

ANALYSIS OF RESONANT CURVE AND PHASE PORTRAIT DUE TO EARTH'S EQUATORIAL ELLIPTICITY IN THE EARTH-MOON SYSTEM USING PERTURBATION TECHNIQUE

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(Received: 11 January 2021, Accepted: 17 August 2022)

ABSTRACT. In this paper, the resonance due to equatorial ellipticity of Earth, spin rate of Earth and angular velocity of bary-center are investigated. Using spherical coordinate system, we have determined equations of motion of the moon. By using perturbations relative to moon, equations of motion are simplified to second order ordinary differential equation containing the equatorial ellipticity parameter of the Earth. Two types of resonant curves due to the frequencies (a) $\dot{\gamma}$, $\dot{\theta}_E$, and (b) $\dot{\alpha}$, $\dot{\theta}_E$ are analyzed. It is observed that the amplitudes of the oscillation are different at different resonant points due to different size of the bandwidth. Finally, the phase portrait of the orbits of the moon and the phase spaces are analyzed by using Poincare section with periodicity $\frac{\pi}{\gamma}$ in time t for the unperturbed system.

AMS Classification: 54H20.

Keywords: Resonance; Earth-Moon System; Earth's equatorial ellipticity; Perturbation; Phase portrait.

1. INTRODUCTION

Resonance problems due to the perturbing influence of the gravitational field of Earth have been studied extensively. There are many forces that effect the motion of a planets like solar gravitational field, solar radiation pressure, resistive force etc. When the orbital periods of the planets become ratio of small integer we can say Resonance occurs.

Many authors have worked on the problems of resonance including the effect of various forces but we have studied the resonant curves, phase portrait and Poincare section in the Earth-Moon system including γ (Earth's equatorial ellipticity parameter).

Shaw and Holmes (1983) studied a single degree of freedom periodically forced piecewise linear oscillator. They analyzed the phase portrait and bifurcation leading to harmonic, subharmonic and chaotic motion [11].

Farmiga and Moraes (2011) studied the motion of artificial satellite under the impact of resonance on artificial satellite orbital elements due to commensurability between the mean orbital motion and Earth's rotation [4].

Sampaio et. al (2012) studied the dynamical behaviour of the critical angles associated to the 2 : 1 resonance problem in the artificial Earth's satellite motion [9].

The problems of resonance investigated by Yadav and Aggarwal (2013) in the motion of the geocentric satellite, taking into account the equatorial ellipticity parameter of Earth [12] and studied the amplitude and time period using Brown and Shook method [3].

Aggarwal et.al (2015) investigated resonance between Earth's equatorial ellipticity parameter and angular velocity of geo-centric satellite [1].

Maria and Eva (2018) investigated the stability of degree of freedom (DOF) models with respect to coefficients of differential system. Also they have drawn the phase portraits with respect to various system coefficients [6].

Shamolin (2018) investigated the complete system of equations of motion including moment of the interaction force and obtained the family of phase portraits [10].

Eduardo and Philippe (2018) investigated resonance in multiplanetary circumbinary system consisting $N - 2$ planets and analyzed the influence of the post-Newtonian interaction [2].

Kaur et. al (2018) determined resonant point including the effect of resistive force [7].

Martin et. al (2019) investigated the n -dimensional phase portrait and provide a tool for higher order phase portrait.

Yadav et. al (2019) investigated the stability of equilibrium points of geo-centric satellite and they found infinitely many equilibrium points for different values of second order tesseral harmonic term Γ [13].

Yadav et. al (2022) have studied resonant curves including the rate of change of the Earth's equatorial ellipticity parameter, the steady-state value of the Moon's angular velocity, and the angular velocity of the bary-center in the Earth-Moon-Sun system by using the unperturbed solution of the system [14].

In our paper, we have investigated the resonance in the Earth-Moon system due to the Earth's equatorial ellipticity and the Moon's latitude, this includes a second order tesseral harmonic.

We organized this paper as: in section 2, the problem statement and configuration have been given. In section 3, using the spherical coordinate relative to the Earth, we have determined the equations of motion of Moon by following the procedure of Frick [5]. In section 4, we defined the perturbation relative to the orbit of Moon and found the solution. In section 5, we have analyzed the effect of Earth's equatorial ellipticity parameter and angular velocity of the Moon on the resonant curves. In Section 6, we investigated the phase portrait and Poincare section in the Earth-Moon system, as well as the Earth's equatorial ellipticity parameter. Finally, in section 7, we discussed and analysed the obtained results.

2. STATEMENT OF THE PROBLEM

We define the coordinate system for this problem. (Fig. 1)

We have taken (X, Y, Z) as an inertial coordinate system with center of the Sun as the origin with XY -plane as the ecliptic plane. The unit vectors are $\vec{x}_1$, $\vec{y}_1$ and $\vec{z}_1$. (X', Y', Z') be the coordinate system of Earth-Moon system with origin at the center of mass. (X'', Y'', Z'') be the coordinate system with origin at the center of the Earth. We assumed that all the axes parallel to the inertial coordinate system (X, Y, Z) .

We take (r_m, θ_m, ϕ_m) as spherical system and center of the Earth as origin, where r_m be the orbital distance of the Moon from the Earth's center, θ_m and ϕ_m are the Moon longitude and latitude in the direction of vernal equinox with unit vectors $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$.

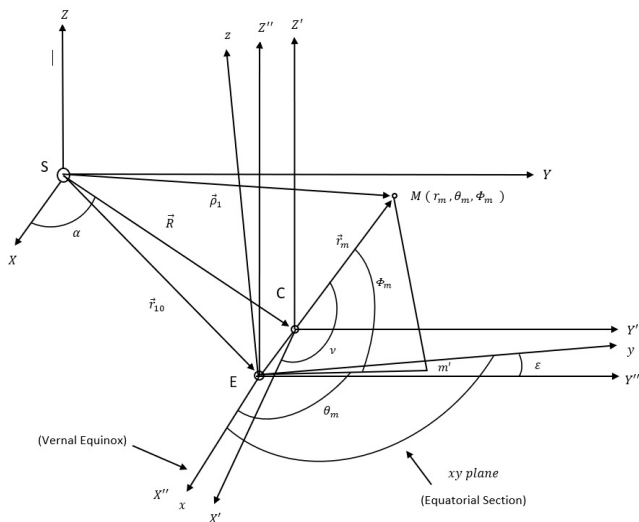


FIGURE 1. Coordinate System

We define the relation between the unit vectors as [5],

$$\vec{n}_2 = c_x \vec{x}_1 + c_y \vec{y}_1 + c_z \vec{z}_1,$$

where

$$\begin{aligned}
 a_x &= \cos \theta_m \cos \phi_m, \\
 a_y &= \sin \theta_m \cos \phi_m \cos \epsilon - \sin \epsilon \sin \phi_m, \\
 a_z &= \sin \theta_m \cos \phi_m \sin \epsilon + \cos \epsilon \sin \phi_m, \\
 b_x &= -\sin \theta_m, \\
 b_y &= \cos \theta_m \cos \epsilon, \\
 b_z &= \cos \theta_m \sin \epsilon, \\
 c_x &= -\cos \theta_m \sin \phi_m, \\
 c_y &= -\sin \theta_m \sin \phi_m \cos \epsilon - \sin \epsilon \cos \phi_m, \\
 c_z &= -\sin \theta_m \sin \phi_m \sin \epsilon + \cos \epsilon \cos \phi_m.
 \end{aligned}$$

