}$$

Then, (φ, ξ, η, g) under conditions (21) is a contact metric structure. Thus, it is a K-contact structure and a Sasakian structure.

Example 3.14. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1,$$

under condition $C_{12}^3 = 0$ *is an almost cosymplectic structure.*

Assume that $X = \Sigma x_i e_i$ is a vector on HLg_5. By applying (18), we have $\delta\eta = 0$ and

$$\delta\Phi(X) = x_1 C^1_{23} b_{32} + x_2 C^2_{13} b_{31} + x_3 C^3_{12} b_{21}.$$

Thus, there is a semi cosymplectic structure on HLg_5 if and only if

$$C^1_{23} b_{32} = 0, \quad C^2_{13} b_{31} = 0, \quad C^3_{12} b_{21} = 0.$$

Example 3.15. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2, \quad \Phi = e^{23}, \quad C^1_{23} = 0,$$

is a semi cosymplectic structure.

By calculating $\nabla\Phi = 0$, we deduce that any almost contact metric structure is a cosymplectic structure if and only if

$$\begin{aligned}
&(C^3_{12} - C^2_{13} - C^1_{23}) b_{13} = 0, \quad (-C^3_{12} + C^2_{13} + C^1_{23}) b_{12} = 0,\\
&(C^3_{12} + C^2_{13} + C^1_{23}) b_{32} = 0, \quad (C^3_{12} + C^2_{13} - C^1_{23}) b_{13} = 0,\\
&(C^3_{12} + C^2_{13} + C^1_{23}) b_{12} = 0, \quad (-C^3_{12} - C^2_{13} + C^1_{23}) b_{32} = 0.
\end{aligned}$$

Example 3.16. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under conditions $C^3_{12} = 0$ and $C^2_{13} = -C^1_{23}$ is a cosymplectic structure.

Example 3.17. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0,$$

under conditions (21) is a K-contact structure and so it must be a Sasakian structure.

3.6. **The Hom-Lie algebra HLg_6.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1} e_2 &= C^1_{12} e_1 + \frac{1}{2}(C^3_{12} - C^1_{23}) e_3, \quad \nabla_{e_1} e_3 = \frac{1}{2}(-C^3_{12} + C^1_{23}) e_2,\\
\nabla_{e_3} e_2 &= \frac{1}{2}(C^3_{12} - C^1_{23}) e_1, \quad \nabla_{e_2} e_1 = -\frac{1}{2}(C^3_{12} + C^1_{23}) e_3,\\
\nabla_{e_3} e_1 &= \frac{1}{2}(-C^3_{12} + C^1_{23}) e_2, \quad \nabla_{e_2} e_3 = \frac{1}{2}(C^3_{12} + C^1_{23}) e_1,\\
\nabla_{e_1} e_1 &= -C^1_{12} e_2, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\end{aligned}$$

Let an almost contact metric structure (φ, ξ, η, g), $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$ on HLg_6. So, there is an almost cosymplectic structure if and only if

$$b_1 C^1_{12} + b_3 C^3_{12} = 0, \quad b_1 C^1_{23} = 0, \quad b_{13} C^1_{12} = 0.$$

In addition, for any vector $X = \Sigma x_i e_i$ on HLg_5, there is an semi cosymplectic structure if and only if

$$C^1_{23} b_{23} = 0, \quad C^1_{12} b_{23} + C^3_{12} b_{12} = 0, \quad b_2 C^1_{12} = 0.$$

Now suppose that $X = a_1e_1 + a_2e_2 + a_3e_3$ is a vector field on HLg_6. Then, it is a Killing vector field if and only if we have

$$a_2C^1_{12} = 0, \quad a_1C^3_{12} = 0, \quad a_2(C^3_{12} - C^1_{23}) = 0, \quad a_3C^1_{23} - a_1C^1_{12} = 0. \tag{22}$$

Let $\xi = a_1e_1 + a_2e_2 + a_3e_3$ be a Killing vector field and (φ, ξ, η, g) is an almost contact metric structure on HLg_6. Then, by (1), (2) and (22), we obtain

$$C^1_{23} = \pm 2, \quad C^3_{12} = 0. \tag{23}$$

Then, (φ, ξ, η, g) under conditions (23) is a contact metric structure. So, it is a K-contact and a Sasakian structure.

Example 3.18. *Let an almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13}, \quad C^1_{12} = 0.$$

It is a semi cosymplectic and an almost cosymplectic structure.

By $\nabla\Phi = 0$, we obtain

$$\begin{aligned} &(C^3_{12} - C^1_{23})b_{23} = 0, \quad (C^3_{12} + C^1_{23})b_{21} = 0, \quad C^1_{12}b_{23} - \frac{1}{2}(-C^3_{12} + C^1_{23})b_{12} = 0,\\ &(C^3_{12} + C^1_{23})b_{32} = 0, \quad (C^3_{12} - C^1_{23})b_{13} = 0, \quad C^1_{12}b_{13} = 0, \end{aligned} \tag{24}$$

Thus, an almost contact metric structure which satisfies in (24) is a cosymplectic structure and vice versa.

Example 3.19. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under conditions $C^1_{12} = 0$ and $C^3_{12} = C^1_{23}$ is a cosymplectic structure.

Example 3.20. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad C^3_{12} = 0, \quad C^1_{23} = 2,$$

is an example of a K-contact structure and also a Sasakian structure.

3.7. **The Hom-Lie algebra HLg_7.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned} \nabla_{e_1}e_2 &= C^1_{12}e_1 + \frac{1}{2}(C^3_{12} - C^1_{23})e_3, \quad \nabla_{e_1}e_3 = \frac{1}{2}(-C^3_{12} + C^1_{23})e_2,\\ \nabla_{e_3}e_2 &= \frac{1}{2}(C^3_{12} - C^1_{23})e_1 - C^3_{23}e_3, \quad \nabla_{e_2}e_1 = -C^2_{12}e_2 - \frac{1}{2}(C^3_{12} + C^1_{23})e_3,\\ \nabla_{e_3}e_1 &= \frac{1}{2}(-C^3_{12} + C^1_{23})e_2, \quad \nabla_{e_2}e_3 = \frac{1}{2}(C^3_{12} + C^1_{23})e_1,\\ \nabla_{e_1}e_1 &= -C^1_{12}e_2, \quad \nabla_{e_2}e_2 = C^2_{12}e_1, \quad \nabla_{e_3}e_3 = C^3_{23}e_2. \end{aligned}$$

Let the almost contact metric structure (φ, ξ, η, g), $\eta = \Sigma b_ie^i$ and $\Phi = \Sigma b_{ij}e^{ij}$ on HLg_7. Then, we have an almost cosymplectic structure if and only if

$$b_1C^1_{12} + b_2C^2_{12} + b_3C^3_{12} = 0, \quad b_1C^1_{23} + b_3C^3_{23} = 0, \quad b_{13}(-C^1_{12} + C^3_{23}) - b_{23}C^2_{12} = 0.$$

Moreover, for a vector $X = \Sigma x_i e_i$ on HLg_7, there is a semi cosymplectic structure if and only if

$$b_2(C_{23}^3 - C_{12}^1) = 0, \quad C_{23}^3 b_{21} + C_{23}^1 b_{32} = 0, \quad -C_{12}^1 b_{23} + C_{12}^3 b_{21} + C_{12}^2 b_{13} = 0.$$

Now, $X = a_1e_1 + a_2e_2 + a_3e_3$ is a Killing vector field on HLg_7 if and only if

$$\begin{aligned} &a_1 C_{12}^2 = 0, \quad a_2 C_{12}^1 = 0, \quad a_2 C_{23}^3 = 0, \quad a_2(C_{12}^3 - C_{23}^1) = 0, \\ &a_3 C_{23}^1 - a_1 C_{12}^1 + a_2 C_{12}^2 = 0, \quad -a_1 C_{12}^3 + a_3 C_{23}^3 = 0. \end{aligned} \tag{25}$$

Let (φ, ξ, η, g) is an almost contact metric structure and $\xi = a_1e_1 + a_2e_2 + a_3e_3$ is a Killing vector field. So, by (1), (2) and (25), we have

$$C_{12}^3 = 2, \quad C_{12}^1 = C_{23}^1 = 0, \quad a_2 = 0. \tag{26}$$

Then, the almost contact metric structure (φ, ξ, η, g) under conditions (26) is a contact metric structure. So, it is a K-contact structure and a Sasakian structure.

Example 3.21. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under conditions $C_{12}^2 = 0$ and $C_{12}^1 = C_{23}^3$ is a semi cosymplectic and an almost cosymplectic structure.

By $\nabla\Phi = 0$, we conclude that any almost contact metric structure is a cosymplectic structure if and only if we have

$$\begin{aligned} &(C_{12}^3 + C_{23}^1)b_{32} = 0, \quad (C_{12}^3 - C_{23}^1)b_{13} = 0, \quad C_{12}^1 b_{23} - \frac{1}{2}(-C_{12}^3 + C_{23}^1)b_{12} = 0, \\ &-C_{12}^2 b_{13} - \frac{1}{2}(C_{12}^3 + C_{23}^1)b_{21} = 0, \quad -C_{23}^3 b_{12} - \frac{1}{2}(-C_{12}^3 + C_{23}^1)b_{23} = 0, \\ &C_{23}^3 b_{13} = 0, \quad C_{12}^1 b_{13} = 0, \quad C_{12}^2 b_{23} = 0. \end{aligned}$$

Example 3.22. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under conditions $C_{12}^1 = C_{12}^2 = C_{23}^3 = 0$ and $C_{12}^3 = C_{23}^1$ is a cosymplectic structure.

Example 3.23. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad C_{12}^3 = 2, \quad C_{23}^1 = C_{12}^1 = 0,$$

is a K-contact and a Sasakian structure.

3.8. **The Hom-Lie algebra HLg_8.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned} &\nabla_{e_1} e_2 = \frac{1}{2}(C_{12}^3 - C_{13}^2 - C_{23}^1)e_3, \quad \nabla_{e_1} e_3 = \frac{1}{2}(-C_{12}^3 + C_{23}^1 + C_{13}^2)e_2, \\ &\nabla_{e_3} e_2 = \frac{1}{2}(C_{12}^3 - C_{23}^1 + C_{13}^2)e_1, \quad \nabla_{e_2} e_3 = \frac{1}{2}(C_{12}^3 + C_{23}^1 + C_{13}^2)e_1, \\ &\nabla_{e_2} e_1 = C_{13}^3 e_2 - \frac{1}{2}(C_{12}^3 + C_{23}^1 + C_{13}^2)e_3, \quad \nabla_{e_3} e_1 = \frac{1}{2}(-C_{12}^3 + C_{23}^1 - C_{13}^2)e_2 - C_{13}^3 e_3, \\ &\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_2 = -C_{13}^3 e_1, \quad \nabla_{e_3} e_3 = C_{13}^3 e_1. \end{aligned}$$

Let $\eta = \Sigma b_i e^i$, $\Phi = \Sigma b_{ij} e^{ij}$ and (φ, ξ, η, g) be an almost contact metric structure on HLg_8. Then, $d\Phi = 0$ and $d\eta = 0$ if and only if

$$b_2 C_{13}^3 - b_3 C_{12}^3 = 0, \quad b_2 C_{13}^2 + b_3 C_{13}^3 = 0, \quad b_1 C_{23}^1 = 0.$$

Also, for any vector $X = \Sigma x_i e_i$ on HLg_8 there is a semi cosymplectic structure if and only if

$$C_{23}^1 b_{32} = 0, \quad C_{13}^3 b_{12} + C_{13}^2 b_{31} = 0, \quad C_{12}^3 b_{21} - C_{13}^3 b_{13} = 0.$$

Now let $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ be a vector field on HLg_8. Then, it is a Killing vector field if and only if we have

$$a_1 C_{13}^3 = 0, \quad a_1(C_{12}^3 + C_{13}^2) = 0, \quad a_2(C_{12}^3 - C_{23}^1) + a_3 C_{13}^3 = 0, \quad a_3(C_{13}^2 + C_{23}^1) - a_2 C_{13}^3 = 0, \tag{27}$$

Suppose that $\xi = a_1 e_1 + a_2 e_2 + a_3 e_3$ be a Killing vector field on HLg_8 and we have an almost contact metric structure (φ, ξ, η, g). Then, by (1), (2) and (27), we obtain

$$C_{12}^3 = \pm 2, \quad C_{23}^1 = \pm 2, \quad C_{13}^2 = \pm 2, \quad C_{23}^1 C_{12}^3 = 4, \quad C_{23}^1 C_{13}^2 = -4. \tag{28}$$

Then, (φ, ξ, η, g) under conditions (28) is a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

Example 3.24. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2, \quad \Phi = e^{23},$$

under condition $C_{23}^1 = 0$ *is a semi cosymplectic structure and an almost cosymplectic structure.*

By calculating $\nabla \Phi = 0$, we obtain

$$\begin{aligned}
&(C_{12}^3 - C_{13}^2 - C_{23}^1) b_{13} = 0, \quad (-C_{12}^3 + C_{13}^2 + C_{23}^1) b_{12} = 0, \quad C_{13}^3 b_{23} = 0,\\
&(C_{12}^3 + C_{13}^2 + C_{23}^1) b_{32} = 0, \quad (-C_{12}^3 - C_{13}^2 + C_{23}^1) b_{23} = 0,\\
&C_{13}^3 b_{13} - \frac{1}{2}(C_{12}^3 + C_{13}^2 + C_{23}^1) b_{21} = 0, \quad C_{13}^3 b_{21} + \frac{1}{2}(C_{12}^3 + C_{13}^2 - C_{23}^1) b_{13} = 0,
\end{aligned} \tag{29}$$

Thus, any almost contact metric structure which satisfies in (29) is a cosymplectic structure and vice versa.

