

TRIHARMONIC CURVES IN HEISENBERG GROUP $\mathcal{H}_3$

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ABSTRACT. In this paper, we study triharmonic curves in Heisenberg group $\mathcal{H}_3$. We give necessary and sufficient conditions for helices to be triharmonic. We characterize the triharmonic curves in terms of their curvature and torsion.

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1. INTRODUCTION

In the theory of curves in a Riemannian manifolds, one of the important and interesting problem is characterizations of a regular curve.

First we should recall some notions and results related to the harmonic and the Polyharmonic (r – harmonic $r \geq 1$) maps between Riemannian manifolds.

Harmonic maps $\psi : (\mathcal{M}, g) \rightarrow (\mathcal{N}, \tilde{g})$ between Riemannian manifolds are the critical points of the energy functional

$$\mathcal{E}_1 : C^\infty(\mathcal{M}, \mathcal{N}) \rightarrow \mathbb{R}, \quad \mathcal{E}_1(\psi) = \frac{1}{2} \int_{\mathcal{M}} |d\psi|^2 v_g,$$

and is characterized by the vanishing of the first tension field

$$\tau_1(\psi) = -d^*\mathrm{d}\psi = \mathrm{trace}\nabla\mathrm{d}\psi,$$

where d is the exterior differentiation and d^* is the codifferentiation.

We remind that the bienergy of ψ is given by

$$\mathcal{E}_2 : C^\infty(\mathcal{M}, \mathcal{N}) \rightarrow \mathbb{R}, \quad \mathcal{E}_2(\psi) = \frac{1}{2} \int_{\mathcal{M}} |\tau(\psi)|^2 v_g,$$

and the bitension field $\tau_2(\psi)$ has the expression

$$\tau_2(\psi) = -\Delta^\psi \tau(\psi) - \text{trace}_g R^{\mathcal{N}}(d\psi, \tau(\psi))d\psi,$$

where $\Delta^\psi = -\text{trace}(\nabla^\psi)^2 = -\text{trace}(\nabla^\psi \nabla^\psi - \nabla_{\nabla^\psi}^\psi)$.

A smooth map ψ is *biharmonic* if it satisfies the following biharmonic equation

$$\tau_2(\psi) = 0.$$

Biharmonic maps are the critical points of the bienergy functional $\mathcal{E}_2$.

We call proper biharmonic the non-harmonic biharmonic maps. Biharmonic curves ψ of a Riemannian manifold are the solutions of the fourth order differential equation

$$(1) \quad \nabla_{\phi'}^3 \phi' - R(\phi', \nabla_{\phi'} \phi') \phi' = 0.$$

Eells and Lemaire [6] proposed the problem to consider the polyharmonic (r -harmonic $r \geq 1$) maps of order r , these are critical points of the r -energy functional defined by

$$(2) \quad \mathcal{E}_r(\psi) = \int_{\mathcal{M}} e_r(\psi) v_g, \quad r \geq 1,$$

where $e_r(\psi) = \frac{1}{2} \|(d + d^*)^r \psi\|^2$ for smooth maps ψ .

A map ψ is r -harmonic if it is a critical point of the functional $\mathcal{E}_r(\psi)$ defined in (2).

Every harmonic map is a solution of the polyharmonic map, see [3] for a recent classification result. In [13], S.B. Wang studied the first variation formula of the k -energy $\mathcal{E}_k$, whose critical maps are called k -harmonic maps. In [10], S. Maeta showed the second variation formula of the k -energy. Triharmonic curves with constant curvature in space forms were studied by Maeta in [10].

The aim of this paper is to continue the study of curves, more precisely triharmonic curves. We characterize the triharmonic curves in terms of their curvature and torsion.

2. PRELIMINARIES

We consider the Heisenberg group

$$\mathcal{H}_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, (x, y, z) \in \mathbb{R}^3 \right\}$$

with the group multiplication induced by the standard matrix product.

