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# THE $\phi$-TOPOLOGICAL CONFORMAL DIMENSION FOR THE SIERPINSKI CARPET

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Abstract. In this paper, a generalized fractal dimension is introduced based on a combination of the topological dimension, conformal dimension, and the $\phi$-conformal dimension. Lower and upper bound estimates of the new variant of dimension are provided for the case of the Sierpinski carpet fractal set. Moreover, the equality of such bounds is proved for a large class of the basic measure $\phi$.

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## 1. INTRODUCTION

Fractals since their appearance have been attractive for scientists in quasi all branches of science, such as theoretical mathematics, they constituted an extension of both set, function, and measure theories. In nature, many phenomena are now comprehensible due to their fractal models. In financial/economic fields, the movements of markets, indexes, prices, and other factors are well explained by fractals. In signal and image processing, we are nowadays speaking about fractal images, especially in bio-domains, where many organs in the body have been re-discovered and their strange geometry has been well explained using fractal models.

Mathematically speaking, the most spectacular character in fractal geometry/analysis is the dimension. Before the invention of fractals, humans thought that the only way is the Euclidean dimension. Every set and phenomenon in the real-life should be embedded in Euclidean space, and thus any essay to understand the object is related to the whole Euclidean space. Moreover, any eventual measure that should be done on these sets is nothing but the Lebesgue measure in the mathematical sense. With the appearance of fractals and other types of measures, researchers understood that these classical measurements are no longer able to explain and reflect the real structure and description of many objects, such as the well-known Cantor type sets and measures. This was the major motivation leading to the invention of fractal dimensions.

From an applied point of view, fractals appear in many forms such as patterns, stochastic, iterated systems, scale-invariant, etc. These forms are related to image processing such as signals, clouds, rocks, complex systems in nature such as climate factors, environment, light emission, chemical reactions, ... etc, [2, 3].

In our present work, we join the topic of fractal dimension of sets and measures, in which we are interested in some type of topological conformal dimension relative to a gauge function. Such dimensions constitute other variants of the original Hausdorff definition, the conformal dimension, the topological variant, the distortion of Hausdorff dimension, etc. Many types of research have been developed to investigate such concepts. In [15], for example, the authors focused on establishing some bounds for the Hausdorff dimension distortion of the whole Euclidean space $\mathbb{R}^d$ relative to $K$-quasi-conformal self-maps. In [1], sharp bounds have been provided for the two-dimensional cases. Bishop in [8] related the positivity of the Hausdorff dimension of compact sets to the possibility of raising such dimension by quasi-conformal maps. However, decreasing the dimension, although being small as in the case of increasing, may not occur as investigated in [9, 23]. These problems are related strongly to the structure of the functions used to increase/decrease the concerning dimension. In the case of homeomorphisms, for example, it is well known that the topological dimension is conserved. For the case of bi-Lipschitz function the Hausdorff dimension is conserved. This in turn leads to a natural question about the eventual and/or possible intermediate maps between the two previous classes and their influence on the dimensions. We speak here about quasi-symmetric maps. These maps shall permit the classification of sets according to their dimensions by introducing suitable equivalence or invariance concepts. Backgrounds and useful ideas about the classification relative to the conformal dimension may be found in [9, 10, 11, 12, 13, 17, 18, 19, 21, 22, 24]. The general idea turns around quasi-symmetric equivalence if and only if quasi-symmetrically invariant dimension.

Already in the same direction of finding and/or constructing a good invariant for sets that permit the characterization, the classification, and equivalence in some sense, another extension of dimension has been introduced in [4]. The two previous concepts of topological and conformal dimensions have been next combined to yield a hybrid type of dimension called the topological conformal dimension developed in [12]. Later, the authors in [5, 20] studied the $\phi$-topological variant of the Hausdorff dimension for a case of self-similar sets such as the Sierpinski carpet. This was one motivation for the present work. on the $\phi$-topological conformal dimension.

The principal aim of the present work is to continue in the same direction by introducing concepts of (fractal) dimensions and their eventual application in characterizing special sets and measures. A function $\phi$ is considered as a control parameter to construct a new variant of (fractal) dimensions called the $\phi$-topological conformal dimension. The framework is considered metric spaces.

The new variant of dimension constitutes a general form compared to the developments in [5, 20]. Indeed, the first reference considered the topological Billingsley dimension as a variant of the fractal dimensions based on an anlogue gage function $\phi$. In the second reference, the authors considered the $\phi$-topological Hausdorff dimension. This confirms effectively that the present work provides a different and precisely larger class of fractal dimensions relatively to the variants introduced in [5, 20].

The structure of the present work will be as follows. In section 2, some preliminaries about the notion of topological conformal dimension on metric spaces will be reviewed briefly. In section 3, the new fractal dimension which we call $\phi$-topological conformal dimension will be introduced relative to the ambient space. Basic properties of these dimensions will be provided. In section 4, estimations of these dimensions for the case of the Sierpinski carpet will be provided for a special case of the function $\phi$, for which the exact computation is possible.

## 2. Preliminaries

Consider a metric space $(X,d)$, where $X$ is assumed to be separable for the whole document, if no exception is mentioned. As usual we denote $B(x,r)=\{y\in X;\ d(x,y)\leq r\}$ the closed ball of center $x\in X$ and radius $r>0$. Next, for $A\subset X$, we write $|A|$ to designate the diameter. We denote also $\partial A$ to designate the boundary of $A$. Next, for two sets $A,B\subseteq X$, the distance between them will be evaluated by means of the Hausdorff sense,

$$\text{dist}(A,B)=\inf\{d(a,b);\ a\in A,\ b\in B\}.$$

For a subset $E\subset X$ and $s,\delta>0$, denote

$$\mathcal{H}^s_\delta(E)=\inf\Big\{\sum_i\mid U_i\mid^s:\ E\subset\bigcup_i U_i,\ \mid U_i\mid<\delta\Big\}$$

and

$$\mathcal{H}^s(E)=\sup_{\delta>0}\mathcal{H}^s_\delta(E).$$

This leads to the Hausdorff dimension defined for any set $E\subset X$ by

$$\dim_H(E)=\inf\Big\{s>0;\ \mathcal{H}^s(E)<+\infty\Big\},$$

with the convention $\dim_H(\emptyset)=-1$ to be adopted.

In [4], the topological Hausdorff dimension of sets has been investigated on metric spaces.