The acceleration components along the direction $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by,

$$(1) \quad a_{r_m} = \ddot{\vec{r}}_m \cdot \vec{r}_1 = \ddot{r}_m - r_m \dot{\theta}_m^2 \cos^2 \phi_m - r_m \dot{\phi}_m^2,$$

$$(2) \quad a_{\theta_m} = \ddot{\vec{r}}_m \cdot \vec{n}_1 = \frac{1}{r_m \cos \phi_m} \frac{d}{dt} (r_m^2 \dot{\theta}_m \cos^2 \phi_m),$$

$$(3) \quad a_{\phi_m} = \ddot{\vec{r}}_m \cdot \vec{n}_2 = \frac{1}{r_m} \frac{d}{dt} (r_m^2 \dot{\phi}_m).$$

3. EQUATIONS OF MOTION IN SPHERICAL COORDINATE SYSTEM

$\ddot{\vec{r}}_m$ is a acceleration of the Moon relative to Earth and it can be expressed as

$$(4) \quad \ddot{\vec{r}}_m = \frac{1}{M_m} (\vec{F}_{ME} + \vec{F}_{MS}) - \frac{1}{M_E} (\vec{F}_{ES} + \vec{F}_{EM}),$$

where M_m and M_E are the masses of the Moon and the Earth, respectively. $\vec{F}_{ME}$ and $\vec{F}_{MS}$ are the gravitation force of the Earth and the Sun on Moon respectively, while $\vec{F}_{EM}$ and $\vec{F}_{ES}$ are gravitational force of the Moon and the Sun on the Earth respectively.

We consider the gravitational potential of Earth to calculate $\vec{F}_{ME}$ as

$$(5) \quad U = \frac{g_0 R_0^2}{r_m} \left[1 - \frac{J_2 R_0^2}{r_m^2} \left(\frac{3 \sin^2 \phi_m - 1}{2} \right) + 3 \frac{J_2^{(2)} R_0^2}{r_m^2} \cos^2 \phi_m \cos 2\gamma \right],$$

where

$$R_0 = 63781 Km = \text{radius of the Earth},$$

$$g_0 = 7.3156608 \times 10^7 \text{ Km/day}^2 = \text{gravitational acceleration at the Earth's surface},$$

$$J_2 = 1.08219 \times 10^{-3} = \text{Earth's oblateness coefficient},$$

$$J_2^{(2)} = 2.32 \times 10^{-6} = \text{equatorial ellipticity coefficient},$$

$$\phi_m = \text{latitude of the Moon}.$$

The force components of $\vec{F}_{ME}$ in the direction of $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given as

$$(6) \quad \frac{\vec{F}_{ME} \cdot \vec{r}_1}{M_m} = -\frac{g_0 R_0^2}{r_m^2} + \frac{3J_2 g_0 R_0^4}{r_m^4} \left(\frac{3 \sin^2 \phi_m - 1}{2} \right) - \frac{9J_2^{(2)} g_0 R_0^4 \cos^2 \phi_m \cos 2\gamma}{r_m^4},$$

$$(7) \quad \frac{\vec{F}_{ME} \cdot \vec{n}_1}{M_m} = -\frac{6J_2^{(2)} g_0 R_0^4 \cos \phi_m \sin 2\gamma}{r_m^4},$$

$$(8) \quad \frac{\vec{F}_{ME} \cdot \vec{n}_2}{M_m} = -\frac{3J_2 g_0 R_0^4 \cos \phi_m \sin \phi_m}{r_m^4} - \frac{6J_2^{(2)} g_0 R_0^4 \cos \phi_m \sin \phi_m \cos 2\gamma}{r_m^4}.$$

$\vec{F}_{MS}$ is the gravitational force of Sun to Moon

$$(9) \quad \frac{\vec{F}_{MS}}{M_m} = -\frac{g_1 R_1^2}{\rho_1^3} \vec{\rho}_1,$$

where $\vec{\rho}_1$ is vector from center of the Sun to the Moon,

g_1 is gravitational acceleration at the surface of Sun,

R_1 is the radius of the Sun.

$\vec{\rho}_1$ can be written as

$$(10) \quad \vec{\rho}_1 = \vec{R} + \vec{r}_m - \frac{\vec{r}_m}{\mu},$$

where $\mu = \frac{M_m + M_E}{M_m}$, $\vec{R}$ is a vector from the Sun to the bary-center of Earth-Moon system and $\frac{\vec{r}_m}{\mu}$ is a vector from the bary-center of Earth-Moon system to the center of the Earth.

Sustituting the value of Eq.(10) in (9), we get

$$(11) \quad \frac{\vec{F}_{MS}}{M_m} = -\frac{g_1 R_1^2}{R^3} \left[1 - \frac{3(\vec{R} \cdot \vec{r}_m)}{R^2} + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] \left(\vec{R} + \vec{r}_m - \frac{\vec{r}_m}{\mu} \right).$$

Now, the components of $\vec{F}_{MS}$ along the direction of $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by

$$(12) \quad \frac{\vec{F}_{MS} \cdot \vec{r}_1}{M_m} = -\frac{g_1 R_1^2}{R^3} \left[1 - \frac{3(\vec{R} \cdot \vec{r}_m)}{R^2} + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] \left(\vec{R} \cdot \vec{r}_1 + r_m - \frac{r_m}{\mu} \right),$$

$$(13) \quad \frac{\vec{F}_{MS} \cdot \vec{n}_1}{M_m} = -\frac{g_1 R_1^2}{R^3} \left[1 - \frac{3(\vec{R} \cdot \vec{r}_m)}{R^2} + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_1),$$

$$(14) \quad \frac{\vec{F}_{MS} \cdot \vec{n}_2}{M_m} = -\frac{g_1 R_1^2}{R^3} \left[1 - \frac{3(\vec{R} \cdot \vec{r}_m)}{R^2} + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_2).$$

Now, the gravitational attraction of the Sun on the Earth $\vec{F}_{ES}$ can be written as

$$(15) \quad \frac{\vec{F}_{ES}}{M_E} = -\frac{g_1 R_1^2}{r_{10}^3} \vec{r}_{10},$$

where $\vec{r}_{10}$ is a vector from the Sun's center to the Earth, it can be be represented by the relation

$$(16) \quad \vec{r}_{10} = \vec{R} - \frac{\vec{r}_m}{\mu},$$

substituting the value of Eq. (16) in (15), we get

$$(17) \quad \frac{\vec{F}_{ES}}{M_E} = -\frac{g_1 R_1^2}{R^3} \left[1 + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] \left(\vec{R} - \frac{\vec{r}_m}{\mu} \right).$$