Example 3.25. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2,$$

under conditions $C_{13}^3 = C_{23}^1 = 0$ *and* $C_{12}^3 = -C_{13}^2$ *is a cosymplectic structure.*

Example 3.26. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = e_3, \quad \varphi e_2 = -e_3, \quad \varphi e_1 = 0, \quad C_{13}^2 = -2, \quad C_{23}^1 = C_{12}^3 = 2,$$

is an example of a K-contact structure and a Sasakian structure.

3.9. **The Hom-Lie algebra HLg_9.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\nabla_{e_1}e_2=\frac{1}{2}(C^3_{12}-C^2_{13}-C^1_{23})e_3,\quad \nabla_{e_2}e_1=C^3_{13}e_2-\frac{1}{2}(C^3_{12}+C^1_{23}+C^2_{13})e_3,$$
$$\nabla_{e_1}e_3=-\frac{a}{b}C^2_{23}e_1+\frac{1}{2}(-C^3_{12}+C^1_{23}+C^2_{13})e_2,\quad \nabla_{e_3}e_2=\frac{1}{2}(C^3_{12}-C^1_{23}+C^2_{13})e_1,$$
$$\nabla_{e_3}e_1=\frac{1}{2}(-C^3_{12}+C^1_{23}-C^2_{13})e_2-C^3_{13}e_3,\quad \nabla_{e_2}e_3=\frac{1}{2}(C^3_{12}+C^1_{23}+C^2_{13})e_1+C^2_{23}e_2,$$
$$\nabla_{e_1}e_1=\frac{a}{b}C^2_{23}e_3,\quad \nabla_{e_2}e_2=-C^3_{13}e_1-C^2_{23}e_3,\quad \nabla_{e_3}e_3=C^3_{13}e_1.$$

Let the almost contact metric structure (φ,ξ,η,g), $\eta=\Sigma b_ie^i$ and $\Phi=\Sigma b_{ij}e^{ij}$ on HLg_9. Then, we have an almost cosymplectic structure if and only if

$$b_2C^3_{13}-b_3C^3_{12}=0,\quad b_1\frac{a}{b}C^2_{23}-b_2C^2_{13}-b_3C^3_{13}=0,$$
$$b_1C^1_{23}+b_2C^2_{23}=0,\quad b_{12}(-\frac{a}{b}C^2_{23}+C^2_{23})=0.$$

Moreover, assume that $X=\Sigma x_ie_i$ is a vector on HLg_9. So, (φ,ξ,η,g) is a semi cosymplectic structure if and only if

$$-C^2_{23}b_{31}+C^1_{23}b_{32}=0,\quad \frac{a}{b}C^2_{23}b_{32}+C^2_{13}b_{31}+C^3_{13}b_{12}=0,$$
$$C^3_{12}b_{21}-C^3_{13}b_{13}=0,\quad b_3(\frac{a}{b}C^2_{23}-C^2_{23})=0.$$

$X=a_1e_1+a_2e_2+a_3e_3$ is a Killing vector field on HLg_9 if and only if

$$a_1C^3_{13}=0,\quad a_3C^2_{23}=0,\quad a_3\frac{a}{b}C^2_{23}=0,\quad -a_1(C^3_{12}+C^2_{13})-a_2C^2_{23}=0,$$
$$a_2(C^3_{12}-C^1_{23})+a_1\frac{a}{b}C^2_{23}+a_3C^3_{13}=0,\quad a_3(C^1_{23}+C^2_{13})-a_2C^3_{13}=0. \tag{30}$$

Let (φ,ξ,η,g) is an almost contact metric structure and $\xi=a_1e_1+a_2e_2+a_3e_3$ is a Killing vector field. So, by (2) and (30) we have

$$\begin{aligned}
\varphi e_1 &= -\frac{1}{2}a_3(C^2_{13}-C^3_{12}+C^1_{23})e_2+(a_3C^3_{13}+\frac{1}{2}a_2(-C^3_{12}+C^1_{23}-C^2_{13}))e_3,\\
\varphi e_2 &= \frac{1}{2}a_3(C^2_{13}-C^3_{12}+C^1_{23})e_1+\frac{1}{2}a_1(-C^3_{12}+C^1_{23}-C^2_{13})e_3,\\
\varphi e_3 &= (-a_3C^3_{13}-\frac{1}{2}a_2(C^2_{13}+C^3_{12}-C^1_{23}))e_1-\frac{1}{2}a_1(-C^3_{12}+C^1_{23}-C^2_{13})e_2.
\end{aligned}$$

Then, if (φ,ξ,η,g) satisfies in $g(e_i,\varphi e_j)=d\eta(e_i,e_j), i,j=1,2,3$, it will be a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

Example 3.27. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_3,\quad \eta=e^3,\quad \varphi e_1=-e_2,\quad \varphi e_2=e_1,$$
$$\Phi=e^{12},\quad C^3_{12}=C^2_{23}=C^3_{13}=0,$$

is a semi cosymplectic structure. It is also an example of an almost cosymplectic structure on HLg_9.

By calculating $\nabla\Phi = 0$, we deduce that we have a cosymplectic structure on HLg_9 if and only if

$$\begin{aligned}
&\frac{a}{b}C^2_{23}b_{32} + \frac{1}{2}(C^3_{12} - C^2_{13} - C^1_{23})b_{13} = 0, \quad (-C^3_{12} + C^2_{13} + C^1_{23})b_{12} = 0,\\
&C^2_{23}b_{13} + \frac{1}{2}(C^3_{12} + C^2_{13} + C^1_{23})b_{32} = 0, \quad (-C^3_{12} - C^2_{13} + C^1_{23})b_{23} = 0,\\
&C^3_{13}b_{13} - \frac{1}{2}(C^3_{12} + C^2_{13} + C^1_{23})b_{21} = 0, \quad C^3_{13}b_{21} + \frac{1}{2}(C^3_{12} + C^2_{13} - C^1_{23})b_{13} = 0,\\
&C^3_{13}b_{32} = 0, \quad \frac{a}{b}C^2_{23}b_{21} = 0, \quad C^3_{13}b_{23} + C^2_{23}b_{12} = 0.
\end{aligned}$$

Example 3.28. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under conditions $C^3_{13} = C^2_{23} = C^3_{12} = 0$ and $C^1_{23} = -C^2_{13}$ is a cosymplectic structure.

Example 3.29. *The almost contact metric structure (φ, ξ, η, g) where*

$$\begin{aligned}
&\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2, \quad \varphi e_1 = 0,\\
&C^1_{23} = 2, \quad C^3_{13} = C^2_{23} = 0, \quad C^3_{12} + C^2_{13} = 0,
\end{aligned}$$

is a contact metric structure. So, it is a K-contact and a Sasakian structure.

3.10. **The Hom-Lie algebra HLg_{10}.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1}e_2 &= -\frac{1}{2}(C^2_{13} + C^1_{23})e_3, \quad \nabla_{e_1}e_3 = C^1_{13}e_1 + \frac{1}{2}(C^1_{23} + C^2_{13})e_2,\\
\nabla_{e_3}e_2 &= \frac{1}{2}(-C^1_{23} + C^2_{13})e_1, \quad \nabla_{e_2}e_1 = -\frac{1}{2}(C^2_{13} + C^1_{23})e_3,\\
\nabla_{e_3}e_1 &= \frac{1}{2}(C^1_{23} - C^2_{13})e_2 - C^3_{13}e_3, \quad \nabla_{e_2}e_3 = \frac{1}{2}(C^1_{23} + C^2_{13})e_1 + C^2_{23}e_2,\\
\nabla_{e_1}e_1 &= -C^1_{13}e_3, \quad \nabla_{e_2}e_2 = -C^2_{23}e_3, \quad \nabla_{e_3}e_3 = C^3_{13}e_1.
\end{aligned}$$

Let $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij}e^{ij}$. Then, for the almost contact metric structure (φ, ξ, η, g) on HLg_{10}, there is an almost cosymplectic structure if and only if

$$b_1C^1_{13} + b_2C^2_{13} + b_3C^3_{13} = 0, \quad b_1C^1_{23} + b_2C^2_{23} = 0, \quad b_{12}(C^1_{13} + C^2_{23}) - b_{23}C^3_{13} = 0.$$

Let $X = a_1e_1 + a_2e_2 + a_3e_3$ be a vector field on HLg_{10}. Then, it will be a Killing vector field if and only if we have the following equations

$$\begin{aligned}
&a_1C^3_{13} = 0, \quad a_3C^1_{13} = 0, \quad a_3C^2_{23} = 0, \quad a_3(C^1_{23} + C^2_{13}) = 0,\\
&-a_1C^1_{13} + a_3C^3_{13} - a_2C^1_{23} = 0, \quad a_2C^2_{23} + a_1C^2_{13} = 0.
\end{aligned} \tag{31}$$

Let $\xi = a_1e_1 + a_2e_2 + a_3e_3$ be a Killing vector field on HLg_{10} and we have an almost contact metric structure (φ, ξ, η, g). So, by (1), (2) and (31) we obtain

$$(C^2_{13} - C^1_{23})^2 = 4, \quad a_3 = 0. \tag{32}$$

Then, (φ, ξ, η, g) under conditions (32) is a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

Example 3.30. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under condition $C_{13}^2 = C_{23}^2 = 0$ *is an almost cosymplectic structure.*

Suppose that $X = \Sigma x_i e_i$ is a vector on HLg_{10}. Therefore, there is a semi cosymplectic structure on HL_{10} if and only if

$$-b_1 C_{13}^3 - b_3(C_{13}^1 + C_{23}^2) = 0, \quad C_{23}^1 b_{32} - C_{23}^2 b_{31} = 0, \quad C_{13}^3 b_{12} + C_{13}^2 b_{31} - C_{13}^1 b_{32} = 0.$$

Example 3.31. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under conditions $C_{13}^1 = -C_{23}^2$ *and* $C_{13}^3 = 0$ *and is a semi cosymplectic structure.*

By calculating $\nabla\Phi = 0$, we obtain

$$\begin{aligned}
&C_{13}^1 b_{32} + \frac{1}{2}(C_{13}^2 + C_{23}^1) b_{13} = 0, \quad (C_{13}^2 + C_{23}^1) b_{12} = 0,\\
&C_{23}^2 b_{13} + \frac{1}{2}(C_{13}^2 + C_{23}^1) b_{32} = 0, \quad (-C_{13}^2 + C_{23}^1) b_{23} = 0,\\
&C_{13}^3 b_{21} + \frac{1}{2}(C_{13}^2 - C_{23}^1) b_{13} = 0, \quad C_{13}^1 b_{21} = 0, \quad C_{23}^2 b_{12} = 0, \quad C_{13}^3 b_{32} = 0,
\end{aligned} \tag{33}$$

So, an almost contact metric structure on HLg_{10} is a cosymplectic structure if and only if it satisfies in (33).

Example 3.32. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under condition $C_{23}^2 = C_{13}^2 = C_{23}^1 = 0$ *is a cosymplectic structure.*

Example 3.33. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\begin{aligned}
&\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2, \quad \varphi e_1 = 0,\\
&C_{13}^2 = C_{13}^1 = C_{13}^3 = 0, \quad C_{23}^1 = 2,
\end{aligned}$$

is an example of a K*-contact structure and a Sasakian structure.*

3.11. **The Hom-Lie algebra** HLg_{11}**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1} e_2 &= \frac{1}{2}(C_{12}^3 - C_{13}^2) e_3, \quad \nabla_{e_1} e_3 = \frac{1}{2}(-C_{12}^3 + C_{13}^2) e_2,\\
\nabla_{e_3} e_2 &= \frac{1}{2}(C_{12}^3 + C_{13}^2) e_1, \quad \nabla_{e_2} e_1 = -C_{12}^2 e_2 - \frac{1}{2}(C_{13}^2 + C_{12}^3) e_3,\\
\nabla_{e_3} e_1 &= -\frac{1}{2}(C_{13}^2 + C_{12}^3) e_2 - C_{13}^3 e_3, \quad \nabla_{e_2} e_3 = \frac{1}{2}(C_{12}^3 + C_{13}^2) e_1,\\
\nabla_{e_1} e_1 &= 0, \quad \nabla_{e_2} e_2 = C_{12}^2 e_1, \quad \nabla_{e_3} e_3 = C_{13}^3 e_1.
\end{aligned}$$

Let the almost contact metric structure (φ, ξ, η, g), $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$ on HLg_{11}. Then, we have an almost cosymplectic structure if and only if

$$b_2 C_{12}^2 + b_3 C_{12}^3 = 0, \quad b_2 C_{13}^2 + b_3 C_{13}^3 = 0, \quad b_{23}(C_{12}^2 + C_{13}^3) = 0.$$

Moreover, let $X = \Sigma x_i e_i$ be a vector on HLg_{11}. Then, there is a semi cosymplectic structure if and only if

$$b_1(C_{12}^2 + C_{13}^3) = 0, \quad C_{13}^2 b_{31} + C_{13}^3 b_{12} = 0, \quad C_{12}^3 b_{21} + C_{12}^2 b_{13} = 0.$$

Now, $X = a_1e_1 + a_2e_2 + a_3e_3$ is a Killing vector field on HLg_{11} if and only if

$$a_1 C_{12}^2 = 0, \quad a_1 C_{13}^3 = 0, \quad a_3 C_{13}^2 + a_2 C_{12}^2 = 0, \quad a_3 C_{13}^3 + a_2 C_{12}^3 = 0, \quad a_1(C_{12}^3 + C_{13}^2) = 0.$$

The conditions of being almost contact metric structure of (φ, ξ, η, g) on HLg_{11} while $\xi = a_1e_1+a_2e_2+a_3e_3$ is a Killing vector field will be

$$(C_{13}^2 - C_{12}^3)^2 = 4, \quad a_1 = 0.$$

Moreover, (φ, ξ, η, g) is a contact metric structure. So, it is a K-contact and a Sasakian structure.