**Definition 2.1.** [4] *Let $(X,d)$ be a separable metric space. Its topological Hausdorff dimension is defined by*

$$\dim_{tH}(X)=\inf\Big\{s:\ \exists\ \textit{a basis}\ \mathcal{U}\ \textit{of}\ X;\ \dim_H(\partial O)\leq s-1,\ \forall O\in\mathcal{U}\Big\},$$

*with as usual* $\dim_H(\emptyset)=-1$.

Notice that Definition 2.1 may be formulated otherwise by setting

$$\dim_{tH}(X)=1+\inf_{\mathcal{U}}\sup_{O\in\mathcal{U}}\dim_H(\partial O),$$

where the lower bound above is evaluated on all the bases $\mathcal{U}$ of $X$.

The concepts of Hausdorff and the topological Hausdorff variants of fractal dimensions are extended to both the conformal and the topological conformal cases. These last ones are defined relative to the concept of quasi-symmetry of maps.

**Definition 2.2.** *Let $(X, d)$ and $(Y, \delta)$ be two metric spaces. Let also $f : X \to Y$ be an embedding homeomorphism. $f$ is said to be quasi-symmetric, if there exists $\eta : [0, \infty[\to [0, \infty[$ a homeomorphism satisfying $\forall a, b, x \in X$ and $\forall t > 0$,*

$$\Big(d(x,a) \leq td(x,b)\Big) \Longrightarrow \Big(\delta(f(x), f(a)) \leq \eta(t)\delta(f(x), f(b))\Big).$$

The following definitions state the conformal dimension.

**Definition 2.3.** *Let $(X, d)$ be a metric space. Its conformal dimension is*

$$C\dim(X) = \inf\Big\{\dim_H f(X);\ f \text{ quasi-symmetric embedding homeomorphism on } X.\Big\}.$$

*The topological conformal dimension of $X$ is*

$$tC\dim(X) = \inf\Big\{s;\ \exists \text{ a basis } \mathcal{U} \text{ of } X;\ C\dim(\partial O) \leq s-1,\ \forall O \in \mathcal{U}\Big\}.$$

## 3. New conformal and topological Hausdorff dimensions

In this section, we propose to introduce new variants of the topological and conformal extension of the Hausdorff dimension of sets taking into account the ambient space in the formulation. The idea reposes on the simple comparison between the union of a set with the boundary of another on one hand, and the boundary of the union of the two sets on the other hand. Let $E, U \subset X$, it is easily noticeable that $E \cap \partial U$ is more flexible to manipulate than the set $\partial_E(E \cap U)$. This simple remark yielded the definition of the new topological variant of the Hausdorff dimension as follows.

**Definition 3.1.** *Let $E \subset X$ be nonempty. Its topological Hausdorff dimension relatively to the ambient space $X$ is defined by*

$$\boldsymbol{dim}_{tH}(E) = \begin{cases} 1 + \inf\limits_{\mathcal{U}} \sup\limits_{U \in \mathcal{U}} \dim_H(E \cap \partial U) & \text{if} \quad \dim_H(E) \geq 1 \\ 0 \quad \text{if} \quad \dim_H(E) < 1, \end{cases}$$

*where the* $\inf$ *is taken over all basis $\mathcal{U}$ of $X$.*

*The topological conformal dimension of $E$ relatively to $X$ is defined by*

$$tC\boldsymbol{dim}(E) = \begin{cases} 1 + \inf\limits_{\mathcal{U}} \sup\limits_{U \in \mathcal{U}} C\dim(E \cap \partial U) & \text{if} \quad C\dim(E) \geq 1 \\ 0 \quad \text{if} \quad C\dim(E) < 1, \end{cases}$$

*where the lower bound is evaluated on all basis $\mathcal{U}$ of $X$. Observing that $\partial_E(E\cap U)\subset E\cap\partial U$, for all $E,U\subset X$, it results that*

$$\boldsymbol{dim}_{tH}(X)=\dim_{tH}(X),\quad tC\boldsymbol{dim}(X)=tC\dim(X)$$

*and*

$$\dim_{tH}(E)\leq\boldsymbol{dim}_{tH}(E),\quad tC\dim(E)\leq tC\boldsymbol{dim}(E).$$

We will now expose some properties of the new dimensions.

**Proposition 3.2.** *Assume that $X$ is a separable metric space. We have,*

*(1)* $\forall E\subset X$, $tC\boldsymbol{dim}(E)\leq C\dim(E)$.
*(2)* $\forall E\subset X$, $C\dim(E)=0\implies tC\boldsymbol{dim}(E\times[0,1])=1$.
*(3)* $\forall E,F\subset X$, $E\subset F\implies tC\boldsymbol{dim}(E)\leq tC\boldsymbol{dim}(F)$.
*(4)* $\forall E_1.E_2\subset X$, *disjoint and compact, it holds that*

$$tC\boldsymbol{dim}\left(E_1\cup E_2\right)=\max\left(tC\boldsymbol{dim}(E_1),tC\boldsymbol{dim}(E_2)\right).$$

*Proof.*

(1) It is a consequence of the inequality $tC\mathbf{dim}(E)\leq tC\dim(E)\leq C\dim(E)$.
(2) Since there exists a segment line contained in $E\times[0,1]$, we deduce that $tC\mathbf{dim}(E\times[0,1])\geq 1$. Otherwise, there exists a basis $\mathcal{U}$ of $X$ composed of open sets. Therefore, as $\dim_t E\leq C\dim E=0$, it results for every $U\in\mathcal{U}$, and $0\leq x\leq y\leq 1$, that

$$\begin{aligned}C\dim\left(E\cap(\partial U\times(x,y))\right) &= C\dim\left(E\cap(U\times\partial(x,y))\right)\\ &= C\dim\left(E\cap(U\times\{x,y\})\right)\\ &= C\dim(E\cap U)\\ &\leq C\dim(E)=0.\end{aligned}$$

(3) Notice that for a basis $\mathcal{U}$ of $X$, and an element $U\in\mathcal{U}$, we get $E\cap\partial U\subset F\cap\partial U$. As the dimensions are monotone, we get

$$C\dim(E\cap\partial U)\leq C\dim(F\cap\partial U).$$

Hence, the result follows immediately.