Now, the components of $\vec{F}_{ES}$ along the direction of $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by

$$(18) \quad \frac{\vec{F}_{ES} \cdot \vec{r}_1}{M_E} = -\frac{g_1 R_1^2}{R^3} \left[1 + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] \left(\vec{R} \cdot \vec{r}_1 - \frac{r_m}{\mu} \right),$$

$$(19) \quad \frac{\vec{F}_{ES} \cdot \vec{n}_1}{M_E} = -\frac{g_1 R_1^2}{R^3} \left[1 + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_1),$$

$$(20) \quad \frac{\vec{F}_{ES} \cdot \vec{n}_2}{M_E} = -\frac{g_1 R_1^2}{R^3} \left[1 + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_2).$$

Now, the gravitational attraction of the Moon on the Earth $\vec{F}_{EM}$ is given by

$$(21) \quad \frac{\vec{F}_{EM}}{M_E} = -\frac{g_2 R_2^2}{r_m^3} \vec{r}_m,$$

where g_2 gravitational attraction at the surface of the Moon, R_2 is radius of the Moon.

The components of $\vec{F}_{EM}$ along the direction $\vec{r}_1$, $\vec{n}_1$ and $\vec{n}_2$ are given by

$$(22) \quad \frac{\vec{F}_{EM} \cdot \vec{r}_1}{M_E} = -\frac{g_2 R_2^2}{r_m^2},$$

$$(23) \quad \frac{\vec{F}_{EM} \cdot \vec{n}_1}{M_E} = 0,$$

$$(24) \quad \frac{\vec{F}_{EM} \cdot \vec{n}_2}{M_E} = 0,$$

also,

$$(25) \quad \vec{R} = R (\vec{x}_1 \cos \alpha + \vec{y}_1 \sin \alpha),$$

$$(26) \quad \vec{r}_m = r_m \vec{r}_1,$$

$$(27) \quad \dot{\alpha}^2 = \frac{g_1 R_1^2}{R^3},$$

$$(28) \quad \dot{\nu}^2 = \frac{g_2 R_2^2}{r_m^3} \mu.$$

Substituting the value of the Eqs. (4), (6), (12), (18), (22), (25), (26), (27) and (28) in Eq (1), we get

$$(29) \quad \begin{aligned} \ddot{r}_m - r_m \dot{\theta}_m^2 \cos^2 \phi_m - r_m \dot{\phi}_m^2 = & -\frac{g_0 R_0^2}{r_m^2} + \frac{3J_2 g_0 R_0^4}{r_m^4} \left(\frac{3 \sin^2 \phi_m - 1}{2} \right) \\ & - \frac{9J_2^{(2)} g_0 R_0^4 \cos^2 \phi_m \cos 2\gamma}{r_m^4} \\ & - \frac{g_1 R_1^2}{R^3} \left[1 - \frac{3(\vec{R} \cdot \vec{r}_m)}{R^2} + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{r}_1 + r_m - \frac{r_m}{\mu}) \\ & + \frac{g_1 R_1^2}{R^3} \left[1 + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{r}_1 - \frac{r_m}{\mu}) \\ & + \frac{g_2 R_2^2}{r_m^2}, \end{aligned}$$

$$\begin{aligned}
&= -\frac{g_0 R_0^2}{r_m^2} + \frac{3J_2 g_0 R_0^4}{r_m^4} \left(\frac{3 \sin^2 \phi_m - 1}{2} \right) - \frac{9J_2^{(2)} g_0 R_0^4 \cos^2 \phi_m \cos 2\gamma}{r_m^4} \\
&+ \frac{3r_m^2 \dot{\alpha}^2}{2R} \left(\frac{\mu - 2}{\mu} \right) \left[\cos(\theta_m + \alpha) K_1 + \cos(\theta_m - \alpha) K_2 - \sin \alpha K_3 \right] \\
&+ \frac{3r_m \dot{\alpha}^2}{4} \left[\cos(2\theta_m + 2\alpha) K_4 + \cos(2\theta_m - 2\alpha) K_5 + \cos 2\theta_m K_6 + \cos 2\alpha K_7 \right. \\
&\quad \left. + \sin(\theta_m + 2\alpha) K_8 + \sin(\theta_m - 2\alpha) K_9 + \sin \theta_m K_{10} + K_{11} \right] \\
(30) \quad &+ \frac{\dot{\nu}^2 r_m}{\mu} - r_m \dot{\alpha}^2,
\end{aligned}$$

where values of K'_i 's are given in Appendix (A).

Now, substitute the values of Eq. (4), (7), (13), 19, (23), (25), (26), (27), (28) in Eq. (2), we get

$$\begin{aligned}
(31) \quad \frac{1}{r_m \cos \phi_m} \frac{d}{dt} (r_m^2 \dot{\theta}_m \cos^2 \phi_m) &= -\frac{6J_2^{(2)} g_0 R_0^4 \cos \phi_m \sin 2\gamma}{r_m^4} \\
&- \frac{g_1 R_1^2}{R^3} \left[1 - \frac{3(\vec{R} \cdot \vec{r}_m)}{R^2} + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_1) \\
&+ \frac{g_1 R_1^2}{R^3} \left[1 + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_1),
\end{aligned}$$

$$\begin{aligned}
(32) \quad &= -\frac{6J_2^{(2)} g_0 R_0^4 \cos \phi_m \sin 2\gamma}{r_m^4} + 3r_m \dot{\alpha}^2 \left[\sin(2\theta_m + 2\alpha) M_1 + \sin 2\theta_m M_2 \right. \\
&\quad + \sin(2\theta_m - 2\alpha) M_3 + \cos(\theta_m - 2\alpha) M_4 \\
&\quad + \cos(\theta_m + 2\alpha) M_5 + \cos 2\theta_m M_6 \\
&\quad + \cos(2\theta_m + 2\alpha) M_7 + \cos(2\theta_m - 2\alpha) M_8 \\
&\quad \left. + \cos \theta_m M_9 \right],
\end{aligned}$$

where values of M'_i 's are given in Appendix (B).