Example 3.34. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under condition $C_{13}^2 = C_{12}^2 = 0$ is a semi cosymplectic and an almost cosymplectic structure.

By $\nabla\Phi = 0$, we deduce that an almost contact metric structure is a cosymplectic structure if and only if

$$\begin{aligned}
&C_{12}^2 b_{13} + \frac{1}{2}(C_{13}^2 + C_{12}^3) b_{21} = 0, \quad C_{12}^2 b_{23} = 0,\\
&C_{13}^3 b_{21} + \frac{1}{2}(C_{13}^2 + C_{12}^3) b_{13} = 0, \quad C_{13}^3 b_{23} = 0,\\
&(C_{12}^3 - C_{13}^2) b_{13} = 0, \quad (C_{13}^2 - C_{12}^3) b_{12} = 0, \quad (C_{12}^3 + C_{13}^2) b_{32} = 0.
\end{aligned}$$

Example 3.35. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13}$$

under condition $C_{12}^2 = C_{13}^2 = C_{12}^3 = 0$ is a cosymplectic structure.

Example 3.36. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad C_{12}^3 = 2, \quad C_{13}^2 = 0,$$

is a K-contact and a Sasakian structure.

3.12. **The Hom-Lie algebra HLg_{12}.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1}e_2 &= -\frac{C^1_{13}C^3_{23}}{C^2_{23}}e_1+\frac{1}{2}(-\frac{C^3_{13}C^3_{23}}{C^2_{23}}+\frac{C^2_{12}C^2_{23}}{C^3_{23}}-C^1_{23})e_3,\\
\nabla_{e_2}e_1 &= -C^2_{12}e_2+\frac{1}{2}(\frac{C^3_{13}C^3_{23}}{C^2_{23}}+\frac{C^2_{12}C^2_{23}}{C^3_{23}}-C^1_{23})e_3,\\
\nabla_{e_3}e_1 &= \frac{1}{2}(\frac{C^3_{13}C^3_{23}}{C^2_{23}}+\frac{C^2_{12}C^2_{23}}{C^3_{23}}+C^1_{23})e_2-C^3_{13}e_3,\\
\nabla_{e_1}e_3 &= C^1_{13}e_1+\frac{1}{2}(-\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})e_2,\\
\nabla_{e_2}e_3 &= \frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^3_{23}}-\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})e_1+C^2_{23}e_2,\\
\nabla_{e_3}e_2 &= \frac{1}{2}(-\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}-C^1_{23})e_1-C^3_{23}e_3,\\
\nabla_{e_1}e_1 &= \frac{C^1_{13}C^3_{23}}{C^2_{23}}e_2-C^1_{13}e_3,\quad \nabla_{e_2}e_2=C^2_{12}e_1-C^2_{23}e_3,\\
\nabla_{e_3}e_3 &= C^3_{13}e_1+C^3_{23}e_2.
\end{aligned}$$

Let $\eta=\Sigma b_ie^i$, $\Phi=\Sigma b_{ij}e^{ij}$ and the almost contact metric structure (φ,ξ,η,g) on HLg_{12}. Then, there is an almost cosymplectic structure if and only if

$$\begin{aligned}
&b_1\frac{C^1_{13}C^3_{23}}{C^2_{23}}-b_2C^2_{12}+b_3\frac{C^3_{13}C^3_{23}}{C^2_{23}}=0,\\
&b_1C^1_{23}+b_2C^2_{23}+b_3C^3_{23}=0,\quad -b_1C^1_{13}+b_2\frac{C^2_{12}C^2_{23}}{C^3_{23}}-b_3C^3_{13}=0,\\
&b_{12}(C^1_{13}+C^2_{23})+b_{13}(\frac{C^1_{13}C^3_{23}}{C^2_{23}}+C^3_{23})-b_{23}(C^2_{12}+C^3_{13})=0.
\end{aligned}$$

Also, for any vector $X=\Sigma x_ie_i$ on HLg_{12}, we have a semi cosymplectic structure if and only if

$$\begin{aligned}
&b_1(C^2_{12}+C^3_{13})+b_2(\frac{C^1_{13}C^3_{23}}{C^2_{23}}+C^3_{23})-b_3(C^2_{23}+C^1_{13})=0,\\
&-C^2_{23}b_{31}+C^1_{23}b_{32}+C^3_{23}b_{21}=0,\quad -C^1_{13}b_{32}+C^1_{23}b_{31}+C^3_{13}b_{12}=0,\\
&\frac{C^1_{13}C^3_{23}}{C^2_{23}}b_{23}+(\frac{C^3_{13}C^3_{23}}{C^2_{23}}-\frac{C^2_{12}C^2_{23}}{C^3_{23}})b_{12}+C^2_{12}b_{13}=0.
\end{aligned}$$

Now let $X=a_1e_1+a_2e_2+a_3e_3$ be a vector field on HLg_{12}. Then, it is a Killing vector field if and only if

$$\begin{aligned}
&-a_1C^1_{13}-a_2C^1_{23}+a_3C^3_{13}=0,\quad -a_1C^2_{12}+a_3C^2_{23}=0,\quad -a_1C^3_{13}-a_2C^3_{23}=0,\\
&a_1\frac{C^1_{13}C^3_{23}}{C^2_{23}}+a_2C^2_{12}+a_3C^1_{23}=0,\quad -a_2\frac{C^1_{13}C^3_{23}}{C^2_{23}}+a_3C^1_{13}=0,\\
&a_1(\frac{C^3_{13}C^3_{23}}{C^2_{23}}+\frac{C^2_{12}C^2_{23}}{C^3_{23}})-a_2C^2_{23}+a_3C^3_{23}=0. \qquad (34)
\end{aligned}$$

Let (φ,ξ,η,g) is an almost contact metric structure on HLg_{12} and $\xi=a_1e_1+a_2e_2+a_3e_3$ is a Killing vector field. So, by (2) and (34) we have

$$\begin{aligned}
\varphi e_1 &= \big(-a_1\frac{C^1_{13}C^3_{23}}{C^2_{23}}-\frac{1}{2}a_3(-\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})\big)e_2\\
&\quad+\big(a_1C^1_{13}-\frac{1}{2}a_2(\frac{C^2_{12}C^2_{23}}{C^3_{23}}-\frac{C^3_{13}C^3_{23}}{C^2_{23}}-C^1_{23})\big)e_3,\\
\varphi e_2 &= \big(-a_2C^2_{12}-\frac{1}{2}a_3(\frac{C^2_{12}C^2_{23}}{C^3_{23}}-\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})\big)e_1\\
&\quad+\big(a_2C^2_{23}-\frac{1}{2}a_1(\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}-C^1_{23})\big)e_3,\\
\varphi e_3 &= \big(-a_3C^3_{13}-\frac{1}{2}a_2(-\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}-C^1_{23})\big)e_1\\
&\quad+\big(-a_3C^3_{23}-\frac{1}{2}a_1(\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})\big)e_2.
\end{aligned}$$

Then, if (φ,ξ,η,g) satisfies in $g(e_i,\varphi e_j)=d\eta(e_i,e_j), i,j=1,2,3$, it will be a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

Example 3.37. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_1,\quad \eta=e^1,\quad \varphi e_2=-e_3,\quad \varphi e_3=e_2,\quad \Phi=e^{23},$$

under conditions $C^2_{12}=-C^3_{13}$ and $C^1_{13}=C^1_{23}=0$ is a semi and an almost cosymplectic structure.

By calculating $\nabla\Phi=0$, we obtain

$$\begin{aligned}
&C^1_{13}b_{32}-\frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^3_{23}}-\frac{C^3_{13}C^3_{23}}{C^2_{23}}-C^1_{23})b_{13}=0,\\
&-C^2_{12}b_{13}-\frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^3_{23}}-\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})b_{21}=0,\\
&-C^3_{23}b_{12}-\frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})b_{23}=0,\\
&-C^3_{13}b_{21}-\frac{1}{2}(-\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}-C^1_{23})b_{13}=0,\\
&-\frac{C^1_{13}C^3_{23}}{C^2_{23}}b_{23}-\frac{1}{2}(-\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}+C^1_{23})b_{12}=0,\\
&C^2_{23}b_{13}-\frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^3_{23}}+\frac{C^3_{13}C^3_{23}}{C^2_{23}}-C^1_{23})b_{32}=0,
\end{aligned}$$

$$\frac{C^1_{13}C^3_{23}}{C^2_{23}}b_{13}-C^1_{13}b_{21}=0,\quad C^2_{12}b_{23}-C^2_{23}b_{12}=0,\quad C^3_{13}b_{32}+C^3_{23}b_{13}=0. \tag{35}$$

Thus, any almost contact metric structure which satisfies in (35) is a cosymplectic structure and vice versa.

Example 3.38. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_1,\quad \eta=e^1,\quad \varphi e_2=-e_3,\quad \varphi e_3=e_2,\quad \Phi=e^{23},$$

under condition $C^1_{13}=C^1_{23}=C^2_{12}=C^3_{13}=0$ is a cosymplectic structure.

Example 3.39. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2, \quad C_{23}^1 = 2, \quad C_{13}^3 = C_{13}^1 = C_{12}^2 = 0,$$

is a contact metric structure. Then, it is a K-contact and a Sasakian structure.

3.13. **The Hom-Lie algebra** HLg_{13}**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
&\nabla_{e_1} e_2 = \frac{1}{2}(C_{12}^3 - C_{13}^2 - C_{23}^1)e_3, \quad \nabla_{e_3} e_1 = \frac{1}{2}(-C_{12}^3 - C_{13}^2 + C_{23}^1)e_2,\\
&\nabla_{e_1} e_3 = -\frac{a}{b}C_{23}^2 e_1 + \frac{1}{2}(-C_{12}^3 + C_{13}^2 + C_{23}^1)e_2,\\
&\nabla_{e_3} e_2 = \frac{1}{2}(C_{12}^3 + C_{13}^2 - C_{23}^1)e_1, \quad \nabla_{e_2} e_1 = -\frac{1}{2}(C_{12}^3 + C_{13}^2 + C_{23}^1)e_3,\\
&\nabla_{e_2} e_3 = \frac{1}{2}(C_{12}^3 + C_{13}^2 + C_{23}^1)e_1 + C_{23}^2 e_2,\\
&\nabla_{e_1} e_1 = \frac{a}{b}C_{23}^2 e_3, \quad \nabla_{e_2} e_2 = -C_{23}^2 e_3, \quad \nabla_{e_3} e_3 = 0.
\end{aligned}$$

Let the almost contact metric structure (φ, ξ, η, g), $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij} e^{ij}$ on HLg_{13}. Then, we have an almost cosymplectic structure if and only if

$$b_3 C_{12}^3 = 0, \quad b_1 \frac{a}{b} C_{23}^2 - b_2 C_{13}^2 = 0, \quad b_1 C_{23}^1 - b_2 C_{23}^2 = 0, \quad b_{12}(-\frac{a}{b}C_{23}^2 + C_{23}^2) = 0.$$

Now, $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a Killing vector field on HLg_{13} if and only if

$$\begin{aligned}
&a_3 \frac{a}{b} C_{23}^2 = 0, \quad a_3 C_{23}^2 = 0, \quad a_3(C_{13}^2 + C_{23}^1) = 0\\
&a_2(C_{12}^3 - C_{23}^1) + a_1 \frac{a}{b} C_{23}^2 = 0, \quad -a_1(C_{12}^3 + C_{13}^2) - a_2 C_{23}^2 = 0,
\end{aligned} \tag{36}$$

Let (φ, ξ, η, g) is an almost contact metric structure on HLg_{13} and $\xi = a_1 e_1 + a_2 e_2 + a_3 e_3$ is a Killing vector field. So, by (1), (2) and (36) we have the following equations

$$C_{12}^3(C_{23}^1 - C_{13}^2 - C_{12}^3) = 4, \quad (C_{23}^1 - C_{13}^2 - C_{12}^3)^2 = 4, \quad C_{23}^1 - C_{13}^2 = \pm 4, \quad C_{12}^3 = \pm 2,$$

Then, (φ, ξ, η, g) is contact metric structure. So, it is a K-contact and a Sasakian structure.