(4) Let $\mathcal{U}_i$ and $d_i$, $i=1,2$, such that $\mathcal{U}_i$ is a basis of $E_i$,

$$C\dim(E_i\cap\partial U_i)\leq d_i-1,\ \forall U_i\in\mathcal{U}_i,$$

and

$$d_i\leq tC\textbf{dim}(E_i)+\varepsilon.$$

Consider by the next the union basis $\mathcal{U}=\mathcal{U}_1\cup\mathcal{U}_2$ of $E=E_1\cup E_2$, and let $d=\max(d_1,d_2)$. For each $U=U_1\cup U_2$ with compact boundaries $\partial U_1$ and $\partial U_2$ being disjoint, we get $\partial U=\partial U_1\cup\partial U_2$, which is also compact. Consequently,

$$\begin{aligned}C\dim(E\cap\partial U) &= \max\big(C\dim(E_1\cap\partial U_1),C\dim(E_2\cap\partial U_2)\big)\\ &\leq \max\big(d_1-1,d_2-1\big)\\ &\leq \max_{i=1,2}tC\textbf{dim}(E_i)+\varepsilon-1.\end{aligned}$$

Finally, the monotonicity of the dimension gives the opposite inequality.

□

One immediate consequence of the lat proposition may state that $\forall n\in\mathbb{N}$ fixed, and $E_i,\ i=1,\dots,n$ pairwise disjoint, and compact sets in $X$, we have

$$tC\textbf{dim}\left(\bigcup_{i=1}^{n}E_i\right)=\max_{1\leq i\leq n}\left(tC\textbf{dim}(E_i)\right).$$

The following part is concerned with a generalization of the Billingsley (Hausdorff) dimension relative to a measure ([6, 7]) to the case of a gauge set function $\phi$, which by the next will be called the $\phi$-topological conformal dimension.

Consider as above a metric space $X$ and a countable collection $\mathscr{F}$ of subsets of $X$. Let also $\phi\geq 0$ be a function on a collection of subsets of the same space $X$, and containing $\mathscr{F}$. Assume further that $\forall x\in X$, and $\forall\varepsilon>0$,

$$\exists\, E\in\mathscr{F}\quad\text{such that}\quad\phi(E)<\varepsilon\quad\text{and}\quad x\in E \tag{1}$$

For $E\subseteq X$ a non-empty subset of $X$, $s\geq 0$, and $\delta>0$, we write

$$\phi\mathcal{H}^s_\delta(E)=\inf\left\{\sum_i\phi(U_i)^s;\ U_i\in\mathscr{F},\ \phi(U_i)<\delta,\ E\subset\bigcup_i U_i\right\},$$

with the convention $0^s=0$. This permits to defined the $\phi$-Hausdorff measure relative to $\mathscr{F}$ by

$$\phi\mathcal{H}^s(E)=\lim_{\delta\to 0}\phi\mathcal{H}^s_\delta(E).$$

To such a measure, we associate in a natural way (as for the different variants of Hausdorff measures), the so-called $\phi$-Hausdorff dimension relative to $\mathscr{F}$, defined for any non-empty set $E \subset X$ by

$$\dim_{\phi H}(E) = \inf\Big\{s > 0 : \ \phi\mathcal{H}^s(E) < +\infty\Big\}.$$

Notice that the original Hausdorff dimension may be obtained from the last definition, which confirms the generalization. Indeed, whenever $X$ is separable, we just take the collection

$$\mathscr{F} = \Big\{B(x_n, r); \ n \in \mathbb{N} \ and \ r \in \mathbb{Q}_+^*\Big\},$$

for a dense set $\big\{x_n, \ n \in \mathbb{N}\big\}$ in $X$, and where $\phi(B(x_n, r)) = 2r$. We immediately observe that $\phi$ satisfies (1). Furthermore, $\forall E \subset X$, we obtain

$$\dim_{\phi H}(E) = \dim_H(E).$$

The following propositions gather some trivial properties of the $\phi$-Hausdorff dimension.

**Proposition 3.3.** [20]

*(1) If $E \subset F$ then $\dim_{\phi H}(E) \le \dim_{\phi H}(F)$.*

*(2) For $(E_i)_{i\in\mathbb{N}} \subset X$,*

$$\dim_{\phi H}(\cup_i E_i) = \sup_i \dim_{\phi H}(E_i).$$

*(3) $\dim_{\phi H}(E) = 0$, for all countable $E \subset X$.*

Now, as for existing cases of dimensions, the $\phi$-Hausdorff dimension will be adopted to introduce the $\phi$-conformal dimension of $X$ relative to the collection $\mathscr{F}$, and the $\phi$-topological conformal dimension according to $\mathscr{F}$ by using similar techniques as in [5, 20].

**Definition 3.4.** *The $\phi$-conformal dimension of $X$ relatively to $\mathscr{F}$ is defined by*

$$C\dim_\phi X = \inf\Big\{\dim_{\phi H} f(X) : \ f \text{ is quasisymmetric}\Big\}.$$

*Let $E \subset X$. The $\phi$-topological conformal dimension of $E$ relatively to $\mathscr{F}$ is defined by*

$$tC\dim_\phi(E) = \inf\Big\{s : \ \exists \text{ a basis } \mathcal{U} \text{ of } X; \ C\dim_\phi(E\cap\partial O) \le s-1, \ \forall\, O\in\mathcal{U}\Big\},$$

*and $tC\dim_\phi(\emptyset) = -1$. Otherwise,*

$$tC\dim_\phi(E) = \inf_{\mathcal{U}} \sup_{O\in\mathcal{U}} C\dim_\phi(E\cap\partial O) + 1,$$

*where the lower bound above is evaluated on all bases $\mathcal{U}$ of $X$.*

## 4. Evaluation of the $\phi$-topological conformal dimension for the Sierpinski carpet

This part of the paper is concerned with the computation of the $\phi$-topological conformal dimension developed previously to the well-known Sierpinski carpet as a concrete case of (fractal) self-similar set, which will be denoted here by $\mathcal{SC}$.

The Sierpinski carpet is a planar fractal set which may be described as follows. Let $I = [0,1] \times [0,1]$ be the unit square. In the first stage, $I$ is subdivided into nine isometric squares and remove the middle one. Next, we subdivide each one of the eight remaining squares into nine isometric squares and remove again the new middle square from each one. We continue recursively the same process. The resulting limit set is the Sierpinski carpet $\mathcal{SC}$. Figure 1 illustrates the process.