We can define the Lie group structure on $\mathbb{R}^3$ by a multiplication law as follows:

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x}, y + \bar{y}, z + \bar{z} + \frac{1}{2}x\bar{y} - \frac{1}{2}y\bar{x}).$$

The orthonormal basis

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of the tangent space at the identity, determines on $\mathcal{H}_3$ a left-invariant metric

$$g_{\mathcal{H}_3} = ds_{\mathcal{H}_3}^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2,$$

where

$$\omega^1 = dx, \quad \omega^2 = dy, \quad \omega^3 = dz + \frac{1}{2}(ydx - xdy)$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame

$$e_1 = \partial_x - \frac{y}{2}\partial_z, \quad e_2 = \partial_y + \frac{x}{2}\partial_z, \quad e_3 = \partial_z,$$

where $\left\{ \partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \partial_z = \frac{\partial}{\partial z} \right\}$ denote the canonical vector fields in $\mathbb{R}^3$.

The corresponding Lie brackets are

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

The Levi-Civita connection $\bar{\nabla}$ of $\mathcal{H}_3$ is given by

$$(3) \quad \begin{aligned} \begin{pmatrix} \nabla_{e_1} e_1 \\ \nabla_{e_1} e_2 \\ \nabla_{e_1} e_3 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ \begin{pmatrix} \nabla_{e_2} e_1 \\ \nabla_{e_2} e_2 \\ \nabla_{e_2} e_3 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \\ \begin{pmatrix} \nabla_{e_3} e_1 \\ \nabla_{e_3} e_2 \\ \nabla_{e_3} e_3 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \end{aligned}$$

Let $m = (x, y, z)$ be a point in $\mathcal{H}_3$. For an arbitrary tangent vector ξ at m , if

$$\xi = \zeta_1 \frac{\partial}{\partial x} + \zeta_2 \frac{\partial}{\partial y} + \zeta_3 \frac{\partial}{\partial z},$$

then

$$\xi = \zeta_1 e_1 + \zeta_2 e_2 + \left(\zeta_3 + \frac{1}{2}(\zeta_1 y - \zeta_2 x)\right) e_3.$$

The complete classification of $\mathcal{H}_3$ spaces is as follows: (see for instance [8]).

We shall adopt the following notation and sign convention. The Riemannian curvature operator is given by

$$(4) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

The Riemannian curvature tensor is given by

$$(5) \quad R(X, Y, Z, W) = g_{\mathcal{H}_3}(R(Y, X)Z, W) = -g_{\mathcal{H}_3}(R(X, Y)Z, W),$$

where X, Y, Z, W are smooth vector fields on $\mathcal{H}_3$.

The nonvanishing components of the Riemann-Christoffel are

$$(6) \quad R_{1212} = -\frac{3}{4}, \quad R_{1313} = \frac{1}{4} = R_{2323}.$$

3. POLYHARMONIC CURVES IN $\mathcal{H}_3$

3.1. Biharmonic curves in $\mathcal{H}_3$. In [5] and [7], the authors studied biharmonic curves in BCV-spaces and they obtained interesting classification results. In [4] the authors proved that any proper biharmonic curve in the Heisenberg group $\mathcal{H}_3$ is a helix and gave its explicit parametrization.

Let $\psi : I \rightarrow \mathcal{H}_3$ be a differentiable curve parametrized by arc length and let $\{T, N, B\}$ be the orthonormal moving Frenet frame along the curve ψ in $\mathcal{H}_3$ such that $T = \psi'$ is the unit vector field tangent to ψ , N is the unit vector field in the direction $\nabla_T T$ normal to ψ (principal normal) and $B = T \wedge N$ (binormal vector). Then we have the following Frenet equations

$$(7) \quad \begin{pmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where

$$k^2 = g_{\mathcal{H}_3}(\nabla_T T, \nabla_T T),$$

is the curvature of ψ , τ is its torsion and

$$g_{\mathcal{H}_3}(T, T) = g_{\mathcal{H}_3}(N, N) = g_{\mathcal{H}_3}(B, B) = 1$$

$$g_{\mathcal{H}_3}(T, N) = g_{\mathcal{H}_3}(T, B) = g_{\mathcal{H}_3}(N, B) = 0.$$

The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are respectively known as the osculating, the rectifying and the normal plane.