Example 3.40. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under condition $C_{12}^3 = C_{23}^2 = 0$ *is an almost cosymplectic structure.*

In addition, assume that $X = \Sigma x_i e_i$ is a vector on HLg_{13}. So, there is a semi cosymplectic structure if and only if

$$b_3(\frac{a}{b}C_{23}^2 - C_{23}^2) = 0, \quad -C_{23}^2 b_{31} + C_{23}^1 b_{32} = 0, \quad C_{13}^2 b_{31} + \frac{a}{b} C_{23}^2 b_{32} = 0, \quad C_{12}^3 b_{21} = 0.$$

Example 3.41. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under condition $C_{23}^2 = C_{12}^3 = 0$ *is a semi cosymplectic structure.*

By calculating $\nabla\Phi = 0$, we conclude that an almost contact metric structure is a cosymplectic structure if and only if we have the following equations

$$
\begin{aligned}
&(C^2_{13} - C^3_{12} + C^1_{23})b_{12} = 0, \quad (C^2_{13} + C^3_{12} + C^1_{23})b_{21} = 0, \quad \frac{a}{b}C^2_{23}b_{21} = 0,\\
&(-C^2_{13} - C^3_{12} + C^1_{23})b_{23} = 0, \quad (C^2_{13} + C^3_{12} - C^1_{23})b_{13} = 0, \quad C^2_{23}b_{12} = 0,\\
&-\frac{a}{b}C^2_{23}b_{32} - \frac{1}{2}(-C^2_{13} + C^3_{12} - C^1_{23})b_{13} = 0, \quad C^2_{23}b_{13} + \frac{1}{2}(C^2_{13} + C^3_{12} + C^1_{23})b_{32} = 0.
\end{aligned}
$$

Example 3.42. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under conditions $C^1_{23} = C^3_{12}$ *and* $C^2_{23} = C^2_{13} = 0$ *is a cosymplectic structure.*

Example 3.43. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$
\begin{aligned}
&\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0,\\
&C^3_{12} = 2, \quad C^2_{23} = 0, \quad C^2_{13} + C^1_{23} = 0,
\end{aligned}
$$

is a K-contact structure. Also, it is a Sasakian structure.

3.14. **The Hom-Lie algebra** HLg_{14}**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$
\begin{aligned}
\nabla_{e_1} e_2 &= C^1_{12}e_1 + \frac{1}{2}(\frac{C^1_{12}C^3_{13}}{C^1_{13}} + \frac{C^1_{13}C^3_{13}}{C^1_{12}} - C^1_{23})e_3,\\
\nabla_{e_1} e_3 &= C^1_{13}e_1 - \frac{1}{2}(\frac{C^1_{13}C^3_{13}}{C^1_{12}} + \frac{C^1_{12}C^3_{13}}{C^1_{13}} + C^1_{23})e_2,\\
\nabla_{e_3} e_2 &= \frac{1}{2}(-\frac{C^1_{13}C^3_{13}}{C^1_{12}} + \frac{C^1_{12}C^3_{13}}{C^1_{13}} - C^1_{23})e_1,\\
\nabla_{e_2} e_1 &= C^3_{13}e_2 + \frac{1}{2}(-\frac{C^1_{12}C^3_{13}}{C^1_{13}} + \frac{C^1_{13}C^3_{13}}{C^1_{12}} - C^1_{23})e_3,\\
\nabla_{e_3} e_1 &= \frac{1}{2}(\frac{C^1_{13}C^3_{13}}{C^1_{12}} - \frac{C^1_{12}C^3_{13}}{C^1_{13}} + C^1_{23})e_2 - C^3_{13}e_3,\\
\nabla_{e_2} e_3 &= \frac{1}{2}(-\frac{C^1_{13}C^3_{13}}{C^1_{12}} + \frac{C^1_{12}C^3_{13}}{C^1_{13}} + C^1_{23})e_1,\\
\nabla_{e_1} e_1 &= -C^1_{12}e_2 - C^1_{13}e_3, \quad \nabla_{e_2} e_2 = -C^3_{13}e_1, \quad \nabla_{e_3} e_3 = C^3_{13}e_1.
\end{aligned}
$$

Let $\eta = \Sigma b_i e^i$, $\Phi = \Sigma b_{ij} e^{ij}$ and the almost contact metric structure (φ, ξ, η, g) on HLg_{14}. Then, we have an almost cosymplectic structure if and only if

$$
\begin{aligned}
&-b_1C^1_{12} + b_2C^3_{13} - b_3\frac{C^1_{12}C^3_{13}}{C^1_{13}} = 0, \quad -b_1C^1_{13} + b_2\frac{C^1_{13}C^3_{13}}{C^1_{12}} - b_3C^3_{13} = 0,\\
&b_1C^1_{23} = 0, \quad b_{12}C^1_{13} - b_{13}C^1_{12} = 0.
\end{aligned}
$$

Moreover, for any vector $X = \Sigma x_i e_i$ on HLg_{14}, there is a semi cosymplectic structure if and only if

$$
\begin{aligned}
&b_2C^1_{12} + b_3C^1_{13} = 0, \quad -C^1_{13}b_{32} + \frac{C^1_{13}C^3_{13}}{C^1_{12}}b_{13} + C^3_{13}b_{12} = 0,\\
&C^1_{12}b_{23} + C^3_{13}b_{13} = 0, \quad C^1_{23}b_{32} = 0.
\end{aligned}
$$

By calculating $\nabla\Phi = 0$, we obtain

$$\begin{aligned}
&C_{13}^1 b_{32} - \frac{1}{2}(\frac{C_{12}^1 C_{13}^3}{C_{13}^1} + \frac{C_{13}^1 C_{13}^3}{C_{12}^1} - C_{23}^1) b_{13} = 0,\\
&C_{12}^1 b_{23} + \frac{1}{2}(\frac{C_{12}^1 C_{13}^3}{C_{13}^1} + \frac{C_{13}^1 C_{13}^3}{C_{12}^1} + C_{23}^1) b_{12} = 0,\\
&C_{13}^3 b_{13} - \frac{1}{2}(\frac{C_{12}^1 C_{13}^3}{C_{13}^1} - \frac{C_{13}^1 C_{13}^3}{C_{12}^1} + C_{23}^1) b_{21} = 0,\\
&C_{13}^3 b_{21} + \frac{1}{2}(\frac{C_{12}^1 C_{13}^3}{C_{13}^1} - \frac{C_{13}^1 C_{13}^3}{C_{12}^1} - C_{23}^1) b_{13} = 0,\\
&(-\frac{C_{12}^1 C_{13}^3}{C_{13}^1} + \frac{C_{13}^1 C_{13}^3}{C_{12}^1} - C_{23}^1) b_{32} = 0,\\
&(-\frac{C_{12}^1 C_{13}^3}{C_{13}^1} + \frac{C_{13}^1 C_{13}^3}{C_{12}^1} + C_{23}^1) b_{23} = 0,\\
&C_{12}^1 b_{13} + C_{13}^1 b_{21} = 0, \quad C_{13}^3 b_{23} = 0,
\end{aligned} \tag{37}$$

Thus, any almost contact metric structure on HLg_{14} is a cosymplectic structure if and only if it satisfies in (37) .

Let $X = a_1e_1 + a_2e_2 + a_3e_3$ be a vector field on HLg_{14}. Then, it will be a Killing vector field if and only if a_1, a_2, a_3 satisfies in the following equations

$$\begin{aligned}
&a_1 C_{13}^3 = 0, \quad a_2 C_{12}^1 + a_3 C_{13}^1 = 0, \quad a_1 C_{12}^1 + a_2 C_{13}^3 + a_3 \frac{C_{13}^1 C_{13}^3}{C_{12}^1} = 0,\\
&-a_1 C_{13}^1 + a_3 C_{13}^3 + a_2(-C_{23}^1 + \frac{C_{12}^1 C_{13}^3}{C_{13}^1}) = 0, \quad a_1(\frac{C_{13}^1 C_{13}^3}{C_{12}^1} - \frac{C_{12}^1 C_{13}^3}{C_{13}^1}) = 0,
\end{aligned} \tag{38}$$

Let (φ, ξ, η, g) is an almost contact metric structure on HLg_{14} and $\xi = a_1e_1 + a_2e_2 + a_3e_3$ is a Killing vector field. So, by (2) and (38) we have

$$\begin{aligned}
\varphi e_1 &= \big(a_1 C_{12}^1 + \frac{1}{2} a_3(\frac{C_{13}^1 C_{13}^3}{C_{12}^1} + \frac{C_{12}^1 C_{13}^3}{C_{13}^1} + C_{23}^1)\big) e_2\\
&\quad + \big(a_1 C_{13}^1 - \frac{1}{2} a_2(\frac{C_{13}^1 C_{13}^3}{C_{12}^1} + \frac{C_{12}^1 C_{13}^3}{C_{13}^1} - C_{23}^1)\big) e_3,\\
\varphi e_2 &= \big(a_2 C_{13}^3 - \frac{1}{2} a_3(-\frac{C_{13}^1 C_{13}^3}{C_{12}^1} + \frac{C_{12}^1 C_{13}^3}{C_{13}^1} + C_{23}^1)\big) e_1\\
&\quad - \frac{1}{2} a_1(-\frac{C_{12}^1 C_{13}^3}{C_{13}^1} + \frac{C_{13}^1 C_{13}^3}{C_{12}^1} - C_{23}^1) e_3,\\
\varphi e_3 &= \big(-a_3 C_{13}^3 - \frac{1}{2} a_2(-\frac{C_{13}^1 C_{13}^3}{C_{12}^1} + \frac{C_{12}^1 C_{13}^3}{C_{13}^1} - C_{23}^1)\big) e_1\\
&\quad - \frac{1}{2} a_1(-\frac{C_{12}^1 C_{13}^3}{C_{13}^1} + \frac{C_{13}^1 C_{13}^3}{C_{12}^1} + C_{23}^1) e_2.
\end{aligned}$$

Then, if (φ, ξ, η, g) satisfies in $g(e_i, \varphi e_j) = d\eta(e_i, e_j), i, j = 1, 2, 3$, it will be a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

3.15. **The Hom-Lie algebra HLg_{15}.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$
\begin{aligned}
\nabla_{e_1}e_2 &= \frac{a}{b}C_{23}^3e_1+\frac{1}{2}\Big(\frac{C_{12}^2C_{23}^3}{C_{23}^2}-\frac{aC_{23}^2C_{23}^3}{bC_{12}^2}-C_{13}^2\Big)e_3,\\
\nabla_{e_1}e_3 &= \frac{\big((a-b)C_{12}^2-bC_{13}^3\big)C_{23}^2}{bC_{12}^2}e_1+\frac{1}{2}\Big(\frac{aC_{23}^2C_{23}^3}{bC_{12}^2}-\frac{C_{12}^2C_{23}^3}{C_{23}^2}+C_{13}^2\Big)e_2,\\
\nabla_{e_3}e_2 &= \frac{1}{2}\Big(-\frac{aC_{23}^2C_{23}^3}{bC_{12}^2}+\frac{C_{12}^2C_{23}^3}{C_{23}^2}+C_{13}^2\Big)e_1-C_{23}^3e_3,\\
\nabla_{e_2}e_1 &= -C_{12}^2e_2-\frac{1}{2}\Big(\frac{C_{12}^2C_{23}^3}{C_{23}^2}+\frac{aC_{23}^2C_{23}^3}{bC_{12}^2}+C_{13}^2\Big)e_3,\\
\nabla_{e_3}e_1 &= \frac{1}{2}\Big(\frac{aC_{23}^2C_{23}^3}{bC_{12}^2}-\frac{C_{12}^2C_{23}^3}{C_{23}^2}-C_{13}^2\Big)e_2-C_{13}^3e_3,\\
\nabla_{e_2}e_3 &= \frac{1}{2}\Big(\frac{aC_{23}^2C_{23}^3}{bC_{12}^2}+\frac{C_{12}^2C_{23}^3}{C_{23}^2}+C_{13}^2\Big)e_1+C_{23}^2e_2,\\
\nabla_{e_1}e_1 &= -\frac{a}{b}C_{23}^3e_2+\frac{\big((a-b)C_{12}^2-bC_{13}^3\big)C_{23}^2}{bC_{12}^2}e_3,\\
\nabla_{e_2}e_2 &= C_{12}^2e_1-C_{23}^2e_3,\quad \nabla_{e_3}e_3=C_{13}^3e_1+C_{23}^3e_2.
\end{aligned}
$$

Let $\eta=\Sigma b_ie^i$, $\Phi=\Sigma b_{ij}e^{ij}$ and the almost contact metric structure (φ,ξ,η,g). Then, we have an almost cosymplectic structure on HLg_{15} if and only if

$$
\begin{aligned}
& b_{12}\Big(-\frac{\big((a-b)C_{12}^2-bC_{13}^3\big)C_{23}^2}{bC_{12}^2}+C_{23}^2\Big)+b_{13}\Big(-\frac{a}{b}C_{23}^3+C_{23}^3\Big)-b_{23}(C_{12}^2+C_{13}^3)=0,\\
& b_1\frac{a}{b}C_{23}^3+b_2C_{12}^2+b_3\frac{C_{12}^2C_{23}^3}{C_{23}^2}=0,\quad b_1\frac{aC_{23}^2C_{23}^3}{bC_{12}^2}+b_2C_{23}^2+b_3C_{23}^3=0.\\
& b_1\frac{\big((a-b)C_{12}^2-bC_{13}^3\big)C_{23}^2}{bC_{12}^2}-b_2C_{13}^2-b_3C_{13}^3=0.
\end{aligned}
$$