Finally, substituting the values of Eq. (4), (8), (14) (20) and (24) in Eq.(3), we get

$$\begin{aligned}
 \frac{1}{r_m} \frac{d}{dt} (r_m^2 \dot{\phi}_m) = & - \frac{3J_2 g_0 R_0^4 \cos \phi_m \sin \phi_m}{r_m^4} - \frac{6J_2^{(2)} g_0 R_0^4 \cos \phi_m \sin \phi_m \cos 2\gamma}{r_m^4} \\
 & - \frac{g_1 R_1^2}{R^3} \left[1 - \frac{3(\vec{R} \cdot \vec{r}_m)}{R^2} + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_2) \\
 (33) \quad & + \frac{g_1 R_1^2}{R^3} \left[1 + \frac{3(\vec{R} \cdot \vec{r}_m)}{\mu R^2} \right] (\vec{R} \cdot \vec{n}_2).
 \end{aligned}$$

4. PERTURBATIONS RELATIVE TO THE MOON

We define perturbations relative to the Moon as follows,

$$\begin{aligned}
 r_m &= r_{em} + \Delta r_m, \\
 \dot{\theta}_m &= \dot{\theta}_E + \Delta \dot{\theta}_m,
 \end{aligned}$$

where r_{em} is the steady state value of orbital radius of Moon.

From Eq. (30), we get

$$\begin{aligned}
 \Delta \ddot{r}_m - (\dot{\theta}_E^2 \Delta r_m + r_{em} \dot{\theta}_E^2 + 2r_{em} \dot{\theta}_E \Delta \dot{\theta}_m) \cos^2 \phi_m = & - \frac{g_0 R_0^2}{r_{em}^2} + \frac{3J_2 g_0 R_0^4 (3 \sin^2 \phi_m - 1)}{2r_{em}^4} \\
 & - \frac{9J_2^{(2)} g_0 R_0^4}{r_{em}^4} \cos^2 \phi_m \cos 2\gamma \\
 & + \frac{3r_{em}^2 \dot{\alpha}^2}{2R} \left(\frac{\mu - 2}{\mu} \right) \left[\cos(\theta_m + \alpha) K_1 \right. \\
 & \left. + \cos(\theta_m - \alpha) K_2 - \sin \alpha K_3 \right] \\
 & + \frac{3r_{em} \dot{\alpha}^2}{4} \left[\cos(2\theta_m + 2\alpha) K_4 \right. \\
 & + \cos(2\theta_m - 2\alpha) K_5 + \cos 2\theta_m K_6 \\
 & + \cos 2\alpha K_7 + \sin(\theta_m + 2\alpha) K_8 \\
 & \left. + \sin(\theta_m - 2\alpha) K_9 + \sin \theta_m K_{10} + K_{11} \right] \\
 (34) \quad & + \frac{\dot{\nu}^2 r_{em}}{\mu} - r_{em} \dot{\alpha}^2.
 \end{aligned}$$

Similarly, from Eq. (32), we get

$$\begin{aligned}
 \frac{d}{dt}[(r_{em}\Delta\dot{\theta}_m + 2\Delta r_m\dot{\theta}_E)\cos^2\phi_m] = & -\frac{6J_2^{(2)}g_0R_0^4\cos^2\phi_m\sin 2\gamma}{r_{em}^4} \\
 & + 3r_{em}\dot{\alpha}^2\cos\phi_m \left[\sin(2\theta_m + 2\alpha) M_1 \right. \\
 & + \sin 2\theta_m M_2 \\
 & + \sin(2\theta_m - 2\alpha) M_3 \\
 & + \cos(\theta_m - 2\alpha) M_4 \\
 & + \cos(\theta_m + 2\alpha) M_5 \\
 & + \cos 2\theta_m M_6 \\
 & + \cos(2\theta_m + 2\alpha) M_7 \\
 & + \cos(2\theta_m - 2\alpha) M_8 \\
 & \left. + \cos \theta_m M_9 \right].
 \end{aligned}
 \tag{35}$$

On integrating Eq. (35), we get

$$\begin{aligned}
 (r_{em}\Delta\dot{\theta}_m + 2\Delta r_m\dot{\theta}_E)\cos^2\phi_m = & \frac{3J_2^{(2)}g_0R_0^4\cos^2\phi_m\cos 2\gamma}{\dot{\gamma}r_{em}^4} \\
 & + 3r_{em}\dot{\alpha}^2\cos\phi_m \left[-\frac{\cos(2(\dot{\theta}_E + \Delta\dot{\theta}_m + \dot{\alpha})t)}{2(\dot{\theta}_E + \Delta\dot{\theta}_m + \dot{\alpha})} M_1 \right. \\
 & - \frac{\cos(2(\dot{\theta}_E + \Delta\dot{\theta}_m)t)}{2(\dot{\theta}_E + \Delta\dot{\theta}_m)} M_2 \\
 & - \frac{\cos(2(\dot{\theta}_E + \Delta\dot{\theta}_m - \dot{\alpha})t)}{2(\dot{\theta}_E + \Delta\dot{\theta}_m - \dot{\alpha})} M_3 \\
 & + \frac{\sin((\dot{\theta}_E + \Delta\dot{\theta}_m - 2\dot{\alpha})t)}{(\dot{\theta}_E + \Delta\dot{\theta}_m - 2\dot{\alpha})} M_4 \\
 & \left. + \frac{\sin((\dot{\theta}_E + \Delta\dot{\theta}_m + 2\dot{\alpha})t)}{(\dot{\theta}_E + \Delta\dot{\theta}_m + 2\dot{\alpha})} M_5 \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\sin(2(\dot{\theta}_E + \Delta\dot{\theta}_m)t)}{2(\dot{\theta}_E + \Delta\dot{\theta}_m)} M_6 \\
& + \frac{\sin(2(\dot{\theta}_E + \Delta\dot{\theta}_m + \dot{\alpha})t)}{2(\dot{\theta}_E + \Delta\dot{\theta}_m + \dot{\alpha})} M_7 \\
& + \frac{\sin(2(\dot{\theta}_E + \Delta\dot{\theta}_m - \dot{\alpha})t)}{2(\dot{\theta}_E + \Delta\dot{\theta}_m - \dot{\alpha})} M_8 \\
& + \frac{\sin((\dot{\theta}_E + \Delta\dot{\theta}_m)t)}{(\dot{\theta}_E + \Delta\dot{\theta}_m)} M_9 \Big] + C,
\end{aligned}
\tag{36}$$

where C is a constant of integration.