From (7) we have

$$(8) \quad \nabla_T^3 T = (-3kk')T + (k'' - k^3 - k\tau^2)N + (2k'\tau + k\tau')B,$$

where $k' = \frac{dk}{ds}$, $k'' = \frac{d^2k}{ds^2}$, $\tau' = \frac{d\tau}{ds}$. Writing

$$(9) \quad \begin{cases} T = T_1 e_1 + T_2 e_2 + T_3 e_3 \\ N = N_1 e_1 + N_2 e_2 + N_3 e_3 \\ B = T \wedge N = B_1 e_1 + B_2 e_2 + B_3 e_3, \end{cases}$$

and using (6), we have that

$$(10) \quad \begin{cases} R(T, N, T, N) = \sum_{1 \leq i, j, k, l \leq 3} T_i N_j T_k N_l R_{ijkl} = -B_3^2 + \frac{1}{4} \\ R(T, N, T, B) = \sum_{1 \leq i, j, k, l \leq 3} T_i N_j T_k B_l R_{ijkl} = N_3 B_3 \\ R(B, N, T, N) = \sum_{1 \leq i, j, k, l \leq 3} B_i N_j T_k N_l R_{ijkl} = T_3 B_3 \\ R(B, N, T, B) = \sum_{1 \leq i, j, k, l \leq 3} B_i N_j T_k B_l R_{ijkl} = -T_3 N_3 \\ R(B, T, T, B) = \sum_{1 \leq i, j, k, l \leq 3} B_i T_j T_k B_l R_{ijkl} = N_3^2 - \frac{1}{4}. \end{cases}$$

Theorem 3.1 ([4]). *Let $\psi : I \rightarrow \mathcal{H}_3$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic biharmonic curve if and only if*

$$(11) \quad \begin{cases} k = \text{constant} \neq 0 \\ k^2 + \tau^2 = -B_3^2 + \frac{1}{4} \\ \tau' = N_3 B_3. \end{cases}$$

3.2. Triharmonic curves in $\mathcal{H}_3$. To study the triharmonic curves in $\mathcal{H}_3$, we shall use their Frenet vector fields and equations.

Let us denote by $\psi : I \rightarrow \mathcal{H}_3$ an arclength parametrized curve in $\mathcal{H}_3$. Assume that ψ is non-geodesic.

If $r = 2t$, $t \geq 1$, then (2) takes the form [9, 13]

$$(12) \quad \mathcal{E}_{2t}(\psi) = \frac{1}{2} \int_{\mathcal{M}} < \underbrace{(d^*d) \dots (d^*d) \psi}_{t \text{ times}}, \underbrace{(d^*d) \dots (d^*d) \psi}_{t \text{ times}} > v_g.$$

If $r = 2t + 1$, then (2) takes the form

$$(13) \quad \mathcal{E}_{2t+1}(\psi) = \frac{1}{2} \int_{\mathcal{M}} < \underbrace{d(d^*d) \dots (d^*d) \psi}_{t \text{ times}}, \underbrace{d(d^*d) \dots (d^*d) \psi}_{t \text{ times}} > v_g.$$

The Euler-Lagrange equations of (12) and (13), reduces to the equation

$$(14) \quad \tau_r(\psi) = \nabla_T^{2r-1} T + \sum_{t=0}^{r-1} (-1)^t R(\nabla_T^{2r-3-t} T, \nabla_T^t T) T, \quad r \geq 1.$$

Solutions of $\tau_r(\psi) = 0$ are called r - harmonic curves.

Remark 3.2. *We say that a r - harmonic curve is proper if it is not harmonic. Any harmonic curve is a r - harmonic curve, for any $r \geq 1$.*

An arc-length parametrized curve $\psi : I \rightarrow \mathcal{M}^n$ from $I \subset \mathbb{R}$ to a Riemannian manifold $\mathcal{M}^n$ of dimension n is called *triharmonic* if [11]

$$(15) \quad \nabla_T^5 T + R(\nabla_T^3 T, T)T - R(\nabla_T^2 T, \nabla_T T)T = 0.$$

Proposition 3.3. *Let $\psi : I \subset \mathbb{R} \rightarrow \mathcal{M}^3$ be a differentiable curve parametrized by arc length. Then ψ is triharmonic curve if and only if*

$$(16) \quad \begin{cases} \rho_1(s) = 0 \\ \rho_2(s) - \rho_4(s)R(N, T, T, N) - \rho_5(s)R(B, T, T, N) + \rho_6(s)R(B, N, T, N) = 0 \\ \rho_3(s) - \rho_4(s)R(N, T, T, B) - \rho_5(s)R(B, T, T, B) + \rho_6(s)R(B, N, T, B) = 0, \end{cases}$$