Moreover, let a vector $X=\Sigma x_ie_i$ on HLg_{15}. So, there is a semi cosymplectic structure if and only if

$$
\begin{aligned}
& \frac{aC_{23}^2C_{23}^3}{bC_{12}^2}b_{32}-C_{23}^2b_{31}+C_{23}^3b_{21}=0,\\
& \frac{\big((a-b)C_{12}^2-bC_{13}^3\big)C_{23}^2}{bC_{12}^2}b_{32}+C_{13}^2b_{31}+C_{13}^3b_{12}=0,\\
& -\frac{a}{b}C_{23}^3b_{23}+\frac{C_{12}^2C_{23}^3}{C_{23}^2}b_{21}+C_{12}^2b_{13}=0,\\
& b_1(C_{12}^2+C_{13}^3)+b_2\Big(-\frac{a}{b}C_{23}^3+C_{23}^3\Big)+b_3\Big(\frac{\big((a-b)C_{12}^2-bC_{13}^3\big)C_{23}^2}{bC_{12}^2}-C_{23}^2\Big)=0.
\end{aligned}
$$

By $\nabla\Phi = 0$, we obtain the following equations

$$\begin{aligned}
&C^2_{23}b_{13} + \frac{1}{2}(\frac{C^2_{12}C^3_{23}}{C^2_{23}} + \frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + C^2_{13})b_{32} = 0,\\
&C^2_{12}b_{13} + \frac{1}{2}(\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + \frac{C^2_{12}C^3_{23}}{C^2_{23}} + C^2_{13})b_{21} = 0,\\
&C^3_{23}b_{12} + \frac{1}{2}(-\frac{C^2_{12}C^3_{23}}{C^2_{23}} + \frac{aC^2_{23}C^3_{23}}{bC^2_{12}} - C^2_{13})b_{23} = 0,\\
&C^3_{13}b_{21} + \frac{1}{2}(\frac{C^2_{12}C^3_{23}}{C^2_{23}} - \frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + C^2_{13})b_{13} = 0,\\
&\frac{a}{b}C^3_{23}b_{23} - \frac{1}{2}(-\frac{C^2_{12}C^3_{23}}{C^2_{23}} + \frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + C^2_{13})b_{12} = 0,\\
&-\frac{a}{b}C^3_{23}b_{13} + \frac{\big((a-b)C^2_{12} - bC^3_{13}\big)C^2_{23}}{bC^2_{12}}b_{21} = 0,\\
&C^2_{12}b_{23} - C^2_{23}b_{12} = 0, \quad C^3_{13}b_{32} + C^3_{23}b_{13} = 0,\\
&-\frac{\big((a-b)C^2_{12} - bC^3_{13}\big)C^2_{23}}{bC^2_{12}}b_{32} - \frac{1}{2}(\frac{C^2_{12}C^3_{23}}{C^2_{23}} - \frac{aC^2_{23}C^3_{23}}{bC^2_{12}} - C^2_{13})b_{13} = 0,
\end{aligned} \tag{39}$$

Therefore, any almost contact metric structure on HLg_{15} is a cosymplectic structure if and only if we have (39).

Now, $X = a_1e_1 + a_2e_2 + a_3e_3$ is a Killing vector field on HLg_{15} if and only if we have

$$\begin{aligned}
&a_2\frac{a}{b}C^3_{23} - a_3\frac{\big((a-b)C^2_{12} - bC^3_{13}\big)C^2_{23}}{bC^2_{12}} = 0, \quad -a_1C^2_{12} + a_3C^2_{23} = 0,\\
&-a_2C^2_{23} + a_3C^3_{23} - a_1(\frac{C^2_{12}C^3_{23}}{C^2_{23}} + C^2_{13}) = 0, \quad a_1C^3_{13} + a_2C^3_{23} = 0,\\
&-a_1\frac{a}{b}C^3_{23} + a_2C^2_{12} + a_3(\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + C^2_{13}) = 0,\\
&a_1\frac{\big((a-b)C^2_{12} - bC^3_{13}\big)C^2_{23}}{bC^2_{12}} + a_2(\frac{C^2_{12}C^3_{23}}{C^2_{23}} - \frac{aC^2_{23}C^3_{23}}{bC^2_{12}}) + a_3C^3_{13} = 0.
\end{aligned} \tag{40}$$

Let $\xi = a_1e_1 + a_2e_2 + a_3e_3$ be a Killing vector field on HLg_{15}. By (2) and (40) we have

$$\begin{aligned}
\varphi e_1 &= \big(a_2C^2_{12} + \frac{1}{2}a_3(\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + \frac{C^2_{12}C^3_{23}}{C^2_{23}} + C^2_{13})\big)e_2\\
&\quad +\big(a_3C^3_{13} + \frac{1}{2}a_2(-\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + \frac{C^2_{12}C^3_{23}}{C^2_{23}} + C^2_{13})\big)e_3\\
\varphi e_2 &= \big(-a_2C^2_{12} - \frac{1}{2}a_3(\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + \frac{C^2_{12}C^3_{23}}{C^2_{23}} + C^2_{13})\big)e_1\\
&\quad +\big(a_2C^2_{23} + \frac{1}{2}a_1(\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + \frac{C^2_{12}C^3_{23}}{C^2_{23}} + C^2_{13})\big)e_3\\
\varphi e_3 &= \big(-a_3C^3_{13} - \frac{1}{2}a_2(-\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} + \frac{C^2_{12}C^3_{23}}{C^2_{23}} + C^2_{13})\big)e_1\\
&\quad +\big(-a_3C^3_{23} - \frac{1}{2}a_1(\frac{aC^2_{23}C^3_{23}}{bC^2_{12}} - \frac{C^2_{12}C^3_{23}}{C^2_{23}} - C^2_{13})\big)e_2.
\end{aligned} \tag{41}$$

Then, if the almost contact metric structure (φ, ξ, η, g) satisfies in $g(e_i, \varphi e_j) = d\eta(e_i, e_j), i, j = 1, 2, 3$, it will be a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

3.16. **The Hom-Lie algebra HLg_{16}.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$
\begin{aligned}
\nabla_{e_1}e_2 &= \frac{a}{b}C^3_{23}e_1 + \frac{1}{2}\Big(\frac{C^3_{13}C^3_{23}}{C^1_{13}} - \frac{aC^1_{13}C^3_{23}}{bC^3_{13}} - C^2_{13}\Big)e_3,\\
\nabla_{e_1}e_3 &= C^1_{13}e_1 + \frac{1}{2}\Big(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} - \frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13}\Big)e_2,\\
\nabla_{e_3}e_2 &= \frac{1}{2}\Big(-\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + \frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13}\Big)e_1 - C^3_{23}e_3,\\
\nabla_{e_2}e_1 &= -\frac{1}{2}\Big(\frac{C^3_{13}C^3_{23}}{C^1_{13}} + \frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + C^2_{13}\Big)e_3,\\
\nabla_{e_3}e_1 &= \frac{1}{2}\Big(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} - \frac{C^3_{13}C^3_{23}}{C^1_{13}} - C^2_{13}\Big)e_2 - C^3_{13}e_3,\\
\nabla_{e_2}e_3 &= \frac{1}{2}\Big(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + \frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13}\Big)e_1,\\
\nabla_{e_1}e_1 &= -\frac{a}{b}C^3_{23}e_2 - C^1_{13}e_3, \quad \nabla_{e_2}e_2 = 0,\\
\nabla_{e_3}e_3 &= C^3_{13}e_1 + C^3_{23}e_2.
\end{aligned}
$$

Let the almost contact metric structure (φ, ξ, η, g), $\eta = \Sigma b_i e^i$ and $\Phi = \Sigma b_{ij}e^{ij}$ on HLg_{15}. Then, there is an almost cosymplectic structure if and only if

$$
\begin{aligned}
&b_1\frac{a}{b}C^3_{23} + b_3\frac{C^3_{13}C^3_{23}}{C^1_{13}} = 0, \quad b_1\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + b_3C^3_{23} = 0, \quad b_1C^1_{13} + b_2C^2_{13} + b_3C^3_{13} = 0,\\
&b_{12}C^1_{13} + b_{13}\Big(-\frac{a}{b}C^3_{23} + C^3_{23}\Big) - b_{23}C^3_{13} = 0.
\end{aligned}
$$

In addition, let $X = \Sigma x_i e_i$ is a vector on HLg_{16}. Then, we have a semi cosymplectic structure if and only if

$$
\begin{aligned}
&\frac{aC^1_{13}C^3_{23}}{bC^3_{13}}b_{32} + C^3_{23}b_{21} = 0, \quad -C^1_{13}b_{32} + C^2_{13}b_{31} + C^1_{13}b_{12} = 0,\\
&-\frac{a}{b}C^3_{23}b_{23} + \frac{C^3_{13}C^3_{23}}{C^1_{13}}b_{21} = 0, \quad b_1C^1_{13} + b_2\Big(-\frac{a}{b}C^3_{23} + C^3_{23}\Big) - b_3C^1_{13} = 0.
\end{aligned}
$$

Also, let $X = a_1e_1 + a_2e_2 + a_3e_3$ be a vector field on HLg_{16}. Then, it is a Killing vector field if and only if

$$
\begin{aligned}
&a_2\frac{a}{b}C^3_{23} + a_3C^1_{13} = 0, \quad a_1C^3_{13} + a_2C^3_{23} = 0, \quad -a_1\Big(\frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13}\Big) + a_3C^3_{23} = 0,\\
&-a_1C^1_{13} + a_3C^3_{13} + a_2\Big(\frac{C^3_{13}C^3_{23}}{C^1_{13}} - \frac{aC^1_{13}C^3_{23}}{bC^3_{13}}\Big) = 0, \quad -a_1\frac{a}{b}C^3_{23} + a_3\Big(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + C^2_{13}\Big) = 0.
\end{aligned}
\tag{42}
$$

Suppose that there is an almost contact metric structure (φ, ξ, η, g) on HLg_{16} while $\xi = a_1e_1+a_2e_2+a_3e_3$ is a Killing vector field. By (2) and (42), we have

$$\begin{aligned}
\varphi e_1 &= \big(a_1\frac{a}{b}C^3_{23} - \frac{1}{2}a_3(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} - \frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13})\big)e_2 \\
&\quad +\big(a_1C^1_{13} - \frac{1}{2}a_2(-\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + \frac{C^3_{13}C^3_{23}}{C^1_{13}} - C^2_{13})\big)e_3, \\
\varphi e_2 &= -\frac{1}{2}a_3(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + \frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13})e_1 \\
&\quad +\frac{1}{2}a_1(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + \frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13})e_3, \\
\varphi e_3 &= \big(-a_3C^3_{13} - \frac{1}{2}a_2(-\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + \frac{C^3_{13}C^3_{23}}{C^1_{13}} + C^2_{13})\big)e_1 \\
&\quad +\big(-a_3C^3_{23} - \frac{1}{2}a_1(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} - \frac{C^3_{13}C^3_{23}}{C^1_{13}} - C^2_{13})\big)e_2.
\end{aligned}$$

Then, if (φ, ξ, η, g) satisfies in $g(e_i, \varphi e_j) = d\eta(e_i, e_j), i, j = 1, 2, 3$, it will be a contact metric structure. So, it is a K-contact and a Sasakian structure.

Example 3.44. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under condition $C^2_{13} = C^3_{23} = 0$ is a semi cosymplectic structure. It is also an almost cosymplectic structure.

By calculating $\nabla\Phi = 0$, we obtain

$$\begin{aligned}
&C^1_{13}b_{32} - \frac{1}{2}(\frac{C^3_{13}C^3_{23}}{C^1_{13}} - \frac{aC^1_{13}C^3_{23}}{bC^3_{13}} - C^2_{13})b_{13} = 0, \quad C^1_{13}b_{21} + \frac{a}{b}C^3_{23}b_{13} = 0, \\
&C^3_{23}b_{12} + \frac{1}{2}(\frac{aC^1_{13}C^3_{23}}{bC^3_{13}} - \frac{C^3_{13}C^3_{23}}{C^1_{13}} - C^2_{13})b_{23} = 0, \quad C^3_{13}b_{32} + C^3_{23}b_{13} = 0, \\
&C^3_{13}b_{21} + \frac{1}{2}(\frac{C^3_{13}C^3_{23}}{C^1_{13}} - \frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + C^2_{13})b_{13} = 0, \\
&\frac{a}{b}C^3_{23}b_{23} - \frac{1}{2}(-\frac{C^3_{13}C^3_{23}}{C^1_{13}} + \frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + C^2_{13})b_{12} = 0,
\end{aligned}$$

$$(\frac{C^3_{13}C^3_{23}}{C^1_{13}} + \frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + C^2_{13})b_{32} = 0, \quad (\frac{C^3_{13}C^3_{23}}{C^1_{13}} + \frac{aC^1_{13}C^3_{23}}{bC^3_{13}} + C^2_{13})b_{21} = 0, \tag{43}$$

Thus, any almost contact metric structure which satisfies in (43) is a cosymplectic structure and vice versa.

Example 3.45. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = -e_3, \quad \varphi e_3 = e_1, \quad \Phi = e^{13},$$

under condition $C^3_{23} = C^2_{13} = 0$ is a cosymplectic structure.

Example 3.46. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_2, \quad \eta = e^2, \quad \varphi e_1 = e_3, \quad \varphi e_3 = -e_1, \quad C^3_{23} = 0, \quad C^2_{13} = 2,$$

is a contact metric structure. So, it will be a K-contact and a Sasakian structure.