Mathematically speaking, the $\mathcal{SC}$ is the result of IFS due to similarities

$$S_\lambda(x) = \frac{1}{3}x + \frac{1}{3}\lambda,\ x \in I,\ \lambda \in \Lambda = \{0,1,2\}^2.$$

Otherwise, $\mathcal{SC}$ is the unique invariant compact satisfying

$$\mathcal{SC} = \bigcup_{\lambda \in \Lambda} S_\lambda(\mathcal{SC}).$$

This leads to the exact computation of its Hausdorff dimension by means of IFS as

$$8\left(\frac{1}{3}\right)^{dim_H \mathcal{SC}} = 1 \iff dim_H \mathcal{SC} = \frac{\log 8}{\log 3}.$$

Due to [4, 12, 21], it is noticeable that all the fractal dimensions such as the conformal, the topological conformal, the topological Hausdorff, and the Hausdorff dimension may be different in general. We have a precise comparison for the $\mathcal{SC}$ set stating that

$$\dim_t(\mathcal{SC}) = 1 = tC\dim(\mathcal{SC}) < \dim_{tH}(\mathcal{SC}) = \frac{\log 6}{\log 3} < \dim_H(\mathcal{SC}) = \frac{\log 8}{\log 3},$$

and

$$tC\dim(\mathcal{SC}) < 1 + \frac{\log 2}{\log 3} \leq C\dim(\mathcal{SC}) < \dim_H(\mathcal{SC}) = \frac{\log 8}{\log 3}.$$

The purpose now is to evaluate the $\phi$-topological conformal dimension of $\mathcal{SC}$ in some special case. We consider for this aim the collection $\mathscr{F} = \cup_{n\geq 1}\mathscr{F}_n$, where for $n \in \mathbb{N}$, $\mathscr{F}_n$ is the set of $n$th level triadic squares. Furthermore, we assume that $\forall \varepsilon > 0,\ \forall n \in \mathbb{N}^*$ and $\forall (x,y) \in \mathbb{R}^2$,

$$\exists\, p > n \text{ and } O \in \mathscr{F}_p \text{ such that}$$

$$(x,y) \in O \text{ and } \phi(f(O)) < \varepsilon, \quad \text{for every quasisymmetric map } f. \tag{2}$$

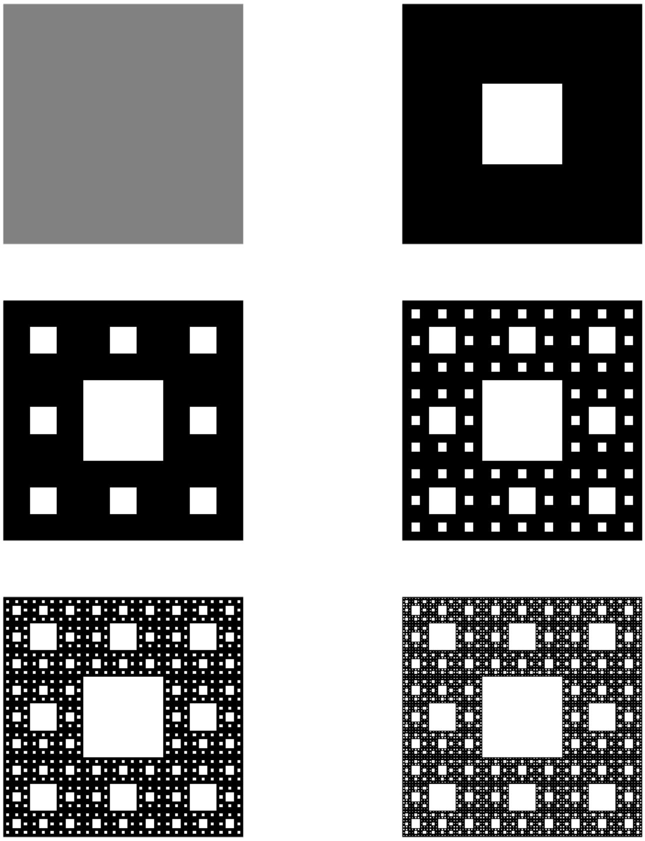

FIGURE 1. The first 6 iterations of $\mathcal{SC}$.

Notice here that (2) $\Longrightarrow$ (1). Besides, if $\phi$ is finite measure, non-atomic on $\mathbb{R}^2$, then equation (2) is satisfied. This confirms that our case covers the class of finite, non-atomic measure such as [5, 6, 7, 20].

We now proceed in establishing that $tC\dim_\phi(\mathcal{SC}) \geq 1$, and thus obtain a lower bound for the $\phi$-topological conformal dimension of $\mathcal{SC}$. This lower bound will be improved more in Theorem 4.2. For this purpose, we will serve of the following resul.

**Lemma 4.1.** *Let $x \in \mathcal{C}$ the middle-$\frac{1}{3}$-Cantor set in $\mathbb{R}$, and $U \subset \mathbb{R}^2$ be open such that $(x, 1) \in U$, then,*

$$\big(\mathcal{C} \times [0, 1]\big) \cap \partial U \neq \emptyset.$$

*Proof.* Denote

$$y = \inf\big\{w \in [0,1] : \ (x,w) \in \big(\mathcal{C} \times [0,1]\big) \cap \partial U\big\}.$$

We shall that $(x,y) \in \partial U$. Indeed, observing firstly that $(x,y) \in \overline{U}$. We have thus to prove that $(x,y) \notin U$. If $(x,y) \in U$, we get $(x,y) \in B(x,\varepsilon)\times ]y-\varepsilon, y+\varepsilon[ \subset U$, for some $\varepsilon > 0$. Hence, $(x,z) \in U$, $\forall z \in ]y-\varepsilon, y[$, which is a contradictory. □

Next, we proceed in proving the inequality $tC\dim_\phi(SC) \geq 1$. Indeed, consider a basis $\mathcal{U}$ of $\mathbb{R}^2$, and a point $x \in \mathcal{C}$. Let also $U_x \in \mathcal{U}$ containing the pair $(x,1)$. Using Lemma 4.1, we obtain

$$\partial U_x \cap \big(\mathcal{C} \times [0,1]\big) \neq \emptyset.$$

It results that

$$tC\dim_\phi\big(\mathcal{C} \times [0,1]\big) \geq 1.$$

Finally, since $\mathcal{C} \times [0,1] \subset \mathcal{SC}$, the results follows immediately as the $\phi$-topological conformal dimension is monotone.