By using the initial condition at $t = t_0$, $\Delta r_m = \Delta r_{m0}$, $\Delta\dot{\theta}_m = 0$, $\theta_m = \theta_{m0}$ and $\alpha = \alpha_0$, then we get the value of C as

$$\begin{aligned}
C = & 2\Delta r_{m0} \dot{\theta}_E \cos^2 \phi_m - \frac{3J_2^{(2)} g_0 R_0^4 \cos^2 \phi_m \cos 2\dot{\gamma} t_0}{\dot{\gamma} r_{em}^4} \\
& - 3r_{em} \dot{\alpha}^2 \cos \phi_m \left[- \frac{\cos(2(\theta_{m0} + \alpha_0))}{2(\dot{\theta}_E + \dot{\alpha})} M_1 \right. \\
& - \frac{\cos(2\theta_{m0})}{2\dot{\theta}_E} M_2 \\
& - \frac{\cos(2(\theta_{m0} - \alpha_0))}{2(\dot{\theta}_E - \dot{\alpha})} M_3 \\
& + \frac{\sin(\theta_{mo} - 2\alpha_0)}{(\dot{\theta}_E - 2\dot{\alpha})} M_4 \\
& + \frac{\sin(\theta_{m0} + 2\alpha_0)}{(\dot{\theta}_E + 2\dot{\alpha})} M_5 \\
& + \frac{\sin(2\theta_{m0})}{2\dot{\theta}_E} M_6 \\
& + \frac{\sin(2(\theta_{m0} + \alpha_0))}{2(\dot{\theta}_E + \dot{\alpha})} M_7 \\
& + \frac{\sin(2(\theta_{m0} - \alpha_0))}{2(\dot{\theta}_E - \dot{\alpha})} M_8 \\
& \left. + \frac{\sin \theta_{mo}}{\dot{\theta}_E} M_9 \right].
\end{aligned}$$

We put the value of C in Eq. (36), we get

$$\begin{aligned}
 r_{em} \Delta \dot{\theta}_m \cos^2 \phi_m &= 2 \dot{\theta}_E (\Delta r_{mo} - \Delta r_m) \cos^2 \phi_m \\
 &+ \frac{3 J_2^{(2)} g_0 R_0^4 \cos^2 \phi_m \left(\cos 2\dot{\gamma} t - \cos 2\dot{\gamma} t_0 \right)}{\dot{\gamma} r_{em}^4} \\
 &+ 3 r_{em} \dot{\alpha}^2 \cos \phi_m \left[\frac{\cos(2(\theta_{m0} + \alpha_0)) - \cos(2(\theta_m + \alpha))}{2(\dot{\theta}_E + \dot{\alpha})} M_1 \right. \\
 &+ \frac{\cos(2\theta_{m0}) - \cos(2\theta_m)}{2\dot{\theta}_E} M_2 \\
 &+ \frac{\cos(2(\theta_{m0} - \alpha_0)) - \cos(2(\theta_m - \alpha))}{2(\dot{\theta}_E - \dot{\alpha})} M_3 \\
 &+ \frac{\sin(\theta_m - 2\alpha) - \sin(\theta_{mo} - 2\alpha_0)}{(\dot{\theta}_E - 2\dot{\alpha})} M_4 \\
 &+ \frac{\sin(\theta_m + 2\alpha) - \sin(\theta_{m0} + 2\alpha_0)}{(\dot{\theta}_E + 2\dot{\alpha})} M_5 \\
 &+ \frac{\sin(2\theta_m) - \sin(2\theta_{m0})}{2\dot{\theta}_E} M_6 \\
 &+ \frac{\sin(2(\theta_m + \alpha)) - \sin(2(\theta_{m0} + \alpha_0))}{2(\dot{\theta}_E + \dot{\alpha})} M_7 \\
 &+ \frac{\sin(2(\theta_m - \alpha)) - \sin(2(\theta_{m0} - \alpha_0))}{2(\dot{\theta}_E - \dot{\alpha})} M_8 \\
 &\left. + \frac{\sin \theta_m - \sin \theta_{mo}}{\dot{\theta}_E} M_9 \right].
 \end{aligned}$$

Substituting the value of $r_{em} \Delta \dot{\theta}_m \cos^2 \phi_m$ in Eq. (34) and ignoring the secular (non-periodic) terms, we get

$$\begin{aligned}
 \Delta \ddot{r}_m + 3 \dot{\theta}_E^2 \cos^2 \phi_m \Delta r_m &= P_1 \cos 2\gamma + P_2 \cos(2\theta_m + 2\alpha) \\
 &+ P_3 \cos(2\theta_m) \\
 &+ P_4 \cos(2\theta_m - 2\alpha) \\
 &+ P_5 \cos(2\alpha) \\
 &+ P_6 \cos(\theta_m + \alpha)
 \end{aligned}$$

$$\begin{aligned}
& + P_7 \cos(\theta_m - \alpha) \\
& + P_8 \sin(\theta_m - 2\alpha) \\
& + P_9 \sin(\theta_m + 2\alpha) \\
& + P_{10} \sin(2\theta_m) \\
& + P_{11} \sin(2\theta_m + 2\alpha) \\
& + P_{12} \sin(2\theta_m - 2\alpha) \\
& + P_{13} \sin \theta_m \\
(37) \quad & + P_{14} \sin \alpha,
\end{aligned}$$

where values of P'_i s are given in Apendix C.

This is a non-homogeneous differential equation of second order.

A particular solution of Equation (37) is given by

$$\begin{aligned}
\Delta r_m = & \frac{P_1}{-4 \dot{\gamma}^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \cos 2\gamma \\
& + \frac{P_2}{-4 (\dot{\theta}_E + \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \cos(2\theta_m + 2\alpha) \\
& + \frac{P_3}{-4 \dot{\theta}_E^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \cos(2\theta_m) \\
& + \frac{P_4}{-4 (\dot{\theta}_E - \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \cos(2\theta_m - 2\alpha) \\
& + \frac{P_5}{-4 \dot{\alpha}^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \cos(2\alpha) \\
& + \frac{P_6}{-(\dot{\theta}_E + \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \cos(\theta_m + \alpha) \\
& + \frac{P_7}{-(\dot{\theta}_E - \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \cos(\theta_m - \alpha) \\
& + \frac{P_8}{-(\dot{\theta}_E - 2\dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \sin(\theta_m - 2\alpha) \\
& + \frac{P_9}{-(\dot{\theta}_E + 2\dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \sin(\theta_m + 2\alpha)
\end{aligned}$$

$$\begin{aligned}
& + \frac{P_{10}}{-4 \dot{\theta}_E^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \sin(2\theta_m) \\
& + \frac{P_{11}}{-4 (\dot{\theta}_E + \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \sin(2\theta_m + 2\alpha) \\
& + \frac{P_{12}}{-4 (\dot{\theta}_E - \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \sin(2\theta_m - 2\alpha) \\
& + \frac{P_{13}}{-\dot{\theta}_E^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \sin \theta_m \\
& + \frac{P_{14}}{-\dot{\alpha}^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m} \sin \alpha.
\end{aligned}$$

The above equation of motion can be deduced from the equation given in Frick [5] by taking $J_2^2 = 0$ and Moon as the satellite of the Earth.

5. RESONANCE

In Fig. 2(a), when $\phi_m = 0$, oscillatory amplitude $A = \frac{P_1}{-4 \dot{\gamma}^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\gamma}^2}{\dot{\theta}_E^2} = \frac{3}{4}$.

So, resonance occurs at the point where $\frac{\dot{\gamma}^2}{\dot{\theta}_E^2} = \frac{3}{4}$.

In Fig. 2(b), when $\phi_m = \frac{\pi}{3}$, oscillatory amplitude $A = \frac{P_1}{-4 \dot{\gamma}^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\gamma}^2}{\dot{\theta}_E^2} = \frac{3}{16}$.

