

## ALMOST KENMOTSU MANIFOLDS ADMITTING CERTAIN CRITICAL METRIC

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**ABSTRACT.** The object of this offering article is to introduce the notion of  $*$ -Miao-Tam critical equation on almost contact metric manifolds and it is studied on almost Kenmotsu manifolds with some nullity condition. It is proved that if the metric of a  $(2n + 1)$ -dimensional  $(k, \mu)'$ -almost Kenmotsu manifold  $(M, g)$  satisfies the  $*$ -Miao-Tam critical equation, then the manifold  $(M, g)$  is  $*$ -Ricci flat and locally isometric to a product space. Finally, the result is verified by an example.

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### 1. INTRODUCTION

In differential geometry and mathematical physics, finding Riemannian metrics on a compact orientable manifold  $M$  that provides constant scalar curvature is a very classical problem. The critical points of the total scalar curvature functional  $\mathcal{C} : \mathcal{M} \rightarrow \mathbb{R}$  plays an important role in this problem, which is defined on a compact

orientable manifold  $(M, g)$  restricted to the set of all Riemannian metrics  $\mathcal{H}$  of unit volume as follows:

$$\mathcal{C}(g) = \int_{\mathcal{H}} r_g dv_g,$$

where  $r_g$  is the scalar curvature of  $g$  and  $dv_g$  denotes the elementary volume form of  $g$ . The critical points of  $\mathcal{C}$  are Einstein (see [1]).

Let  $(M, g)$  be a compact orientable Riemannian manifold of unit volume and  $g^*$  be any symmetric bilinear form on  $M$ . Then the linearization of the scalar curvature  $\mathcal{L}_g(g^*)$  is given by

$$\mathcal{L}_g(g^*) = -\Delta_g(\text{tr}_g g^*) + \text{div}(\text{div}(g^*)) - g(g^*, S_g),$$

where  $\Delta_g$  denotes the Laplacian of  $g$  and  $S_g$  is its Ricci tensor. The  $L^2$ -adjoint  $\mathcal{L}_g^*$  of  $\mathcal{L}_g$  is defined as

$$\mathcal{L}_g^*(\lambda) = -(\Delta_g \lambda)g + \nabla_g^2 \lambda - \lambda S_g,$$

where  $(\nabla_g^2 \lambda)(E, F) = \text{Hess}_g \lambda(E, F) = g(\nabla_E D\lambda, F)$  and  $D$  denotes the gradient operator.

**Definition 1.1.** *On a Riemannian manifold  $(M, g)$  of dimension  $n > 2$  if there exist a non-zero smooth function  $\lambda$  such that  $\mathcal{L}_g^*(\lambda) = g$ , that is,*

$$(1.1) \quad -(\Delta_g \lambda)g + \nabla_g^2 \lambda - \lambda S_g = g,$$

*then  $g$  is said to satisfy the Miao-Tam critical equation.*

In the above definition, the solution of the Miao-Tam critical equation are Einstein metrics when the potential function  $\lambda$  is a non-zero constant. In [13], the authors proved that any Riemannian metric  $g$  satisfying (1.1) have constant scalar curvature. In [14], Patra and Ghosh considered the Miao-Tam critical equation on a complete  $K$ -contact manifold. In 2017, Wang [17] studied this notion on three dimensional almost Kenmotsu manifold and proved that if the metric of a  $(k, \mu)'$ -almost Kenmotsu 3-manifold satisfies the Miao-Tam critical equation, then the manifold is locally isometric to  $\mathbb{H}^3(-1)$ .

In 1959, the notion of  $\ast$ -Ricci tensor was introduced by Tachibana[16] on a class of almost complex manifolds. Later, Hamada [10] defined the  $\ast$ -Ricci tensor of certain space form by

$$(1.2) \quad S^*(E, F) = g(Q^*E, F) = \frac{1}{2}(\text{trace}\{\varphi \circ R(E, \varphi F)\})$$

for any vector fields  $E, F$  on  $M$ , where  $Q^*$  denotes the  $(1, 1)$   $\ast$ -Ricci operator. We denote the  $\ast$ -scalar curvature by  $r^*$  and is given by  $r^* = \text{trace}(Q^*)$ .