4.1. **A $\phi$-topological conformal dimension lower bound for $\mathcal{SC}$.** For convenience, and clearness, we will restrict in the present part to the triadic squares in $I$, coded by the usual triadic representation. Denote $\mathcal{A} = \big\{0,1,2\big\}$ the essential alphabet, and $\mathcal{A}^n$ the sub-alphabet of $\mathcal{A}$ composed of words with length $n$, $n \in \mathbb{N}^*$. We write $|a| = n$ to designate the length of $a \in \mathcal{A}^*$. Denote also $\mathcal{A}^* = \bigcup_{n\geq 1}\mathcal{A}^n$. Two words $a, b \in \mathcal{A}^*$ are concatenated by juxtaposing the word $b$ just at the end of the word $a$. The concatenation result will be written $ab$. Next, for $n \in \mathbb{N}^*$, and a multi-index $i = i_1 i_2 ... i_n \in \mathcal{A}^n$, we write

$$I_i = \Big[\sum_{k=1}^{n}\frac{i_k}{3^k}, \sum_{k=1}^{n}\frac{i_k}{3^k} + \frac{1}{3^n}\Big) \subset [0,1)$$

the triadic interval associated. Next, for $i, j \in \mathcal{A}^n$, we write

$$\mathcal{C}_{i,j} = I_i \times I_j$$

the triadic square of $\mathscr{F}_n$ associated. We next associate to the function $\phi$ two functions $\mu_1$ and $\mu_2$ on the triadic intervals $J \subset \mathbb{R}$ which are written as $J = f(I')$ for some quasisymmetric map $f : \mathbb{R} \to \mathbb{R}$, where $I'$ is also a triadic interval. Now, whenever $J_i = f(I'_i)$ is an $n$-th triadic interval contained in $[0,1)$, we put

$$\mu_1(J_i) := \mu_1^f(J_i) = \inf_{j\in\mathcal{A}^n}\phi(f(\mathcal{C}_{i,j})) \quad \text{and} \quad \mu_2(J_i) := \mu_2^f(J_i) = \inf_{j\in\mathcal{A}^n}\phi(f(\mathcal{C}_{j,i})),$$

(3)

Else, we put

$$\mu_1(J) = \mu_2(J) = 0.$$

Since $\phi$ satisfies (1), then the $\mu_1$ (respectively $\mu_2$)-conformal dimension is well-defined. In the following we provide for the fractal set $\mathcal{SC}$ a lower bound of its $\phi$-topological conformal dimension.

**Theorem 4.2.** *The following inequality holds,*

$$tC\dim_\phi(\mathcal{SC}) \geq 1 + \sup\Big(C\dim_{\mu_1}(\mathcal{C}), C\dim_{\mu_2}(\mathcal{C})\Big).$$

*Proof.* To prove the inequality we need firstly this result.

**Lemma 4.3.** *For $s < C\dim_{\mu_1}(\mathcal{C})$, there exists a point $x_s \in \mathcal{C}$ for which*

$$C\dim_{\mu_1}\big(B(x_s, r) \cap \mathcal{C}\big) > s,\ \forall r > 0. \tag{4}$$

*Proof.* We proceed by the converse. For $x \in \mathcal{C}$, let $r_x > 0$ be such that

$$C\dim_{\mu_1}\big(B(x_s, r) \cap \mathcal{C})\big) \leq s.$$

We immediately observe that $\mathcal{C} \subseteq \bigcup_{x\in\mathcal{C}} B(x_s, r)$. As $\mathcal{C}$ is compact, we deduce that

$$\mathcal{C} \subseteq \bigcup_{i=1}^{p} B(x_{s_i}, r_i),$$

for some integer $p \in \mathbb{N}$. It follows that

$$\mathcal{C} = \bigcup_{i=1}^{p} B(x_{s_i}, r_i) \cap \mathcal{C},$$

which in turn implies that

$$C\dim_{\mu_1}(\mathcal{C}) = \sup_{1\leq i\leq p} C\dim_{\mu_1}(B(x_{s_i}, r_i) \cap \mathcal{C}) \leq s.$$

This last results is in contradiction with the inequality $C\dim_{\mu_1}(\mathcal{C}) > s$. □

Now, we return to the proof of Theorem 4.2, and consider an open basis $\mathcal{U}$ of $\mathbb{R}^2$ and $s < C\dim_{\mu_1}(\mathcal{C})$. By Lemma 4.3, let $0 \neq x_s \in \mathcal{C}$, satisfying

$$C\dim_{\mu_1}\big(\mathcal{C} \cap B(x_s, r)\big) > s,\ \forall\, r > 0.$$

Then, for every quasisymmetric map $f_1 : [0,1] \longrightarrow [0,1]$, we have

$$\dim_{\mu_1 H}\big(f_1(\mathcal{C} \cap B(x_s, r))\big) > s,$$

and

$$\mu_1\mathcal{H}_\varepsilon^s(f_1(B(x_s,r_s)\cap\mathcal{C}))=+\infty.$$

Let now $(x_s,1)\in V_s\in\mathcal{U}$, for which

$$B(x_s,r_s)\times]1-r_s,1+r_s[\subset V_s.$$

Let also $x\in B(x_s,r_s)\cap\mathcal{C}$. Due to Lemma 4.1, we have

$$\partial V_s\cap\mathcal{C}\times[0,1]\neq\emptyset,$$

which leads to

$$(x,y)\in\partial V_s\cap\mathcal{C}\times[0,1]\quad\text{where}\quad y=\inf\big\{w:\ (x,w)\in V_s\cap\mathcal{C}\times[0,1]\big\}.$$

As a result,

$$B(x_s,r_s)\cap\mathcal{C}\subset P\big(\partial V_s\cap\mathcal{C}\times[0,1]\big),$$

where $P(x,y)=x$, for $(x,y)\in\mathbb{R}^2$. Now, fix $\varepsilon>0$, and let $\Big(C_q\Big)_q$ be a covering of $\partial V_s\cap\mathcal{SC}$ by triadic squares satisfying $\phi(f(C_q))<\varepsilon$. Since $\mathcal{C}\times[0,1]\subset\mathcal{SC}$, one gets

$$B(x_s,r_s)\cap\mathcal{C}\subset\bigcup_q P(C_q).$$

Observe next that

$$\phi(f(C_q))\geq\mu_1(f_1(P(C_q))),\ \forall C_q\subset\mathbb{R}^2.$$

Therefore, we get

$$\sum_q\phi(f(C_q))^s\geq\sum_q\mu_1(f_1(P(C_q)))^s.$$

and that

$$\phi\mathcal{H}_\varepsilon^s(f(\partial V_s\cap\mathcal{SC}))\geq\mu_1\mathcal{H}_\varepsilon^s(f_1(B(x_s,r_s)\cap\mathcal{C})).$$