So, resonance occurs at the point where $\frac{\dot{\gamma}^2}{\dot{\theta}_E^2} = \frac{3}{16}$.

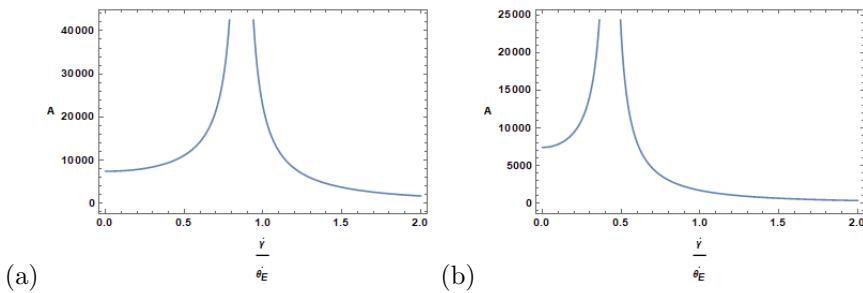


FIGURE 2. (a) Resonant curve at the point $\frac{\dot{\gamma}^2}{\dot{\theta}_E^2} = \frac{3}{4}$ and $\phi_m = 0$ (b) Resonant curve at the point $\frac{\dot{\gamma}^2}{\dot{\theta}_E^2} = \frac{3}{16}$ and $\phi_m = \frac{\pi}{3}$

In Fig. 2, We have drawn the resonant curve due to the frequencies $\dot{\gamma}$ and $\dot{\theta}_E$.

In Fig. 3(a), when $\phi_m = 0$, oscillatory amplitude $A = \frac{P_4}{-4(\dot{\theta}_E - \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = 1$.

So, resonance occurs at the point where $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = 1$.

In Fig. 3(b), when $\phi_m = \frac{\pi}{3}$, oscillatory amplitude $A = \frac{P_4}{-4(\dot{\theta}_E - \dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = 1$.

So, resonance occurs at the point where $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = 1$.

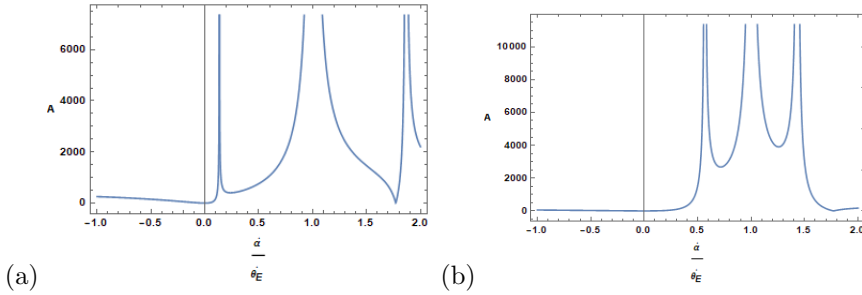


FIGURE 3. (a) Resonant curve at the point $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = 1$ and $\phi_m = 0$

(b) Resonant curve at the point $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = 1$ and $\phi_m = \frac{\pi}{3}$

In Fig. 3, we have drawn resonant curve due to the frequencies $\dot{\alpha}$ and $\dot{\theta}_E$. It is also clear that the sharpness of the resonance are different at the different resonant points and different values of ϕ_m due to the size of the bandwidth.

In Fig. 4(a), when $\phi_m = 0$, oscillatory amplitude $A = \frac{P_8}{-(\dot{\theta}_E - 2\dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = \frac{1}{4}$.

So, resonance occurs at the point where $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = \frac{1}{4}$.

In Fig. 4(b), when $\phi_m = \frac{\pi}{3}$, oscillatory amplitude $A = \frac{P_8}{-(\dot{\theta}_E - 2\dot{\alpha})^2 + 3\dot{\theta}_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = \frac{1}{4}$.

So, resonance occurs at the point where $\frac{\dot{\alpha}^2}{\dot{\theta}_E^2} = \frac{1}{4}$.

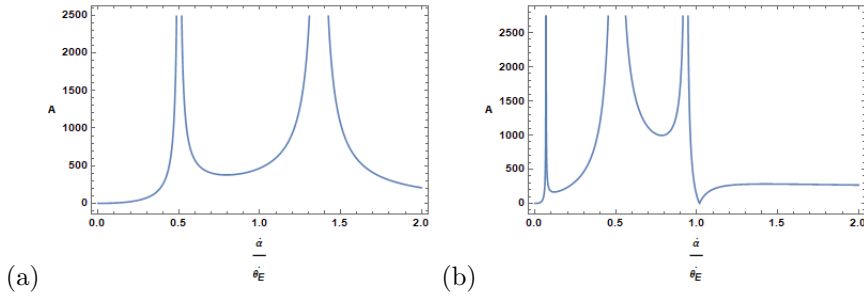


FIGURE 4. (a) Resonant curve at the point $\frac{\dot{\alpha}^2}{\theta_E^2} = \frac{1}{4}$ and $\phi_m = 0$
 (b) Resonant curve at the point $\frac{\dot{\alpha}^2}{\theta_E^2} = \frac{1}{4}$ and $\phi_m = \frac{\pi}{3}$

From Fig. (4), it is clear that amplitude becomes indeterminate at different points and sharpness of the resonance are also different due to different bandwidth.

In Fig. 5(a), when $\phi_m = 0$, oscillatory amplitude $A = \frac{P_{12}}{-4(\theta_E - \dot{\alpha})^2 + 3\theta_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\alpha}^2}{\theta_E^2} = 1$.

So, resonance occurs at the point where $\frac{\dot{\alpha}^2}{\theta_E^2} = 1$.

In Fig. 5(b), when $\phi_m = \frac{\pi}{3}$, oscillatory amplitude $A = \frac{P_{12}}{-4(\theta_E - \dot{\alpha})^2 + 3\theta_E^2 \cos^2 \phi_m}$ becomes indeterminate if $\frac{\dot{\alpha}^2}{\theta_E^2} = 1$.

So, resonance occurs at the point where $\frac{\dot{\alpha}^2}{\theta_E^2} = 1$.

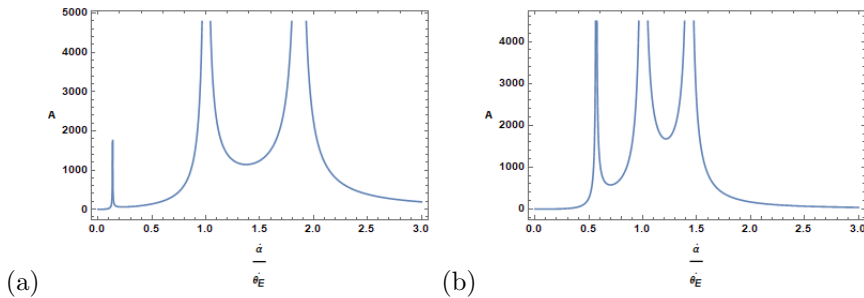


FIGURE 5. (a) Resonant curve at the point $\frac{\dot{\alpha}^2}{\theta_E^2} = 1$ and $\phi_m = 0$
 (b) Resonant curve at the point $\frac{\dot{\alpha}^2}{\theta_E^2} = 1$ and $\phi_m = \frac{\pi}{3}$

From Fig. (5), it is clear that amplitude becomes indeterminate at different points and sharpness of the resonance are also different due to different bandwidth.