3.17. **The Hom-Lie algebra HLg_{17}.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
\nabla_{e_1}e_2 &= -\frac{C_{13}^1C_{23}^3}{C_{23}^2}e_1-\frac{1}{2}(\frac{C_{13}^3C_{23}^3}{C_{23}^2}+\frac{C_{13}^3C_{23}^2}{C_{23}^3}+C_{23}^1)e_3,\\
\nabla_{e_1}e_3 &= C_{13}^1e_1+\frac{1}{2}(\frac{C_{13}^3C_{23}^3}{C_{23}^2}+\frac{C_{13}^3C_{23}^2}{C_{23}^3}+C_{23}^1)e_2,\\
\nabla_{e_3}e_2 &= \frac{1}{2}(\frac{C_{13}^3C_{23}^2}{C_{23}^3}-\frac{C_{13}^3C_{23}^3}{C_{23}^2}-C_{23}^1)e_1-C_{23}^3e_3,\\
\nabla_{e_2}e_1 &= C_{13}^3e_2+\frac{1}{2}(\frac{C_{13}^3C_{23}^3}{C_{23}^2}-\frac{C_{13}^3C_{23}^2}{C_{23}^3}-C_{23}^1)e_3,
\end{aligned}$$

$$\begin{aligned}
\nabla_{e_3}e_1 &= \frac{1}{2}(-\frac{C_{13}^3C_{23}^2}{C_{23}^3}+\frac{C_{13}^3C_{23}^3}{C_{13}^2}+C_{23}^1)e_2-C_{13}^3e_3,\\
\nabla_{e_2}e_3 &= \frac{1}{2}(\frac{C_{13}^3C_{23}^2}{C_{23}^3}-\frac{C_{13}^3C_{23}^3}{C_{23}^2}+C_{23}^1)e_1+C_{23}^2e_2,\\
\nabla_{e_1}e_1 &= \frac{C_{13}^1C_{23}^3}{C_{23}^2}e_2-C_{13}^1e_3, \quad \nabla_{e_2}e_2=-C_{13}^3e_1-C_{23}^2e_3,\\
\nabla_{e_3}e_3 &= C_{13}^3e_1+C_{23}^3e_2.
\end{aligned}$$

Let the almost contact metric structure (φ,ξ,η,g), $\eta=\Sigma b_ie^i$ and $\Phi=\Sigma b_{ij}e^{ij}$ on HLg_{17}. Then, we have an almost cosymplectic structure if and only if

$$\begin{aligned}
&b_1\frac{C_{13}^1C_{23}^3}{C_{23}^2}+b_2C_{13}^3+b_3\frac{C_{13}^3C_{23}^3}{C_{23}^2}=0, \quad b_1C_{13}^1+b_2\frac{C_{13}^3C_{23}^2}{C_{23}^3}+b_3C_{13}^3=0,\\
&b_1C_{23}^1+b_2C_{23}^2+b_3C_{23}^3=0, \quad b_{12}(C_{13}^1+C_{23}^2)+b_{13}(\frac{C_{13}^1C_{23}^3}{C_{23}^2}+C_{23}^3)=0.
\end{aligned}$$

Also, for any vector $X=\Sigma x_ie_i$ on HLg_{17}, there is a semi cosymplectic structure if and only if

$$\begin{aligned}
&-C_{23}^2b_{31}+C_{23}^1b_{32}+C_{23}^3b_{21}=0, \quad -C_{13}^1b_{32}+\frac{C_{13}^3C_{23}^2}{C_{23}^3}b_{31}+C_{13}^3b_{12}=0,\\
&\frac{C_{13}^1C_{23}^3}{C_{23}^2}b_{23}+\frac{C_{13}^3C_{23}^3}{C_{23}^2}b_{12}-C_{13}^3b_{13}=0, \quad b_2(\frac{C_{13}^1C_{23}^3}{C_{23}^2}+C_{23}^3)-b_3(C_{13}^1+C_{23}^2)=0.
\end{aligned}$$

We Know $X=a_1e_1+a_2e_2+a_3e_3$ is a Killing vector field on HLg_{17} if and only if we have the following equations

$$\begin{aligned}
&-a_2\frac{C_{13}^1C_{23}^3}{C_{23}^2}+a_3C_{13}^1=0, \quad a_1C_{13}^3+a_2C_{23}^3=0, \quad a_3C_{23}^2+a_1C_{13}^3=0,\\
&-a_2C_{23}^2+a_3C_{23}^3+a_1(\frac{C_{13}^3C_{23}^3}{C_{23}^2}-\frac{C_{13}^3C_{23}^2}{C_{23}^3})=0,\\
&-a_1C_{13}^1+a_3C_{13}^3-a_2(C_{23}^1+\frac{C_{13}^3C_{23}^3}{C_{23}^2})=0,\\
&a_1\frac{C_{13}^1C_{23}^3}{C_{23}^2}-a_2C_{13}^3+a_3(C_{23}^1+\frac{C_{13}^3C_{23}^2}{C_{23}^3})=0. \qquad (44)
\end{aligned}$$

Suppose that $\xi = a_1e_1 + a_2e_2 + a_3e_3$ is a Killing vector field on HLg_{17}. By applying (2) and (44) we have

$$\begin{aligned}
\varphi e_1 &= \big(- a_1 \frac{C^1_{13}C^3_{23}}{C^2_{23}} - \frac{1}{2}a_3(\frac{C^3_{13}C^2_{23}}{C^3_{23}} + \frac{C^3_{13}C^3_{23}}{C^2_{23}} + C^1_{23})\big)e_2 \\
&\quad +\big(a_1C^1_{13} + \frac{1}{2}a_2(\frac{C^3_{13}C^2_{23}}{C^3_{23}} + \frac{C^3_{13}C^3_{23}}{C^2_{23}} + C^1_{23})e_3, \\
\varphi e_2 &= \big(a_2C^3_{13} - \frac{1}{2}a_3(\frac{C^3_{13}C^2_{23}}{C^3_{23}} - \frac{C^3_{13}C^3_{23}}{C^2_{23}} + C^1_{23})\big)e_1 \\
&\quad +\big(a_2C^2_{23} - \frac{1}{2}a_1(-\frac{C^3_{13}C^2_{23}}{C^3_{23}} + \frac{C^3_{13}C^3_{23}}{C^2_{23}} - C^1_{23})e_3, \\
\varphi e_3 &= \big(- a_3C^3_{13} - \frac{1}{2}a_2(\frac{C^3_{13}C^2_{23}}{C^3_{23}} - \frac{C^3_{13}C^3_{23}}{C^2_{23}} - C^1_{23})\big)e_1 \\
&\quad +\big(- a_3C^3_{23} - \frac{1}{2}a_1(-\frac{C^3_{13}C^2_{23}}{C^3_{23}} + \frac{C^3_{13}C^3_{23}}{C^2_{23}} + C^1_{23})e_2.
\end{aligned}$$

Then, the almost contact metric structure (φ, ξ, η, g) which satisfies in $g(e_i, \varphi e_j) = d\eta(e_i, e_j)$, $i, j = 1, 2, 3$, is contact metric structure. So, it is a K-contact and a Sasakian structure.

Example 3.47. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2, \quad \Phi = e^{23},$$

under condition $C^1_{23} = C^1_{13} = 0$ is a semi cosymplectic and an almost cosymplectic structure.

By $\nabla\Phi = 0$, we conclude that an almost contact metric structure is a cosymplectic structure if and only if we have

$$\begin{aligned}
&C^1_{13}b_{32} + \frac{1}{2}(\frac{C^3_{13}C^3_{23}}{C^2_{23}} + \frac{C^3_{13}C^2_{23}}{C^3_{23}} + C^1_{23})b_{13} = 0, \\
&\frac{C^1_{13}C^3_{23}}{C^2_{23}}b_{13} - C^1_{13}b_{21} = 0, \quad C^3_{13}b_{32} + C^3_{23}b_{13} = 0, \\
&C^3_{23}b_{12} + \frac{1}{2}(\frac{C^3_{13}C^3_{23}}{C^2_{23}} - \frac{C^3_{13}C^2_{23}}{C^3_{23}} + C^1_{23})b_{23} = 0, \\
&C^3_{13}b_{21} + \frac{1}{2}(\frac{C^3_{13}C^2_{23}}{C^3_{23}} - \frac{C^3_{13}C^3_{23}}{C^2_{23}} - C^1_{23})b_{13} = 0, \\
&C^3_{13}b_{23} + C^2_{23}b_{12} = 0, \\
&C^2_{23}b_{13} - \frac{1}{2}(\frac{C^3_{13}C^3_{23}}{C^2_{23}} - \frac{C^3_{13}C^2_{23}}{C^3_{23}} - C^1_{23})b_{32} = 0, \\
&\frac{C^1_{13}C^3_{23}}{C^2_{23}}b_{23} + \frac{1}{2}(\frac{C^3_{13}C^2_{23}}{C^3_{23}} + \frac{C^3_{13}C^3_{23}}{C^2_{23}} + C^1_{23})b_{12} = 0, \\
&C^3_{13}b_{13} - \frac{1}{2}(-\frac{C^3_{13}C^3_{23}}{C^2_{23}} + \frac{C^3_{13}C^2_{23}}{C^3_{23}} + C^1_{23})b_{21} = 0.
\end{aligned}$$

Example 3.48. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = e_2, \quad \Phi = e^{23},$$

under condition $C^3_{13} = C^1_{13} = C^1_{23} = 0$ is a cosymplectic structure.

Example 3.49. *The almost contact metric structure* (φ, ξ, η, g) *where*

$$\xi = e_1, \quad \eta = e^1, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2, \quad C^1_{13} = C^3_{13} = 0, \quad C^1_{23} = 2,$$

is an example of a K*-contact and a Sasakian structure.*

3.18. **The Hom-Lie algebra** HLg_{18}**.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$\begin{aligned}
&\nabla_{e_1} e_2 = C^1_{12} e_1 + \frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^1_{12}} - \frac{aC^1_{12}C^2_{23}}{bC^2_{12}} + C^3_{12})e_3,\\
&\nabla_{e_1} e_3 = -\frac{a}{b}C^2_{23} e_1 + \frac{1}{2}(\frac{aC^1_{12}C^2_{23}}{bC^2_{12}} - \frac{C^2_{12}C^2_{23}}{C^1_{12}} - C^3_{12})e_2,\\
&\nabla_{e_3} e_2 = \frac{1}{2}(-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}} - \frac{C^2_{12}C^2_{23}}{C^1_{12}} + C^3_{12})e_1,\\
&\nabla_{e_2} e_1 = -C^2_{12} e_2 + \frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^1_{12}} - \frac{aC^1_{12}C^2_{23}}{bC^2_{12}} - C^3_{12})e_3,\\
&\nabla_{e_3} e_1 = \frac{1}{2}(\frac{aC^1_{12}C^2_{23}}{bC^2_{12}} + \frac{C^2_{12}C^2_{23}}{C^1_{12}} - C^3_{12})e_2,\\
&\nabla_{e_2} e_3 = \frac{1}{2}(\frac{aC^1_{12}C^2_{23}}{bC^2_{12}} - \frac{C^2_{12}C^2_{23}}{C^1_{12}} + C^3_{12})e_1 + C^2_{23}e_2,\\
&\nabla_{e_1} e_1 = -C^1_{12} e_2 + \frac{a}{b}C^2_{23} e_3, \quad \nabla_{e_2} e_2 = C^2_{12} e_1 - C^2_{23} e_3,\\
&\nabla_{e_3} e_3 = 0.
\end{aligned}$$

Let $\eta = \Sigma b_i e^i$, $\Phi = \Sigma b_{ij} e^{ij}$ and the almost contact metric structure (φ, ξ, η, g). Then, we have an almost cosymplectic structure on HLg_{18} if and only if

$$\begin{aligned}
&b_1 C^1_{12} + b_2 C^2_{12} + b_3 C^3_{12} = 0, \quad b_1 \frac{a}{b} C^2_{23} + b_2 \frac{C^1_{12}C^2_{23}}{C^1_{12}} = 0, \quad b_1 \frac{aC^1_{12}C^2_{23}}{bC^2_{12}} + b_2 C^2_{23} = 0,\\
&b_{12}(-\frac{a}{b}C^2_{23} + C^2_{23}) - b_{13}C^1_{12} - b_{23}C^2_{12} = 0.
\end{aligned} \tag{45}$$

Moreover, let a vector $X = \Sigma x_i e_i$ on HLg_{18}. So, there is a semi cosymplectic structure on if and only if

$$\begin{aligned}
&\frac{aC^1_{12}C^2_{23}}{bC^2_{12}} b_{32} - C^2_{23} b_{31} = 0, \quad \frac{a}{b} C^2_{23} b_{32} + \frac{C^2_{12}C^2_{23}}{C^1_{12}} b_{13} = 0,\\
&-C^1_{12} b_{23} + C^3_{12} b_{21} + C^2_{12} b_{13} = 0, \quad b_1 C^2_{12} - b_2 C^1_{12} + b_3(\frac{a}{b}C^2_{23} - C^2_{23}) = 0.
\end{aligned} \tag{46}$$