Then, we deduce that

$$\phi\mathcal{H}_\varepsilon^s\big(f(\partial V_s\cap\mathcal{SC})\big)=+\infty,$$

$$\dim_{\phi H}\big(f(\partial V_s\cap\mathcal{SC})\big)\geq s,$$

and

$$C\dim_\phi(\partial V_s\cap\mathcal{SC})\geq s.$$

Consequently

$$tC\dim_\phi(\mathcal{SC})\geq 1+s.$$

The proof of the case where $x_s=0$ may follow by analogue arguments as in the first part. $\square$

4.2. **A $\phi$-topological conformal dimension upper bound for $\mathcal{SC}$.** In the present part, we propose to provide an upper bound for the $\phi$-topological conformal dimension of the fractal set $\mathcal{SC}$. To do it, we consider for $i,j\in\mathcal{A}^*$, $|i|=|j|$, the functions previously introduced, $\nu_{i,j}^1$ and $\nu_{i,j}^2$ on the triadic intervals $J\subset\mathbb{R}$ for which $J=f(I')$ for some quasisymmetric map $f:\mathbb{R}\to\mathbb{R}$, and where $I'$ is also a triadic interval, as follows: For an $n$-th triadic interval $J_k=f_1(I'_k)\subset[0,1)$, we put

$$\nu_{i,j}^1(J_k):=\nu_{i,j}^{1,f}(J_k)=\phi(f(C_{i\mathbf{1},jk})),$$

and

$$\nu_{i,j}^2(J_k):=\nu_{i,j}^{2,f}(J_k)=\phi(f(C_{ik,j\mathbf{1}})),$$

where $\mathbf{1}=11...1\in\mathcal{A}^n$. Else, we put

$$\nu_{i,j}^m(J)=0,\ m=1,2.$$

Observing immediately $\nu_{i,j}^1$ and $\nu_{i,j}^2$ satisfy (1). Indeed, let $i=i_1i_2...i_{n_0}$ and $j=j_1j_2...j_{n_0}$ in $\mathcal{A}^*$. Let $x\in[0,1[$ written as

$$x=\sum_{i\geq1}\frac{x_i}{3^i}\quad\text{where}\quad x\in I_{x_1x_2...x_n},\quad\text{for all}\quad n\in\mathbb{N}^*,$$

in its triadic representation. Put

$$y'=\sum_{k=1}^{n_0}\frac{j_k}{3^k}+\sum_{k=n_0+1}^{\infty}\frac{x_k}{3^k}\quad\text{and}\quad x'=\sum_{k=1}^{n_0}\frac{i_k}{3^k}+\frac{1}{2\cdot3^{n_0}}.$$

It holds from (2) that

$$(x',y')\in C_{i\mathbf{1},jx_1x_2...x_n}\quad\text{and}\quad\phi(f(C_{i\mathbf{1},jx_1x_2...x_n}))<\varepsilon,$$

for $\varepsilon>0$, and some $n\in\mathbb{N}$. It follows that

$$\nu_{i,j}^1(f_1(I_{x_1x_2...x_n}))=\phi(f(C_{i\mathbf{1},jx_1x_2...x_n}))<\varepsilon.$$

As a consequence, since $x\in I_{x_1x_2...x_n}$, we deduce that $\nu_{i,j}^1$ satisfies (1). Similarly, we show that $\nu_{i,j}^2$ also satisfies the assumption (1).

In the rest of this section, we propose to provide a special case for $\phi$ for which the assumption (2) holds, while the function $\phi$ is null for any triadic square not contained in the unit square $I$. In such a case, we obtain an upper bound for the $\phi$-topological conformal dimension as stated here-after for the set $\mathcal{SC}$.

**Theorem 4.4.** *It holds that*

$$tC\dim_{\phi}(\mathcal{SC}) \leq 1 + \liminf_{n\to\infty} \sup_{|a|=|b|=n} \sup_{m=1,2} C\dim_{\nu^m_{a,b}}(\mathcal{C}).$$

*Proof.* Denote for $u, v \in \mathbb{Z}$, and $n \in \mathbb{N}^*$, $(z^u_n, z^v_n) = (\frac{u+1/2}{3^n}, \frac{v+1/2}{3^n})$ to designate the center point of $\left[\frac{u}{3^n}, \frac{u+1}{3^n}\right) \times \left[\frac{v}{3^n}, \frac{v+1}{3^n}\right)$ in the collection $\mathscr{F}_n$. Denote also

$$\mathcal{H}_n = \left\{ ]z^u_n, z^{u+2}_n[,\ u \in \mathbb{Z} \right\},$$

and $\mathcal{U}_n = \left\{ I' \times J :\ I', J \in \mathcal{H}_n \right\}$. It is straightforward that $\mathcal{U} = \bigcup\limits_n \mathcal{U}_n$ constitutes a basis of $\mathbb{R}^2$, and it is further open and countable. Therefore, to prove our result, we shall show that

$$C\dim_{\phi}(\mathcal{SC} \cap \partial U) \leq \inf_{n\geq p} \sup_{|a|=|b|=n} \sup_{m=1,2} C\dim_{\nu^m_{a,b}}(\mathcal{C}),\ \forall\, U \in \mathcal{U}_p.$$

Therefore, for $O \in \mathcal{U}_p$, we observe that $\partial O$ is composed of four edges. Denote $C$ one of them, and assume without loss of the generality that it is vertical. For $n \geq p$, consider an $n$-th triadic square $C_{i,j}$ satisfying $C_{i,j} \cap \mathcal{SC} \cap C \neq \emptyset$. Let next $(I'_k)_k$ be a covering of $\mathcal{C} \cap [0, 1)$ by elements of $\mathcal{F}$, such that, for all $k$, we have $\nu^1_{i,j}(f(I'_k)) < \varepsilon$. It follows that

$$C_{i,j} \cap \mathcal{SC} \cap C \subset \bigcup_k C_{i\mathbf{1},jk},\ \text{where } |\mathbf{1}| = k.$$

Since $\nu^1_{i,j}(f(I'_k)) = \phi(f\,(C_{i\mathbf{1},jk}))$, then for all $k$, we get $\phi(f(C_{i\mathbf{1},jk})) < \varepsilon$. Moreover, for $s > 0$, one has

$$\sum_k \nu^1_{i,j}(f_1(I'_k))^s = \sum_k \phi(f\,(C_{i\mathbf{1},jk}))^s.$$

This yields that

$$\phi\mathcal{H}^s_{\varepsilon}\left(f\left(C_{i,j} \cap \mathcal{SC} \cap C\right)\right) \leq \nu^1_{i,j}\mathcal{H}^s_{\varepsilon}\left(f_1\left(\mathcal{C} \cap [0, 1)\right)\right).$$