6. PHASE PORTRAIT AND POINCARÉ SECTION

6.1. Phase Portrait. Phase portrait is a continuous family of closed curves. The parametric curves on the plane for the differential equation (37) $\Delta \dot{s}_m = -3\dot{\theta}_E^2 \Delta r_m$, (where $\Delta s_m = \Delta \dot{r}_m$) trace the path of the particular solution. This plane is called phase plane.

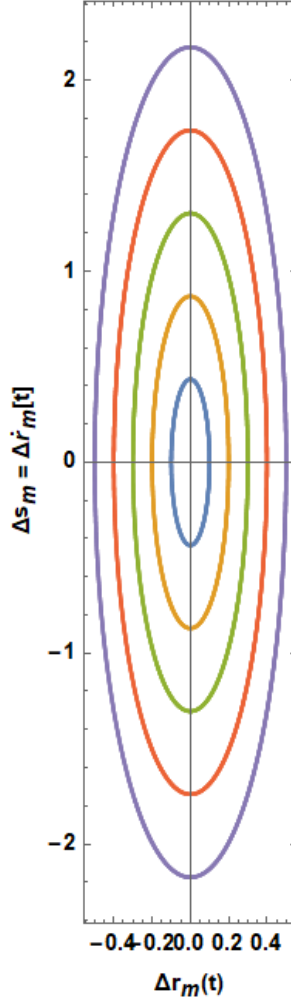
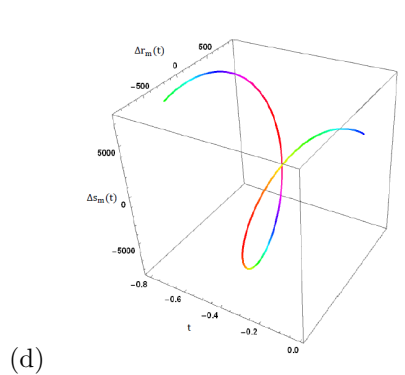
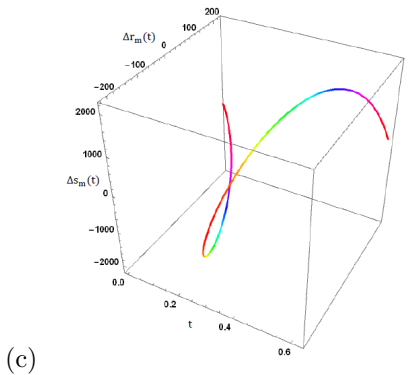
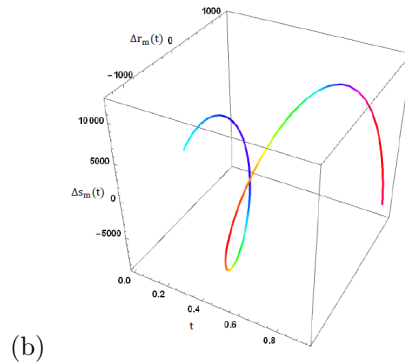
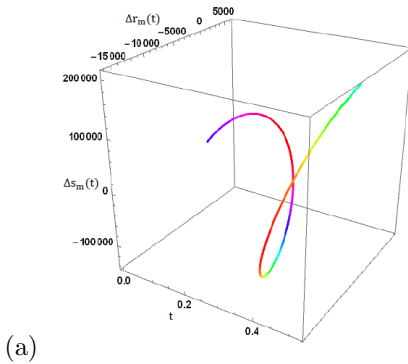


FIGURE 6. Phase Portrait (undamped, unforced)

In Fig. 6, the unforced, undamped phase plane has been shown for Eq. (37).

6.2. Poincare Section. We will study Equation (37) by using Poincare section method. The 3-dimentional extended phase space with periodically forced oscillations has the coordinates $(t, \Delta r_m, \Delta s_m)$ (where $\Delta s_m = \Delta \dot{r}_m$). It is observed that vector field equation (37) is periodic in t with period $\frac{\pi}{\gamma}$. Orbits in the phase space $(t, \Delta r_m, \Delta s_m)$ are explained by the mapping $P : \Sigma \rightarrow \Sigma$ deduced by the solution of Equation (37),

where $\Sigma = \{(t, \Delta r_m, \Delta s) | \Delta r_m = 0, \Delta s_m > 0\}$, is poincare section.



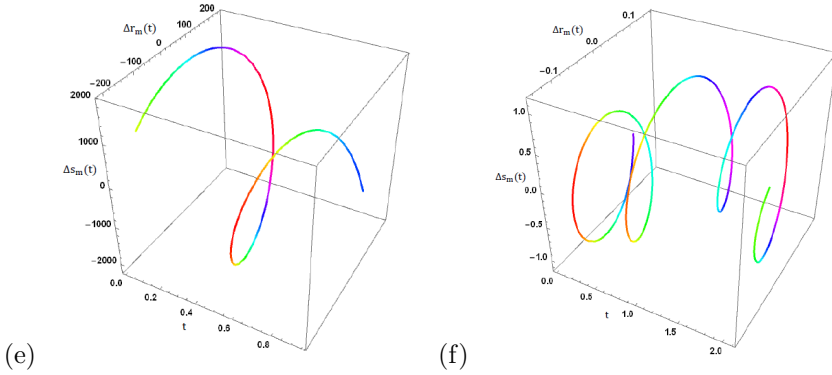


FIGURE 7. Phase space $(t, \Delta r_m, \Delta s_m)$ showing the Poincare section Σ with $\frac{\pi}{\gamma}$ periodicity in t .

Fig. (7) (a - f) describes the phase space $(t, \Delta r_m, \Delta s_m)$ showing poincare section Σ with periodicty $\frac{\pi}{\gamma}$ in t . It is also observed that the flow of differential equation

$$\begin{aligned}
 \Delta \dot{r}_m &= \Delta s_m, & \Delta \dot{s}_m &= P_1 \cos(2\dot{\gamma}t), \\
 \Delta \dot{s}_m &= P_4 \cos(2(\dot{\theta}_E - \dot{\alpha})t), \\
 \Delta \dot{s}_m &= P_5 \cos(2\dot{\alpha}t), \\
 \Delta \dot{s}_m &= P_8 \cos((\dot{\theta}_E - 2\dot{\alpha})t), \\
 \Delta \dot{s}_m &= P_{12} \cos(2(\dot{\theta}_E - \dot{\alpha})t), \\
 \Delta \dot{s}_m &= P_{14} \cos(\dot{\alpha}t),
 \end{aligned}
 \tag{38}$$

for $\Delta s_m > 0$, $\Delta \dot{r}_m \neq 0$ is everywhere transverse to Σ .