Let $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ be a vector field on HLg_{18}. Then, it is a Killing vector field if and only if

$$\begin{aligned}
&-a_3 \frac{a}{b} C^2_{23} + a_2 C^1_{12} = 0, \quad -a_1 C^2_{12} + a_3 C^2_{23} = 0, \quad a_1(\frac{C^2_{12}C^2_{23}}{C^1_{12}} - C^3_{12}) - a_2 C^2_{23} = 0,\\
&-a_1 C^1_{12} + a_2 C^2_{12} + a_3(-\frac{C^2_{12}C^2_{23}}{C^1_{12}} + \frac{aC^1_{12}C^2_{23}}{bC^2_{12}}) = 0,\\
&a_1 \frac{a}{b} C^2_{23} + a_2(-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}} + C^3_{12}) = 0.
\end{aligned} \tag{47}$$

Let an almost contact metric structure (φ,ξ,η,g) on HLg_{18} while $\xi=a_1e_1+a_2e_2+a_3e_3$ is a Killing vector field. By using (2) and (47) we have

$$\begin{aligned}
\varphi e_1 &= \big(a_1C^1_{12}-\frac{1}{2}a_3(-\frac{C^2_{12}C^2_{23}}{C^1_{12}}+\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}-C^3_{12})\big)e_2\\
&\quad -\big(a_1\frac{a}{b}C^2_{23}+\frac{1}{2}a_2(\frac{C^2_{12}C^2_{23}}{C^1_{12}}-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}+C^3_{12})\big)e_3,\\
\varphi e_2 &= \big(-a_2C^2_{12}-\frac{1}{2}a_3(-\frac{C^2_{12}C^2_{23}}{C^1_{12}}+\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}+C^3_{12})\big)e_1\\
&\quad +\big(a_2C^2_{23}-\frac{1}{2}a_1(\frac{C^2_{12}C^2_{23}}{C^1_{12}}-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}-C^3_{12})\big)e_3,\\
\varphi e_3 &= -\frac{1}{2}a_2(-\frac{C^2_{12}C^2_{23}}{C^1_{12}}-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}+C^3_{12})e_1\\
&\quad -\frac{1}{2}a_1(\frac{C^2_{12}C^2_{23}}{C^1_{12}}+\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}-C^3_{12})e_2.
\end{aligned}$$

Then, if (φ,ξ,η,g) satisfies in $g(e_i,\varphi e_j)=d\eta(e_i,e_j), i,j=1,2,3$, it will be a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

Example 3.50. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_3,\quad \eta=e^3,\quad \varphi e_1=-e_2,\quad \varphi e_2=e_1,\quad \Phi=e^{12},\quad C^2_{23}=C^3_{12}=0,$$

satisfies in (45) and (46). So, it is a semi cosymplectic and an cosymplectic structure.

By calculating $\nabla\Phi=0$, we obtain

$$\begin{aligned}
&C^1_{12}b_{23}-\frac{1}{2}(-\frac{C^2_{12}C^2_{23}}{C^1_{12}}+\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}-C^3_{12})b_{12}=0,\\
&-C^1_{12}b_{13}+\frac{a}{b}C^2_{23}b_{21}=0,\quad C^2_{12}b_{23}-C^2_{23}b_{12}=0,\\
&C^2_{23}b_{13}-\frac{1}{2}(-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}+\frac{C^2_{12}C^2_{23}}{C^1_{12}}-C^3_{12})b_{32}=0,\\
&C^2_{12}b_{13}+\frac{1}{2}(-\frac{C^2_{12}C^2_{23}}{C^1_{12}}+\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}+C^3_{12})b_{21}=0,\\
&-\frac{a}{b}C^2_{23}b_{32}-\frac{1}{2}(\frac{C^2_{12}C^2_{23}}{C^1_{12}}-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}+C^3_{12})b_{13}=0,
\end{aligned}$$

$$(\frac{C^2_{12}C^2_{23}}{C^1_{12}}+\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}-C^3_{12})b_{23}=0,\quad (-\frac{C^2_{12}C^2_{23}}{C^1_{12}}-\frac{aC^1_{12}C^2_{23}}{bC^2_{12}}+C^3_{12})b_{13}=0, \tag{48}$$

Thus, any almost contact metric structure which satisfies in (48) is a cosymplectic structure and vice versa.

Example 3.51. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_3,\quad \eta=e^3,\quad \varphi e_1=-e_2,\quad \varphi e_2=e_1,\quad \Phi=e^{12},$$

under condition $C^3_{12}=C^2_{23}=0$ is a cosymplectic structure.

Example 3.52. *The almost contact metric structure (φ,ξ,η,g) where*

$$\xi=e_3,\quad \eta=e^3,\quad \varphi e_1=e_2,\quad \varphi e_2=-e_1,\quad C^2_{23}=0,\quad C^3_{12}=2,$$

is a contact metric structure. So, it is a K-contact and a Sasakian structure.

3.19. **The Hom-Lie algebra HLg_{19}.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$
\begin{aligned}
\nabla_{e_1}e_2 &= -\frac{\big(bC_{12}^2+(b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}e_1+\frac{1}{2}(-\frac{C_{13}^3C_{23}^2}{C_{23}^3}+\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}+C_{12}^3)e_3,\\
\nabla_{e_1}e_3 &= -\frac{a}{b}C_{23}^2e_1+\frac{1}{2}(-\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}+\frac{C_{13}^3C_{23}^2}{C_{23}^3}-C_{12}^3)e_2,\\
\nabla_{e_3}e_2 &= \frac{1}{2}(\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}+\frac{C_{13}^3C_{23}^2}{C_{23}^3}+C_{12}^3)e_1-C_{23}^3e_3,\\
\nabla_{e_2}e_1 &= -C_{12}^2e_2-\frac{1}{2}(\frac{C_{13}^3C_{23}^2}{C_{23}^3}-\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}+C_{12}^3)e_3,\\
\nabla_{e_3}e_1 &= \frac{1}{2}(-\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}-\frac{C_{13}^3C_{23}^2}{C_{23}^3}-C_{12}^3)e_2-C_{13}^3e_3,\\
\nabla_{e_2}e_3 &= \frac{1}{2}(-\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}+\frac{C_{13}^3C_{23}^2}{C_{23}^3}+C_{12}^3)e_1+C_{23}^2e_2,\\
\nabla_{e_1}e_1 &= \frac{\big(bC_{12}^2+(b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}e_2+\frac{a}{b}C_{23}^2e_3,\\
\nabla_{e_2}e_2 &= C_{12}^2e_1-C_{23}^2e_3,\quad \nabla_{e_3}e_3=C_{13}^3e_1+C_{23}^3e_2.
\end{aligned}
$$

Let the almost contact metric structure (φ,ξ,η,g), $\eta=\Sigma b_ie^i$ and $\Phi=\Sigma b_{ij}e^{ij}$ on HLg_{19}. Then, we have an almost cosymplectic structure if and only if

$$
\begin{aligned}
&b_1\frac{\big(bC_{12}^2+(b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}-b_2C_{12}^2-b_3C_{12}^3=0,\\
&b_1\frac{a}{b}C_{23}^2-b_2\frac{C_{13}^3C_{23}^2}{C_{23}^3}-b_3C_{13}^3=0,\quad b_1\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}-b_2C_{23}^2-b_3C_{23}^3=0,\\
&b_{12}(-\frac{a}{b}C_{23}^2+C_{23}^2)+b_{13}(\frac{\big(bC_{12}^2+(b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}+C_{23}^3)-b_{23}(C_{12}^2+C_{13}^3)=0.
\end{aligned}
$$

In addition, for any vector $X=\Sigma x_ie_i$ on HLg_{19}, there is a semi cosymplectic structure if and only if we have

$$
\begin{aligned}
&-\frac{aC_{23}^2C_{23}^3}{bC_{13}^3}b_{32}-C_{23}^2b_{31}+C_{23}^3b_{21}=0,\quad \frac{a}{b}C_{23}^2b_{32}+\frac{C_{13}^3C_{23}^2}{C_{23}^3}b_{31}+C_{13}^3b_{12}=0,\\
&\frac{\big(bC_{12}^2+(b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}b_{23}+C_{12}^3b_{21}+C_{12}^2b_{13}=0,\\
&b_1(C_{12}^2+C_{13}^3)+b_2(\frac{\big(bC_{12}^2+(b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}+C_{23}^3)+b_3(\frac{a}{b}C_{23}^2-C_{23}^2)=0.
\end{aligned}
$$

By calculating $\nabla\Phi = 0$, we deduce that an almost contact metric structure is a cosymplectic structure if and only if we have

$$
\begin{aligned}
&C_{23}^2 b_{13} + \frac{1}{2}\Big(\frac{C_{13}^3 C_{23}^2}{C_{23}^3} - \frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + C_{12}^3\Big)b_{32} = 0,\\
&C_{13}^3 b_{21} + \frac{1}{2}\Big(\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} + C_{12}^3\Big)b_{13} = 0,\\
&C_{12}^2 b_{13} + \frac{1}{2}\Big(\frac{C_{13}^3 C_{23}^2}{C_{23}^3} - \frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + C_{12}^3\Big)b_{21} = 0,\\
&C_{23}^3 b_{12} + \frac{1}{2}\Big(-\frac{C_{13}^3 C_{23}^2}{C_{23}^3} - \frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} - C_{12}^3\Big)b_{23} = 0,\\
&\frac{a}{b}C_{23}^2 b_{32} + \frac{1}{2}\Big(-\frac{C_{13}^3 C_{23}^2}{C_{23}^3} + \frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + C_{12}^3\Big)b_{13} = 0,\\
&\frac{a}{b}C_{23}^2 b_{21} + \frac{\big(bC_{12}^2 + (b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}b_{13} = 0,\\
&C_{12}^2 b_{23} - C_{23}^2 b_{12} = 0, \quad C_{13}^3 b_{32} + C_{23}^3 b_{13} = 0,\\
&\frac{\big(bC_{12}^2 + (b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3}b_{23} + \frac{1}{2}\Big(\frac{C_{13}^3 C_{23}^2}{C_{23}^3} - \frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} - C_{12}^3\Big)b_{12} = 0.
\end{aligned}
$$

One could verify that $X = a_1e_1 + a_2e_2 + a_3e_3$ is a Killing vector field on HLg_{19} if and only if we have

$$
\begin{aligned}
&-a_2\frac{\big(bC_{12}^2 + (b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3} - a_3\frac{a}{b}C_{23}^2 = 0, \quad -a_1C_{12}^2 + a_3C_{23}^2 = 0,\\
&-a_2C_{23}^2 + a_3C_{23}^3 - a_1\Big(\frac{C_{13}^3 C_{23}^2}{C_{23}^3} + C_{12}^3\Big) = 0, \quad a_1C_{13}^3 + a_2C_{23}^3 = 0,\\
&a_1\frac{a}{b}C_{23}^2 + a_3C_{13}^3 + a_2\Big(\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + C_{12}^3\Big) = 0,\\
&a_1\frac{\big(bC_{12}^2 + (b-a)C_{13}^3\big)C_{23}^3}{bC_{13}^3} + a_3\Big(\frac{C_{13}^3 C_{23}^2}{C_{23}^3} - \frac{aC_{23}^2 C_{23}^3}{bC_{13}^3}\Big) + a_2C_{12}^2 = 0.
\end{aligned}
\tag{49}
$$

Let ξ be a Killing vector field on HLg_{19}. By (2) and (49) we have

$$
\begin{aligned}
\varphi e_1 &= \Big(a_2C_{12}^2 - \frac{1}{2}a_3\Big(\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} - \frac{C_{13}^3 C_{23}^2}{C_{23}^3} - C_{12}^3\Big)\Big)e_2\\
&\quad + \Big(a_3C_{13}^3 + \frac{1}{2}a_2\Big(\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} + C_{12}^3\Big)\Big)e_3,\\
\varphi e_2 &= \Big(-a_2C_{12}^2 - \frac{1}{2}a_3\Big(-\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} + C_{12}^3\Big)\Big)e_1\\
&\quad + \Big(a_2C_{23}^2 + \frac{1}{2}a_1\Big(-\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} + C_{12}^3\Big)\Big)e_3,\\
\varphi e_3 &= -\Big(a_3C_{13}^3 + \frac{1}{2}a_2\Big(\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} + C_{12}^3\Big)\Big)e_1\\
&\quad - \Big(a_3C_{23}^3 - \frac{1}{2}a_1\Big(\frac{aC_{23}^2 C_{23}^3}{bC_{13}^3} + \frac{C_{13}^3 C_{23}^2}{C_{23}^3} + C_{12}^3\Big)\Big)e_2.
\end{aligned}
$$

Then, the almost contact metric structure (φ, ξ, η, g) which satisfies in $g(e_i, \varphi e_j) = d\eta(e_i, e_j)$, $i, j = 1, 2, 3$ is a contact metric structure. So, it is a K-contact and a Sasakian structure.