Whenever $\varepsilon$ tends to zero, we obtain

$$\phi\mathcal{H}^s(f\left(C_{i,j} \cap \mathcal{SC} \cap C\right)) \leq \nu^1_{i,j}\mathcal{H}^s(f_1\left(\mathcal{C} \cap [0, 1)\right)),$$

which implies that

$$\dim_{\phi H}\left(f\left(C_{i,j} \cap \mathcal{SC} \cap C\right)\right) \leq \dim_{\nu^1_{i,j} H}\left(f_1\left(\mathcal{C} \cap [0, 1)\right)\right).$$

Finally, since $\nu^1_{i,j}(\{1\}) = 0$, we deduce that

$$\dim_{\phi H}\left(f\left(C_{i,j} \cap \mathcal{SC} \cap C\right)\right) \leq \dim_{\nu^1_{i,j} H}\left(f_1(\mathcal{C})\right).$$

Clearly, by the convention that $\dim_{\phi H}(\emptyset) = -1$, the result is true when $C_{i,j} \cap \mathcal{SC} \cap C = \emptyset$. Otherwise, we have

$$I \cap \mathcal{SC} \cap C \subset \bigcup_{|i|=|j|=n} C_{i,j}.$$

Therefore,

$$\begin{aligned}\dim_{\phi H}\left(f\left(I \cap \mathcal{SC} \cap C\right)\right) &= \sup_{|i|=|j|=n} \dim_{\phi H}\left(f\left(C_{i,j} \cap \mathcal{SC} \cap C\right)\right) \\ &\leq \sup_{|i|=|j|=n} \dim_{\nu^1_{i,j} H}\left(f_1(\mathcal{C})\right).\end{aligned}$$

Hence, as $\phi \equiv 0$ for any triadic squares non intersecting $I$, we deduce that

$$\dim_{\phi H}\left(f(I \cap \mathcal{SC} \cap C)\right) = \dim_{\phi H}\left(f(\mathcal{SC} \cap \mathcal{C})\right).$$

It follows that

$$\dim_{\phi H}\left(f(\mathcal{SC} \cap C)\right) \leq \sup_{|i|=|j|=n} \dim_{\nu^1_{i,j} H}\left(f_1(\mathcal{C})\right).$$

As a result,

$$\dim_{\phi H}(f(SC \cap C)) \leq \inf_{n \geq p} \sup_{|i|=|j|=n} \dim_{\nu^1_{i,j} H}(f_1(\mathcal{C})).$$

The case of a horizontal edge $\mathcal{C}$ satisfies similarly

$$\dim_{\phi H}\left(f(\mathcal{SC} \cap C)\right) \leq \inf_{n \geq p} \sup_{|i|=|j|=n} \dim_{\nu^2_{i,j} H}\left(f_1(\mathcal{C})\right).$$

Finally, it results that

$$\dim_{t\phi H}\left(f(\mathcal{SC} \cap \partial U)\right) \leq \inf_{n \geq p} \sup_{|i|=|j|=n} \sup_{m=1,2} \dim_{\nu^m_{i,j} H}\left(f_1(\mathcal{C})\right)$$

and

$$C\dim_{\phi}(\mathcal{SC} \cap \partial U) \leq \inf_{n \geq p} \sup_{|i|=|j|=n} \sup_{m=1,2} C\dim_{\nu^m_{i,j}}(\mathcal{C}).$$

□

4.3. **A case of equality.** The present part is concerned to the equality of the upper and lower bounds previously established, and thus, we obtain an exact formula for the $\phi$-topological conformal dimension for $\mathcal{SC}$. So, consider a real valued matrix $(P_{i,j})_{0 \leq i,j \leq 2}$ for which

$$P_{i,j} > 0, \ \forall i, j \in \mathcal{A} \quad \text{and} \quad \sum_{i,j} P_{i,j} = 1.$$

Let next $\phi$ the Bernoulli measure defined on $\mathbb{R}^2$ for every quasisymmetric map $f$, by

$$\phi(f(C_{i,j})) = \prod_{k=1}^{n} P_{i_k,j_k}, \quad \text{where} \quad i = i_1 i_2 \ldots i_n \quad \text{and} \quad j = j_1 j_2 \ldots j_n.$$

Next, denote for $k = 0, 2$,

$$l_k = \min_{0 \leq j \leq 2} P_{k,j} \quad \text{and} \quad t_k = \min_{0 \leq j \leq 2} P_{j,k},$$

and let $\alpha, \rho > 0$ be the unique numbers for which

$$l_0^\alpha + l_2^\alpha = 1 \quad \text{and} \quad t_0^\rho + t_2^\rho = 1.$$

The following lemma holds.

**Lemma 4.5.** *We have*

$$C\dim_{\mu_1}(\mathcal{C}) = \alpha \quad \textit{and} \quad C\dim_{\mu_2}(\mathcal{C}) = \rho.$$

*Proof.* Denote by $\widetilde{\mathcal{C}}$ the middle-$\frac{1}{3}$-Cantor set from which we excluded the extremities of triadic intervals. Then, for every quasisymmetric map $f$, we have

$$\dim_{\mu_1 H}(f(\mathcal{C})) = \dim_{\mu_1 H}(f(\widetilde{\mathcal{C}})).$$

Observe now that, if $I_{i_1 i_2 \ldots i_n}$ is a triadic interval that crosses $\widetilde{\mathcal{C}}$, then $i_k \in \big\{0, 2\big\}$, for all $k \in \{1, 2, ..., n\}$. Therefore, we get

$$\mu_1\left(f_1(I_{i_1 i_2 \ldots i_n})\right)^\alpha = \prod_{k=1}^{n} l_{i_k}^\alpha, \quad \text{for every quasisymmetric map} \quad f_1 : \mathbb{R} \longrightarrow \mathbb{R}.$$

It results that for any covering $(I_i)_i$ of $\widetilde{\mathcal{C}}$ composed of disjoint triadic intervals,

$$\sum_i \mu_1(f_1(I_i))^\alpha = 1.$$

Consequently, we get $\mu_1 \mathcal{H}^\alpha(f_1(\widetilde{\mathcal{C}})) = 1$, and thus,

$$\dim_{\mu_1 H}(f_1(\widetilde{\mathcal{C}})) = \dim_{\mu_1 H}(f_1(\mathcal{C})) = \alpha.$$