7. DISCUSSIONS AND CONCLUSIONS

In this paper, by following the procedure of Frick [5] and taking Moon as the satellite of the Earth, we determined the equations of motion of Moon with the help of gravitational potential of Earth in spherical coordinate system. By defining the following perturbations relative to Moon,

$$\begin{aligned}
 r_m &= r_{em} + \Delta r_m, \\
 \dot{\theta}_m &= \dot{\theta}_E + \Delta \dot{\theta}_m,
 \end{aligned}$$

(r_{em} is the steady state value of orbital radius of Moon), the equations of motion of the Moon are deduced to a non-homogeneous second order differential Equation (37) including equatorial ellipticity of Earth. From the solution, we analyzed that two types of resonance occur,

(1) resonance due to the frequencies $\dot{\gamma}$ and $\dot{\theta}_E$, (2) resonance due to the frequencies $\dot{\alpha}$ and $\dot{\theta}_E$. We have drawn the resonant curve due to the frequencies $\dot{\gamma}$ and $\dot{\theta}_E$ (Fig. 2).

In Fig. 3, we have drawn resonant curve due to the frequencies $\dot{\alpha}$ and $\dot{\theta}_E$. It is clear that the sharpness of the resonance are different at the different resonant points due to the different size of the bandwidth.

In Fig. 4 and 5, we have shown resonant curves at other resonant points by considering different values of ϕ_m (latitude of the Moon). In Fig. 6, we have shown the phase portrait of the orbits of the Moon for unforced and undamped phase plane corresponding to Eq. (37). The orbits of the Moon are elliptical.

In Fig. 7, we have shown the phase space $(t, \Delta r_m, \Delta s_m)$ where $\Delta s_m = \Delta \dot{r}_m$. By analysing the Poincare section $\sum$ with periodicity $\frac{\pi}{\dot{\gamma}}$ in time t . These phase spaces are drawn corresponding to different oscillatory terms of Eq. (37).

APPENDIX A

$$K_1 = \cos \phi_m - \cos \phi_m \cos \epsilon,$$

$$K_2 = \cos \phi_m + \cos \phi_m \cos \epsilon,$$

$$K_3 = 2 \sin \phi_m,$$

$$K_4 = \frac{\cos^2 \phi_m}{2} - \cos^2 \phi_m \cos \epsilon + \frac{\cos^2 \phi_m \cos^2 \epsilon}{2},$$

$$K_5 = \frac{\cos^2 \phi_m}{2} + \cos^2 \phi_m \cos \epsilon + \frac{\cos^2 \phi_m \cos^2 \epsilon}{2},$$

$$K_6 = \cos^2 \phi_m - \cos^2 \phi_m \cos^2 \epsilon,$$

$$K_7 = \cos^2 \phi_m - \cos^2 \phi_m \cos^2 \epsilon - 2 \sin^2 \epsilon \sin^2 \phi_m,$$

$$K_8 = -2 \sin \epsilon \sin \phi_m \cos \phi_m (1 - \cos \epsilon),$$

$$K_9 = 2 \sin \epsilon \sin \phi_m \cos \phi_m (1 + \cos \epsilon),$$

$$K_{10} = -4 \sin \epsilon \sin \phi_m \cos \phi_m \cos \epsilon,$$

$$K_{11} = \cos^2 \phi_m + \cos^2 \phi_m \cos^2 \epsilon + \sin^2 \epsilon \sin^2 \phi_m.$$

APPENDIX B

$$\begin{aligned} M_1 &= \frac{-\cos \phi_m}{8} + \frac{\cos \phi_m \cos \epsilon}{4}, \\ M_2 &= \frac{-\cos \phi_m}{4}, \\ M_3 &= \frac{-\cos \phi_m}{8} - \frac{\cos \phi_m \cos \epsilon}{4}, \\ M_4 &= \frac{\sin \epsilon \cos \epsilon}{4} (1 + \sin \phi_m), \\ M_5 &= \frac{\sin \epsilon \cos \epsilon}{4} (\sin \phi_m - 1), \\ M_6 &= \frac{-3 \cos \phi_m \cos^2 \epsilon}{8}, \\ M_7 &= \frac{\cos \phi_m \cos^2 \epsilon}{8}, \\ M_8 &= \frac{\cos \phi_m \cos^2 \epsilon}{8}, \\ M_9 &= \frac{-\sin \epsilon \cos \epsilon \sin \phi_m}{2}. \end{aligned}$$

APPENDIX C

$$\begin{aligned} P_1 &= \frac{3J_2^{(2)} g_0 R_0^4 \cos^2 \phi_m}{r_{em}^4} \left(\frac{2\dot{\theta}_E}{\dot{\gamma}} - 3 \right) \\ P_2 &= \frac{-3 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_1}{(\dot{\theta}_E + \dot{\alpha})} + \frac{3 r_{em} \dot{\alpha}^2 K_4}{4}, \\ P_3 &= \frac{-3 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_2}{\dot{\theta}_E} + \frac{3 r_{em} \dot{\alpha}^2 K_6}{4}, \\ P_4 &= \frac{-3 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_3}{(\dot{\theta}_E - \dot{\alpha})} + \frac{3 r_{em} \dot{\alpha}^2 K_5}{4}, \end{aligned}$$

$$\begin{aligned}
P_5 &= \frac{3 r_{em} \dot{\alpha}^2 K_7}{4}, \\
P_6 &= \frac{3 r_{em}^2 \dot{\alpha}^2 K_1}{2 R} \left(\frac{\mu - 2}{\mu} \right), \\
P_7 &= \frac{3 r_{em}^2 \dot{\alpha}^2 K_2}{2 R} \left(\frac{\mu - 2}{\mu} \right), \\
P_8 &= \frac{6 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_4}{(\dot{\theta}_E - 2 \dot{\alpha})} + \frac{3 r_{em} \dot{\alpha}^2 K_9}{4}, \\
P_9 &= \frac{6 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_5}{(\dot{\theta}_E + 2 \dot{\alpha})} + \frac{3 r_{em} \dot{\alpha}^2 K_8}{4}, \\
P_{10} &= \frac{3 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_6}{\dot{\theta}_E}, \\
P_{11} &= \frac{3 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_7}{(\dot{\theta}_E + \dot{\alpha})}, \\
P_{12} &= \frac{3 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_8}{(\dot{\theta}_E - \dot{\alpha})}, \\
P_{13} &= \frac{6 r_{em} \dot{\theta}_E \dot{\alpha}^2 \cos \phi_m M_9}{\dot{\theta}_E} + \frac{3 r_{em} \dot{\alpha}^2 K_{10}}{4}, \\
P_{14} &= \frac{-3 r_{em}^2 \dot{\alpha}^2 K_3}{2 R} \left(\frac{\mu - 2}{\mu} \right).
\end{aligned}$$

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