where

$$\begin{aligned} \rho_1(s) &= -10k'k'' - 5kk^{(3)} + 5kk'(2k^2 + \tau^2) + 5k^2\tau\tau' \\ \rho_2(s) &= k^5 + k^{(4)} - 15kk'^2 - 10k^2k'' + k\tau^2(2k^2 + \tau^2) - 6\tau^2k'' \\ &\quad - 12k'\tau\tau' - 3k\tau'^2 - 4k\tau\tau'' \\ \rho_3(s) &= 4\tau k^{(3)} + k\tau^{(3)} - 9k^2k'\tau - 4k'(\tau^3 - \tau'') + 6\tau'(k'' - k\tau^2) - \tau'k^3 \\ \rho_4(s) &= k'' - k(2k^2 + \tau^2) \\ \rho_5(s) &= 2k'\tau + k\tau', \quad \rho_6(s) = k^2\tau. \end{aligned}$$

Proof. From (7) we have

$$\begin{aligned} \nabla_T^2 T &= (-k^2)T + (k')N + (k\tau)B, \\ \nabla_T^3 T &= (-3kk')T + (k'' - k(k^2 + \tau^2))N + (2k'\tau + k\tau')B, \\ \nabla_T^5 T &= \rho_1(s)T + \rho_2(s)N + \rho_3(s)B. \end{aligned}$$

By (15) we see that ψ is a triharmonic curve if and only if

$$(17) \quad \rho_1(s)T + \rho_2(s)N + \rho_3(s)B + \rho_4(s)R(N, T)T + \rho_5(s)R(B, T)T - \rho_6(s)R(B, N)T = 0.$$

Using (5), we have (16).

□

Theorem 3.4. *Let $\psi : I \rightarrow \mathcal{H}_3$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic triharmonic curve if and only if*

$$(18) \quad \begin{cases} \rho_1(s) = 0 \\ \rho_2(s) - \rho_4(s)(B_3^2 - \frac{1}{4}) + \rho_5(s)N_3B_3 + \rho_6(s)T_3B_3 = 0 \\ \rho_3(s) + \rho_4(s)N_3B_3 - \rho_5(s)(N_3^2 - \frac{1}{4}) - \rho_6(s)T_3N_3 = 0. \end{cases}$$

Proof. Combining (10) and (16), it is obtained (18). $\square$

Corollary 3.5. *If $\tau = 0$ and $N_3B_3 \neq 0$. Then, $k = 0$.*

Proof. From (18), we obtain

$$\begin{cases} \rho_1(s) = 0 \\ \rho_2(s) - (B_3^2 - \frac{1}{4})(k'' - 2k^3) = 0 \\ N_3B_3(k'' - 2k^3) = 0. \end{cases}$$

Then

$$(19) \quad \begin{cases} 2k'k'' + kk^{(3)} - 2k^3k' = 0 \\ k^5 + k^{(4)} - 15kk'^2 - 10k^2k'' = 0 \\ k'' = 2k^3. \end{cases}$$

Solving system of equations (19) we have $k = 0$. $\square$

4. TRIHARMONIC HELICES IN $\mathcal{H}_3$

Circular helix is a geometric curve with non-vanishing constant curvature and non-vanishing constant torsion. Now, for any helix in $\mathcal{H}_3$, the system (18) becomes

$$(20) \quad \begin{cases} (k^2 + \tau^2)^2 + (2k^2 + \tau^2)(B_3^2 - \frac{1}{4}) + k\tau T_3B_3 = 0 \\ (\tau k T_3 + (2k^2 + \tau^2)B_3)N_3 = 0. \end{cases}$$

Corollary 4.1. *Let $\psi : I \rightarrow \mathcal{H}_3$ be a non-geodesic triharmonic curve parametrized by arc length. If $k = \text{constant} \neq 0$, then ψ is a circular helix.*

Proof. From the first equation of (18), we obtain $\tau = \text{constant}$. Then ψ is a circular helix. $\square$

Theorem 4.2. *Let $\psi : I \rightarrow \mathcal{H}_3$ be a non-geodesic triharmonic helix parametrized by arc length. If $N_3 = 0$, then $T_3 = \text{constant}$.*