**Definition 1.2.** *If the  $\ast$ -Ricci tensor  $S^*$  of an almost contact metric manifold  $(M, g)$  vanishes identically, then  $(M, g)$  is called  $\ast$ -Ricci flat.*

Over the last decade, geometers and mathematical physicists developed several notions related to the  $\ast$ -Ricci tensor. In 2016, the notion of  $\ast$ -Ricci soliton ([11]) was introduced. Later in 2019, the notion of  $\ast$ -critical point equation [6] was introduced. In 2021, the present author together with Majhi introduced the notion of  $\ast$ -gradient  $\rho$ -Einstein soliton [8] and  $\ast$ -conformal Ricci soliton [12]. In this paper, we introduce the notion of  $\ast$ -Miao-Tam critical equation on an almost contact metric manifold as follows:

**Definition 1.3.** *The metric of an almost contact metric manifold  $(M, g)$  of dimension  $n \geq 3$  is said to satisfy the  $\ast$ -Miao-Tam critical equation if there is a non-zero smooth function  $\lambda : M \rightarrow \mathbb{R}$  satisfying*

$$(1.3) \quad -(\Delta_g \lambda)g + \nabla_g^2 \lambda - \lambda S^* = g,$$

where  $S^*$  is the  $(0, 2)$   $\ast$ -Ricci tensor of  $g$ , provided  $S^*$  is symmetric.

Note that, the  $\ast$ -Ricci tensor is not symmetric in general. In a  $(k, \mu)'$ -almost Kenmotsu manifold, the  $\ast$ -Ricci tensor  $S^*$  is symmetric(given later) and hence the above definition is well defined in  $(k, \mu)'$ -almost Kenmotsu manifolds. Further, the  $\ast$ -Ricci tensor is also symmetric in different kind of almost contact metric manifolds. This gives us a wide range to study this notion in contact geometry.

The paper is constructed as follows: In section 2, we give some basic concept of  $(k, \mu)'$ -almost Kenmotsu manifolds. Section 3 deals with  $\ast$ -Miao-Tam critical equations on  $(k, \mu)'$ -almost Kenmotsu manifolds. In section 4, an example is presented to support the result.

## 2. PRELIMINARIES

A differentiable manifold  $M$  of odd dimension is said to have an almost contact structure  $(\varphi, \xi, \eta)$  if it satisfies([2], [3]),

$$(2.1) \quad \varphi^2 E = -E + \eta(E)\xi, \quad \eta(\xi) = 1,$$

where  $\varphi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field and  $\eta$  is a 1-form on  $M$ . Here  $\varphi\xi = 0$  and  $\eta \circ \varphi = 0$ ; both can be derived from (2.1) easily.

A manifold  $M$  with an almost contact structure is said to be an almost contact metric manifold if it admits a Riemannian metric  $g$  such that

$$g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F)$$

for any vector fields  $E, F$  on  $M$ . Irrespective of the structure, we simply denote it as  $(M, g)$ . An almost Kenmotsu manifold is an almost contact metric manifold with closed  $\eta$  and  $d\Phi = 2\eta \wedge \Phi$  (see [9], [15]), where  $\Phi$  is the fundamental 2-form of  $(M, g)$  defined by  $\Phi(E, F) = g(E, \varphi F)$  for any vector fields  $E, F$  on  $M$ .

Let  $\mathcal{D}$  denote the distribution orthogonal to  $\xi$  and is defined by  $\mathcal{D} = \text{Kernel}(\eta) = \text{Image}(\varphi)$ .

Let  $(M, g)$  be a  $(2n + 1)$ -dimensional almost Kenmotsu manifold. We denote by  $h = \frac{1}{2}\mathcal{L}_\xi\varphi$  and  $l = R(\cdot, \xi)\xi$  on  $(M, g)$ . The operators  $l$  and  $h$  are symmetric. For a  $(2n + 1)$ -dimensional almost Kenmotsu manifold, the following relations are well known[15]:

$$(2.2) \quad h\xi = 0, \quad l\xi = 0, \quad \text{tr}(h) = 0, \quad \text{tr}(h \circ \varphi) = 0, \quad h \circ \varphi + \varphi \circ h = 0,$$

$$(2.3) \quad \nabla_E \xi = E - \eta(E)\xi - \varphi hE (\Rightarrow \nabla_\xi \xi = 0),$$

$$(2.4) \quad \varphi l\varphi E - lE = 2(h^2 - \varphi^2)E,$$

$$(2.5) \quad R(E, F)\xi = \eta(E)(F - \varphi hF) - \eta(F)(E - \varphi hE) + (\nabla_F \varphi h)E - (\nabla_E \varphi h)F,$$

for any vector fields  $E, F$ .