3.20. **The Hom-Lie algebra HLg_{20}.** By using Koszul's formula, the covariant derivatives of the basis elements are as follows:

$$
\begin{aligned}
\nabla_{e_1}e_2 &= C_{12}^1e_1+\frac{1}{2}\Big(\frac{-aC_{13}^3C_{23}^3+b(C_{12}^1C_{13}^3+(C_{12}^2+C_{13}^3)C_{23}^3)}{bC_{13}^1+aC_{23}^2}\\
&\quad -\frac{bC_{12}^2C_{13}^1+(a-b)C_{12}^2C_{23}^2-bC_{13}^3C_{23}^2}{bC_{12}^1-aC_{23}^3}-\frac{a(C_{12}^1C_{23}^2+C_{13}^1C_{23}^3)}{b(C_{12}^2+C_{13}^3)}\Big)e_3,\\
\nabla_{e_1}e_3 &= C_{13}^1e_1+\frac{1}{2}\Big(-\frac{-aC_{13}^3C_{23}^3+b(C_{12}^1C_{13}^3+(C_{12}^2+C_{13}^3)C_{23}^3)}{bC_{13}^1+aC_{23}^2}\\
&\quad +\frac{bC_{12}^2C_{13}^1+(a-b)C_{12}^2C_{23}^2-bC_{13}^3C_{23}^2}{bC_{12}^1-aC_{23}^3}+\frac{a(C_{12}^1C_{23}^2+C_{13}^1C_{23}^3)}{b(C_{12}^2+C_{13}^3)}\Big)e_2,\\
\nabla_{e_3}e_2 &= \frac{1}{2}\Big(\frac{-aC_{13}^3C_{23}^3+b(C_{12}^1C_{13}^3+(C_{12}^2+C_{13}^3)C_{23}^3)}{bC_{13}^1+aC_{23}^2}\\
&\quad +\frac{bC_{12}^2C_{13}^1+(a-b)C_{12}^2C_{23}^2-bC_{13}^3C_{23}^2}{bC_{12}^1-aC_{23}^3}-\frac{a(C_{12}^1C_{23}^2+C_{13}^1C_{23}^3)}{b(C_{12}^2+C_{13}^3)}\Big)e_1-C_{23}^3e_3,\\
\nabla_{e_2}e_1 &= -C_{12}^2e_2+\frac{1}{2}\Big(-\frac{-aC_{13}^3C_{23}^3+b(C_{12}^1C_{13}^3+(C_{12}^2+C_{13}^3)C_{23}^3)}{bC_{13}^1+aC_{23}^2}\\
&\quad -\frac{bC_{12}^2C_{13}^1+(a-b)C_{12}^2C_{23}^2-bC_{13}^3C_{23}^2}{bC_{12}^1-aC_{23}^3}-\frac{a(C_{12}^1C_{23}^2+C_{13}^1C_{23}^3)}{b(C_{12}^2+C_{13}^3)}\Big)e_3,\\
\nabla_{e_3}e_1 &= \frac{1}{2}\Big(-\frac{-aC_{13}^3C_{23}^3+b(C_{12}^1C_{13}^3+(C_{12}^2+C_{13}^3)C_{23}^3)}{bC_{13}^1+aC_{23}^2}\\
&\quad +\frac{bC_{12}^2C_{13}^1+(a-b)C_{12}^2C_{23}^2-bC_{13}^3C_{23}^2}{bC_{12}^1-aC_{23}^3}+\frac{a(C_{12}^1C_{23}^2+C_{13}^1C_{23}^3)}{b(C_{12}^2+C_{13}^3)}\Big)e_2-C_{13}^3e_1,\\
\nabla_{e_2}e_3 &= \frac{1}{2}\Big(\frac{-aC_{13}^3C_{23}^3+b(C_{12}^1C_{13}^3+(C_{12}^2+C_{13}^3)C_{23}^3)}{bC_{13}^1+aC_{23}^2}\\
&\quad +\frac{bC_{12}^2C_{13}^1+(a-b)C_{12}^2C_{23}^2-bC_{13}^3C_{23}^2}{bC_{12}^1-aC_{23}^3}+\frac{a(C_{12}^1C_{23}^2+C_{13}^1C_{23}^3)}{b(C_{12}^2+C_{13}^3)}\Big)e_1+C_{23}^2e_2,\\
\nabla_{e_1}e_1 &= -C_{12}^1e_2-C_{13}^1e_3,\quad \nabla_{e_2}e_2=C_{12}^2e_1-C_{23}^2e_3,\quad \nabla_{e_3}e_3=C_{13}^3e_1+C_{23}^3e_2.
\end{aligned}
$$

Let $\eta=\Sigma b_ie^i$, $\Phi=\Sigma b_{ij}e^{ij}$ and the almost contact metric structure (φ,ξ,η,g) on HLg_{20}. Then, we have an almost cosymplectic structure if and only if

$$
\begin{aligned}
&b_1C_{12}^1+b_2C_{12}^2+b_3\frac{-aC_{13}^3C_{23}^3+b(C_{12}^1C_{13}^3+(C_{12}^2+C_{13}^3)C_{23}^3)}{bC_{13}^1+aC_{23}^2}=0,\\
&b_1C_{13}^1+b_2\frac{bC_{12}^2C_{13}^1+(a-b)C_{12}^2C_{23}^2-bC_{13}^3C_{23}^2}{bC_{12}^1-aC_{23}^3}+b_3C_{13}^3=0,\\
&b_1\frac{a(C_{12}^1C_{23}^2+C_{13}^1C_{23}^3)}{b(C_{12}^2+C_{13}^3)}+b_2C_{23}^2+b_3C_{23}^3=0,\\
&b_{12}(C_{13}^1+C_{23}^2)+b_{13}(C_{23}^3-C_{12}^1)-b_{23}(C_{13}^3+C_{12}^2)=0.
\end{aligned}
$$

Let $X = a_1e_1 + a_2e_2 + a_3e_3$ be a vector field on HLg_{20}. Then, it is a Killing vector field if and only if

$$
\begin{aligned}
&a_3C_{13}^1 + a_2C_{12}^1 = 0, \quad -a_1C_{12}^2 + a_3C_{23}^2 = 0, \quad a_1C_{13}^3 + a_2C_{23}^3 = 0,\\
&-a_1C_{12}^1 + a_2C_{12}^2 + a_3\big(\frac{a(C_{12}^1C_{23}^2 + C_{13}^1C_{23}^3)}{b(C_{12}^2 + C_{13}^3)} + \frac{bC_{12}^2C_{13}^1 + (a-b)C_{12}^2C_{23}^2 - bC_{13}^3C_{23}^2}{bC_{12}^1 - aC_{23}^3}\big) = 0,\\
&-a_1C_{13}^1 + a_3C_{13}^3 + a_2\big(-\frac{a(C_{12}^1C_{23}^2 + C_{13}^1C_{23}^3)}{b(C_{12}^2 + C_{13}^3)} + \frac{-aC_{13}^3C_{23}^3 + b(C_{12}^1C_{13}^3 + (C_{12}^2 + C_{13}^3)C_{23}^3)}{bC_{13}^1 + aC_{23}^2}\big) = 0,\\
&-a_2C_{23}^2 + a_3C_{23}^3\\
&+a_1\big(-\frac{-aC_{13}^3C_{23}^3 + b(C_{12}^1C_{13}^3 + (C_{12}^2 + C_{13}^3)C_{23}^3)}{bC_{13}^1 + aC_{23}^2} - \frac{bC_{12}^2C_{13}^1 + (a-b)C_{12}^2C_{23}^2 - bC_{13}^3C_{23}^2}{bC_{12}^1 - aC_{23}^3}\big) = 0.
\end{aligned}
$$

Suppose that we have an almost contact metric structure (φ, ξ, η, g) on HLg_{20} which ξ be a Killing vector field and $\varphi e_i, i = 1, 2, 3$ come from (2). If it satisfies in $g(e_i, \varphi e_j) = d\eta(e_i, e_j), i, j = 1, 2, 3$, it will be a contact metric structure. Thus, it is a K-contact and a Sasakian structure.

Example 3.53. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under conditions $C_{13}^3 = C_{23}^3 = 0$ and $C_{13}^1 = -C_{23}^2$ is an almost cosymplectic structure.

Moreover, let $X = \Sigma x_ie_i$ is a vector on HLg_{20}. Then, there is a semi cosymplectic structure if and only if

$$
\begin{aligned}
&-C_{23}^2b_{31} + \frac{a(C_{12}^1C_{23}^2 + C_{13}^1C_{23}^3)}{b(C_{12}^2 + C_{13}^3)}b_{32} + C_{23}^3b_{21} = 0,\\
&-C_{13}^1b_{32} + \frac{bC_{12}^2C_{13}^1 + (a-b)C_{12}^2C_{23}^2 - bC_{13}^3C_{23}^2}{bC_{12}^1 - aC_{23}^3}b_{31} + C_{13}^3b_{12} = 0,\\
&-C_{12}^1b_{23} + \frac{-aC_{13}^3C_{23}^3 + b(C_{12}^1C_{13}^3 + (C_{12}^2 + C_{13}^3)C_{23}^3)}{bC_{13}^1 + aC_{23}^2}b_{21} + C_{12}^2b_{13} = 0,\\
&b_1(C_{12}^2 + C_{13}^3) + b_2(C_{23}^3 - C_{12}^1) - b_3(C_{13}^1 + C_{23}^2) = 0.
\end{aligned}
$$

Example 3.54. *The almost contact metric structure (φ, ξ, η, g) where*

$$\xi = e_3, \quad \eta = e^3, \quad \varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \Phi = e^{12},$$

under conditions $C_{13}^3 = C_{23}^3 = 0$ and $C_{13}^1 = -C_{23}^2$ is a semi cosymplectic structure.

By calculating $\nabla\Phi = 0$, we find that there is a cosymplectic structure on HLg_{20} if and only if we have the following equations

$$
\begin{aligned}
&C^1_{13}b_{32} - \frac{1}{2}\Big(\frac{-aC^3_{13}C^3_{23} + b(C^1_{12}C^3_{13} + (C^2_{12} + C^3_{13})C^3_{23})}{bC^1_{13} + aC^2_{23}} \\
&-\frac{bC^2_{12}C^1_{13} + (a-b)C^2_{12}C^2_{23} - bC^3_{13}C^2_{23}}{bC^1_{12} - aC^3_{23}} - \frac{a(C^1_{12}C^2_{23} + C^1_{13}C^3_{23})}{b(C^2_{12} + C^3_{13})}\Big)b_{13} = 0, \\
&C^1_{12}b_{23} - \frac{1}{2}\Big(- \frac{-aC^3_{13}C^3_{23} + b(C^1_{12}C^3_{13} + (C^2_{12} + C^3_{13})C^3_{23})}{bC^1_{13} + aC^2_{23}} \\
&+\frac{bC^2_{12}C^1_{13} + (a-b)C^2_{12}C^2_{23} - bC^3_{13}C^2_{23}}{bC^1_{12} - aC^3_{23}} + \frac{a(C^1_{12}C^2_{23} + C^1_{13}C^3_{23})}{b(C^2_{12} + C^3_{13})}\Big)b_{12} = 0, \\
&C^2_{23}b_{13} + \frac{1}{2}\Big(\frac{-aC^3_{13}C^3_{23} + b(C^1_{12}C^3_{13} + (C^2_{12} + C^3_{13})C^3_{23})}{bC^1_{13} + aC^2_{23}} \\
&+\frac{bC^2_{12}C^1_{13} + (a-b)C^2_{12}C^2_{23} - bC^3_{13}C^2_{23}}{bC^1_{12} - aC^3_{23}} + \frac{a(C^1_{12}C^2_{23} + C^1_{13}C^3_{23})}{b(C^2_{12} + C^3_{13})}\Big)b_{32} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&C^2_{12}b_{13} + \frac{1}{2}\Big(\frac{-aC^3_{13}C^3_{23} + b(C^1_{12}C^3_{13} + (C^2_{12} + C^3_{13})C^3_{23})}{bC^1_{13} + aC^2_{23}} \\
&+\frac{bC^2_{12}C^1_{13} + (a-b)C^2_{12}C^2_{23} - bC^3_{13}C^2_{23}}{bC^1_{12} - aC^3_{23}} + \frac{a(C^1_{12}C^2_{23} + C^1_{13}C^3_{23})}{b(C^2_{12} + C^3_{13})}\Big)b_{21} = 0, \\
&C^3_{23}b_{12} + \frac{1}{2}\Big(- \frac{-aC^3_{13}C^3_{23} + b(C^1_{12}C^3_{13} + (C^2_{12} + C^3_{13})C^3_{23})}{bC^1_{13} + aC^2_{23}} \\
&-\frac{bC^2_{12}C^1_{13} + (a-b)C^2_{12}C^2_{23} - bC^3_{13}C^2_{23}}{bC^1_{12} - aC^3_{23}} + \frac{a(C^1_{12}C^2_{23} + C^1_{13}C^3_{23})}{b(C^2_{12} + C^3_{13})}\Big)b_{23} = 0, \\
&C^3_{13}b_{21} + \frac{1}{2}\Big(\frac{-aC^3_{13}C^3_{23} + b(C^1_{12}C^3_{13} + (C^2_{12} + C^3_{13})C^3_{23})}{bC^1_{13} + aC^2_{23}} \\
&+\frac{bC^2_{12}C^1_{13} + (a-b)C^2_{12}C^2_{23} - bC^3_{13}C^2_{23}}{bC^1_{12} - aC^3_{23}} - \frac{a(C^1_{12}C^2_{23} + C^1_{13}C^3_{23})}{b(C^2_{12} + C^3_{13})}\Big)b_{13} = 0, \\
&C^1_{12}b_{13} + C^1_{13}b_{21} = 0, \quad C^2_{12}b_{23} - C^2_{23}b_{12} = 0, \quad C^3_{13}b_{32} + C^3_{23}b_{13} = 0.
\end{aligned}
$$

4. Conclusions

In this manuscript, we considered all 3-dimensional Hom-Lie algebras and gained three systems of equations for each 3-dimensional Hom-Lie algebra which characterized semi-cosymplectic, almost cosymplectic and cosymplectic structures on them.

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