Proof. Let ψ be a non geodesic curve parametrized by arc length. Put $B = B_1e_1 + B_2e_2 + B_3e_3$. Then

$$\begin{aligned}\nabla_TB &= (B'_1 + \frac{1}{2}(T_3B_2 + T_2B_3))e_1 \\ &+ (B'_2 - \frac{1}{2}(T_1B_3 + T_3B_1))e_2 \\ &+ (B'_3 + \frac{1}{2}(T_1B_2 - T_2B_1))e_3.\end{aligned}$$

Using the definition of covariant derivative, we get

$$g_{\mathcal{H}_3}(\nabla_TB, e_3) = B'_3 + \frac{1}{2}(T_1B_2 - T_2B_1).$$

Then from the Frenet equations (7) we obtain

$$(21) \quad g_{\mathcal{H}_3}(\nabla_TB, e_3) = -\tau N_3.$$

Since $N_3 = 0$ and $T_1B_2 - T_2B_1 = -N_3$, we have

$$(22) \quad g_{\mathcal{H}_3}(\nabla_TB, e_3) = B'_3.$$

These, together with (21) and (22) give $B'_3 = 0$. From the definition of helix and (20), we have $T_3 = \text{constant}$. □

Theorem 4.3. *Let $\psi : I \rightarrow \mathcal{H}_3$ be a non-geodesic triharmonic helix parametrized by arc length. If $N_3 \neq 0$, then*

$$(23) \quad \tau^2 = \frac{1 - 8k^2 - \sqrt{1 + 16k^2}}{8}, \quad \text{or,} \quad \tau^2 = \frac{1 - 8k^2 + \sqrt{1 + 16k^2}}{8}.$$

Proof. Since $N_3 \neq 0$, from (20), we obtain

$$(24) \quad (4(k^2 + \tau^2)^2 - (2k^2 + \tau^2))N_3 = 0.$$

Hence (24) implies (23). □

5. GENERAL HELIX IN $\mathcal{H}_3$

In this note we deal with general helices in Heisenberg group $\mathcal{H}_3$. In 1845, de Saint Venant first proved that a space curve is a general helix if and only if the ratio of curvature to torsion be constant (see [1, 2, 12] for details).

Definition 5.1. *Let ψ be a curve in $\mathcal{H}_3$ and $\{T, N, B\}$ be the Frenet frame on $\mathcal{H}_3$ along ψ .*

- 1) *If both k and τ are constant along ψ , then is called circular helix with respect to Frenet frame.*
- 2) *A curve ψ such that*

$$(25) \quad \frac{\tau}{k} = c, \quad c \in \mathbb{R},$$

is called a general helix with respect to Frenet frame.

If $k = \text{constant} \neq 0$ and $\tau = 0$, then the curve ϕ is a circle.

Theorem 5.2. *Let $\psi : I \rightarrow \mathcal{H}_3$ be a non-geodesic triharmonic general helix parametrized by arc length. If $N_3 = 0$, then ψ is a circular helix.*

Proof. From (25), we have

$$(26) \quad \begin{cases} \rho_1(s) = -10k'k'' - 5kk^{(3)} + 10k^3k'(c^2 + 1) \\ \rho_2(s) = k^5(c^2 + 1)^2 + k^{(4)} - 15kk'^2(c^2 + 1) - 10k^2k''(c^2 + 1) \\ \rho_3(s) = -c\xi_1(s) \\ \rho_4(s) = k'' - k^3(c^2 + 1) \\ \rho_5(s) = 3ck'k \\ \rho_6(s) = ck^3. \end{cases}$$

By using equations (26) in (18), equation (18), we can obtain a system of three differential equations characterizing triharmonic general helix in $\mathcal{H}_3$

$$(27) \quad \begin{cases} \rho_1(s) = -10k'k'' - 5kk^{(3)} + 10k^3k'(c^2 + 1) = 0 \\ \rho_2(s) + (-B_3^2 + \frac{1}{4})\rho_4(s) + B_3(N_3\rho_5 + T_3\rho_6) = 0 \\ -\rho_5(s)(N_3^2 - \frac{1}{4}) + N_3(B_3\rho_4(s) - \rho_6(s)T_3) = 0. \end{cases}$$

Substituting $N_3 = 0$ into the third equation in (27) we have $k'k = 0$, which implies $k = \text{constant}$ and hence $\tau = \text{constant}$. Then ψ is a circular helix.

□

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