In 2009, Dileo and Pastore [9] introduced the notion of  $(k, \mu)'$ -nullity distribution

on an almost Kenmotsu manifold  $(M, g)$  as follows:

$$(2.6) \quad \begin{aligned} N_p(k, \mu)' = \{Z \in T_p(M) : R(E, F)Z &= k[g(F, Z)E - g(E, Z)F] \\ &+ \mu[g(F, Z)h'E - g(E, Z)h'F]\}, \end{aligned}$$

where  $p \in M$ ,  $k, \mu \in \mathbb{R}$  and  $h' = h \circ \varphi$ . If the characteristic vector field  $\xi$  of an almost Kenmotsu manifold belongs to the  $(k, \mu)'$ -nullity distribution, then it is called a  $(k, \mu)'$ -almost Kenmotsu manifold. For a  $(k, \mu)'$ -almost Kenmotsu manifold, we have from (2.6)

$$(2.7) \quad R(E, F)\xi = k[\eta(F)E - \eta(E)F] + \mu[\eta(F)h'E - \eta(E)h'F],$$

where  $k, \mu \in \mathbb{R}$ . Again from (2.7), we get

$$(2.8) \quad R(\xi, E)F = k[g(E, F)\xi - \eta(F)E] + \mu[g(h'E, F)\xi - \eta(F)h'E].$$

Using (2.3), we have

$$(2.9) \quad (\nabla_E \eta)F = g(E, F) - \eta(E)\eta(F) + g(h'E, F).$$

The symmetric tensor field  $h' = h \circ \varphi$  is anti-commuting with  $\varphi$  and  $h'\xi = 0$ . Also it is clear that ([9], [18])

$$(2.10) \quad h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k+1)\varphi^2 (\Leftrightarrow h^2 = (k+1)\varphi^2).$$

Let  $E \in \mathcal{D}$  be such that  $h'E = \alpha E$ . Then from (2.10), it is clear that  $\alpha^2 = -(k+1)$ , a constant. Therefore  $k \leq -1$  and  $\alpha = \pm\sqrt{-k-1}$ . Let  $[\alpha]'$  and  $[-\alpha]'$  denotes the eigen spaces related to the non-zero eigenvalues  $\alpha$  and  $-\alpha$  of  $h'$ , respectively. In [9], Dileo and Pastore proved for a  $(2n+1)$  dimensional  $(k, \mu)'$ -almost Kenmotsu manifold  $M$  with  $h' \neq 0$  that  $k < -1$ ,  $\mu = -2$  and  $\text{Spectrum}(h') = \{0, \alpha, -\alpha\}$ , where 0 is a simple eigenvalue and  $\alpha = \sqrt{-k-1}$ .

We refer the reader to go through the references ([5], [9], [15]) for further details on  $(k, \mu)'$ -almost Kenmotsu manifolds.

### 3. \*-Miao-Tam critical equation

In this section, we study the notion of \*-Miao-Tam critical equation in the framework of  $(k, \mu)'$ -almost Kenmotsu manifolds. To prove the main theorem, we need the following lemmas:

**Lemma 3.1.** ([4]) *On a  $(k, \mu)'$ -almost Kenmotsu manifold with  $k < -1$ , the  $*$ -Ricci tensor is given by*

$$(3.1) \quad S^*(E, F) = -(k+2)(g(E, F) - \eta(E)\eta(F))$$

for any vector fields  $E, F$ .

Note that,  $S^*$  is symmetric here and hence the notion of  $*$ -Miao-Tam critical equation is well defined in this setting.

**Lemma 3.2.** *If the metric of an almost contact metric manifold  $(M, g)$  of dimension  $n \geq 3$  satisfies the  $*$ -Miao-Tam critical equation, then the curvature tensor  $R$  can be expressed as*

$$(3.2) \quad \begin{aligned} R(E, F)D\lambda &= (Ef)F - (Ff)E + (E\lambda)Q^*F - (F\lambda)Q^*E \\ &\quad + \lambda[(\nabla_E Q^*)F - (\nabla_F Q^*)E] \end{aligned}$$

for any vector fields  $E$  and  $F$  on  $M$ , where  $f = -\frac{r^*\lambda+n}{n-1}$ .

*Proof.* Tracing (1.3), we obtain  $\Delta_g \lambda = -\frac{r^*\lambda+n}{n-1} = f$ .

Therefore, equation (1.3) can be written as

$$(3.3) \quad \nabla_E D\lambda = (1+f)E + \lambda Q^*E.$$

Differentiating the above equation covariantly along any vector field  $F$ , we obtain

$$(3.4) \quad \nabla_F \nabla_E D\lambda = (Ff)E + (1+f)\nabla_F E + (F\lambda)Q^*E + \lambda \nabla_F Q^*E.$$

Interchanging  $E$  and  $F$  in (3.4) yields

$$(3.5) \quad \nabla_E \nabla_F D\lambda = (Ef)F + (1+f)\nabla_E F + (E\lambda)Q^*F + \lambda \nabla_E Q^*F.$$

Also, from (3.3), we have

$$(3.6) \quad \nabla_{[E, F]} D\lambda = (1+f)(\nabla_E F - \nabla_F E) + \lambda Q^*(\nabla_E F - \nabla_F E).$$

It is well known that,

$$R(E, F)D\lambda = \nabla_E \nabla_F D\lambda - \nabla_F \nabla_E D\lambda - \nabla_{[E, F]} D\lambda.$$

We now complete the proof by substituting (3.4)-(3.6) in the foregoing equation.  $\square$

**Lemma 3.3.** *The  $*$ -Ricci operator  $Q^*$  of a  $(2n + 1)$ -dimensional  $(k, \mu)'$ -almost Kenmotsu manifold  $M$  satisfy the following relation:*

$$(3.7) \quad \begin{aligned} (\nabla_E Q^*)F - (\nabla_F Q^*)E &= (k + 2)[\eta(F)(E - \eta(E)\xi - \varphi hE) \\ &\quad - \eta(E)(F - \eta(F)\xi - \varphi hF)] \end{aligned}$$

for any vector fields  $E, F$  on  $M$ .

*Proof.* From (3.1), we have

$$(3.8) \quad Q^*E = -(k + 2)(E - \eta(E)\xi).$$

Differentiating (3.8) covariantly along any vector field  $F$  and using (2.3), we obtain

$$(3.9) \quad \nabla_F Q^*E = -(k + 2)[\nabla_F E - (\nabla_F \eta(E))\xi - \eta(E)(F - \eta(F)\xi - \varphi hF)].$$

Now, with the help of (3.8), (3.9) and (2.9), we calculate

$$(3.10) \quad \begin{aligned} (\nabla_F Q^*)E &= \nabla_F Q^*E - Q^*(\nabla_F E) \\ &= (k + 2)[(g(E, F) - \eta(E)\eta(F) + g(h'E, F))\xi \\ &\quad + \eta(E)(F - \eta(F)\xi - \varphi hF)]. \end{aligned}$$

In a similar manner, we obtain

$$(3.11) \quad \begin{aligned} (\nabla_E Q^*)F &= (k + 2)[(g(E, F) - \eta(E)\eta(F) + g(h'E, F))\xi \\ &\quad + \eta(F)(E - \eta(E)\xi - \varphi hE)]. \end{aligned}$$

Now, subtracting (3.10) from (3.11), we obtain our desired result.  $\square$

**Theorem 3.4.** *If the metric of a  $(2n + 1)$ -dimensional  $(k, \mu)'$ -almost Kenmotsu manifold  $M$  satisfies the  $*$ -Miao-Tam critical equation, then the manifold  $M$  is  $*$ -Ricci flat and locally isometric to  $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ .*

*Proof.* Tracing (3.1), we have  $r^* = -2n(k + 2)$ .

Using lemma 3.2 and lemma 3.3, for a  $(2n + 1)$ -dimensional  $(k, \mu)'$ -almost Kenmotsu manifold  $M$ , we can write

$$\begin{aligned} R(E, F)D\lambda &= (Ef)F - (Ff)E + (E\lambda)Q^*F - (F\lambda)Q^*E \\ &\quad + (k + 2)[\eta(F)(E - \eta(E)\xi - \varphi hE) - \eta(E)(F - \eta(F)\xi - \varphi hF)], \end{aligned}$$

where  $f = -\frac{2n+1-2n(k+2)\lambda}{2n}$ .

Replacing  $E$  by  $\xi$  in the above equation yields

$$R(\xi, F)D\lambda = (\xi f)F - (Ff)\xi + (\xi\lambda)Q^*F - (k+2)(F - \eta(F)\xi - \varphi hF).$$

Taking inner product of the foregoing equation with  $E$ , we have

$$\begin{aligned} g(R(\xi, F)D\lambda, E) &= (\xi f)g(E, F) - (Ff)\eta(E) + (\xi\lambda)S^*(E, F) \\ (3.12) \quad &\quad - (k+2)[g(E, F) - \eta(E)\eta(F) + g(h'E, F)]. \end{aligned}$$

Again, since  $g(R(\xi, F)D\lambda, E) = -g(R(\xi, F)E, D\lambda)$ , then using (2.8), we obtain

$$\begin{aligned} g(R(\xi, F)D\lambda, E) &= -kg(E, F)(\xi\lambda) + k\eta(E)(F\lambda) \\ (3.13) \quad &\quad + 2g(h'E, F)(\xi\lambda) - 2\eta(E)((h'F)\lambda). \end{aligned}$$

Equating (3.12) and (3.13) and then antisymmetrizing yields

$$\begin{aligned} (Ef)\eta(F) - (Ff)\eta(E) &= k\eta(E)(F\lambda) - k\eta(F)(E\lambda) \\ (3.14) \quad &\quad - 2\eta(E)((h'F)\lambda) + 2\eta(F)((h'E)\lambda). \end{aligned}$$

Since  $f = -\frac{2n+1-2n(k+2)\lambda}{2n}$ , then  $(Ef) = (k+2)(E\lambda)$  for any vector field  $E$ . Hence, equation (3.14) reduces to

$$(3.15) \quad 2(k+1)[\eta(F)(E\lambda) - \eta(E)(F\lambda)] + 2\eta(E)((h'F)\lambda) - 2\eta(F)((h'E)\lambda) = 0.$$

Setting  $F = \xi$  in (3.15), we infer that

$$(k+1)[(E\lambda) - (\xi\lambda)\eta(E)] - ((h'E)\lambda) = 0,$$

which implies

$$(3.16) \quad (k+1)[D\lambda - (\xi\lambda)\xi] - h'(D\lambda) = 0.$$

Operating  $h'$  on the above equation and using (2.10), we obtain

$$(3.17) \quad h'(D\lambda) = -[D\lambda - (\xi\lambda)\xi].$$

Substituting (3.17) in (3.16) yields

$$(k+2)[D\lambda - (\xi\lambda)\xi] = 0,$$

which implies either  $k = -2$  or  $D\lambda = (\xi\lambda)\xi$ .

Case 1: : If  $k = -2$ , then using  $\alpha^2 = -k - 1$ , we get  $\alpha = \pm 1$ . Hence, we can say that the  $(2n + 1)$ -dimensional  $(k, \mu)'$ -almost Kenmotsu manifold  $M$  is locally isometric to  $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$  (see [8]). Again from (3.1),  $k = -2$  implies  $S^* = 0$  and therefore the manifold is  $*$ -Ricci flat.

Case 2: If  $D\lambda = (\xi\lambda)\xi$ , then differentiating this covariantly along any vector field  $E$ , we obtain

$$(3.18) \quad \nabla_E D\lambda = (E(\xi\lambda))\xi + (\xi\lambda)(E - \eta(E)\xi - \varphi hE).$$

Equating (3.3) and (3.18), we obtain

$$(3.19) \quad Q^*E = \frac{1}{\lambda}((\xi\lambda) - 1 - f)E + \frac{1}{\lambda}(E(\xi\lambda) - (\xi\lambda)\eta(E))\xi - \frac{1}{\lambda}(\xi\lambda)\varphi hE.$$

Now, comparing the coefficients of  $E$ ,  $\xi$  and  $\varphi hE$  from (3.19) and (3.8), we get

$$(3.20) \quad (\xi\lambda) - 1 - f = -(k + 2)\lambda.$$

$$(3.21) \quad E(\xi\lambda) - (\xi\lambda)\eta(E) = (k + 2)\lambda.$$

$$(3.22) \quad (\xi\lambda) = 0.$$

Using (3.22) in (3.21), we get  $k = -2$  as  $\lambda$  is a non-zero smooth function. Therefore, from (3.20), we have  $f = -1$ . Using  $k = -2$  in  $f = -1$ , we will arrive at a contradiction.

This completes the proof.  $\square$

#### 4. Example of a $(k, \mu)'$ -almost Kenmotsu manifold

We consider the example of a  $(2n + 1)$ -dimensional  $(k, \mu)'$ -almost Kenmotsu manifold presented in [9]. For 5-dimensional case,  $k = -2$  (see [7]). This implies that the manifold is locally isometric to  $\mathbb{H}^3(-4) \times \mathbb{R}^2$ .

Again since  $k = -2$ , by (3.1), the manifold is  $*$ -Ricci flat, that is,  $S^* = 0$ . Now tracing (1.3), we obtain  $\Delta_g \lambda = -\frac{5}{4}$ . Therefore,  $(g, \lambda)$  is a solution of the  $*$ -Miao-Tam critical equation, where  $\lambda$  is given by the above Poisson equation. Hence, the theorem is verified.

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