The proof of $\dim_{\mu_2 H}(f_1(\mathcal{C})) = \rho$ is similar. In particular, we have $C\dim_{\mu_1}(\mathcal{C}) = \alpha$ and $C\dim_{\mu_2}(\mathcal{C}) = \rho$. □

**Lemma 4.6.** *Let $\beta, \delta > 0$ be such that*

$$P_{0,1}^\beta + P_{2,1}^\beta = P_{1,0}^\delta + P_{1,2}^\delta = 1$$

*Let also $i, j \in \mathcal{A}^*$ be with the same length. It holds that*

$$C\dim_{\nu_{i,j}^1 H}(\mathcal{C}) = \delta \quad \textit{and} \quad C\dim_{\nu_{i,j}^2 H}(\mathcal{C}) = \beta.$$

*Proof.* Let $i = i_1 i_2 ... i_q$ and $j = j_1 j_2 ... j_q$ in $\mathcal{A}^*$. Observing that, if $I_{k_1 k_2 \dots k_n}$ is a triadic interval crosses $\widetilde{\mathcal{C}}$, we deduce that, for all $l \in \{1, 2, ..., n\}$, and every quasisymmetric $f_1 : [0, 1] \longrightarrow [0, 1]$, $k_l \in \big\{0, 2\big\}$ and

$$\nu^1_{i,j}(f_1(I_{k_1 k_2 \dots k_n}))^\delta = \Big(\prod_{k=1}^{q} P_{i_k, j_k}\Big)^\delta \prod_{l=1}^{n} \Big(P_{1,k_l}\Big)^\delta.$$

Now, given a disjoint covering $(I_i)_i$ of $\widetilde{\mathcal{C}}$ by triadic intervals, we get

$$\sum_{k_1 k_2 \dots k_n} \nu^1_{i,j}(f_1(I_{k_1 k_2 \dots k_n}))^\delta = \Big(\prod_{k=1}^{q} P_{i_k, j_k}\Big)^\delta.$$

It follows that $\dim_{\nu^1_{i,j} H}(f_1(\mathcal{C})) = \delta$. Similarly, we prove that $\dim_{\nu^2_{i,j} H}(f_1(\mathcal{C})) = \beta$, and we deduce that

$$C\dim_{\nu^1_{i,j} H}(\mathcal{C}) = \delta \quad \text{and} \quad C\dim_{\nu^2_{i,j} H}(\mathcal{C}) = \beta.$$

□

**Theorem 4.7.** *Assume that the matrix $(P_{i,j})_{i,j \in \mathcal{A}}$ satisfies*

$$\begin{cases} P_{0,1} \leq \min\Big(P_{0,0}, P_{0,2}\Big) \\ P_{2,1} \leq \min\Big(P_{2,0}, P_{2,2}\Big) \end{cases} \quad \textit{and} \quad \begin{cases} P_{1,0} \leq \min\Big(P_{0,0}, P_{2,0}\Big) \\ P_{1,2} \leq \min\Big(P_{0,2}, P_{2,2}\Big), \end{cases}$$

*then, one has*

$$tC\dim_\phi(\mathcal{SC}) = 1 + \sup(\alpha, \rho).$$

*Proof.* Observing that

$$1 + \sup(\alpha, \rho) \leq tC\dim_\phi(\mathcal{SC}) \leq 1 + \sup(\beta, \delta),$$

we obtain from Lemmas 4.5 and 4.6,

$$\begin{cases} l_0 = P_{0,1} = \min\Big(P_{0,0}, P_{0,2}\Big) \\ l_2 = P_{2,1} = \min\Big(P_{2,0}, P_{2,2}\Big) \end{cases} \quad \text{and} \quad \begin{cases} t_0 = P_{1,0} = \min\Big(P_{0,0}, P_{2,0}\Big) \\ t_2 = P_{1,2} = \min\Big(P_{0,2}, P_{2,2}\Big). \end{cases}$$

Now, since

$$\begin{cases} l_0^\alpha + l_2^\alpha = 1 \\ t_0^\rho + t_2^\rho = 1 \end{cases} \quad \text{and} \quad \begin{cases} P_{1,0}^\delta + P_{1,2}^\delta = 1 \\ P_{0,1}^\beta + P_{2,1}^\beta = 1, \end{cases}$$

we deduce that

$$\alpha = \beta \quad \text{and} \quad \rho = \delta.$$

□

## 5. Remarks and discussion

Unlike the Hausdorff/packing/box-counting variants of fractal dimensions, it is noticeable that the topological Hausdorff one is in fact smaller than the Hausdorff dimension, while the $\phi$-topological conformal variant generalizes the topological conformal one. Otherwise, it is always not easy to estimate these dimensions directly from their pure mathematical definitions, it is thus motivating to try to give estimates of the new variants such as the $\phi$-topological conformal one, and discuss. First, we discuss the eventual links between the different dimensions. In [4], it has been proved that

$$\dim_t(X) \leq \dim_{tH}(X) \leq \dim_H(X),$$

where $\dim_t$ stands for the topological dimension (for more information on this concept see [14, 16]). Also, we may notice that that

$$\dim_t(X) \leq C\dim(X) \leq \dim_H(X),$$

$$\dim_t(X) \leq tC\dim(X) \leq \dim_{tH}(X) \leq \dim_H(X),$$

$$\dim_t(X) \leq tC\dim(X) \leq C\dim(X),$$

and that these dimensions vanish simultaneously. Besides, Balka et al. [4] introduced the concept of the topological Hausdorff dimension of sets by considering the set in hand as the metric space itself.

Notice also that whenever we work in a separable metric space $X$, the topological Hausdorff and the topological conformal dimensions maybe both reproduced via the $\phi$-conformal and the $\phi$-topological conformal variants. Indeed, let

$$\mathscr{F} = \Big\{B(x_n, r) : \ n \in \mathbb{N}, \ r \in \mathbb{Q}_+^*\Big\},$$

where the set $\big\{x_n, \ n \in \mathbb{N}\big\}$ is assumed to be dense in $X$, and $\phi(B(x_n, r)) = 2r$. It holds that for $E \subset X$ with $C\dim(E) > 1$,

$$\dim_{\phi H}(E) = \dim_H(E) \quad \text{and} \quad tC\dim_\phi(E) = tC\mathbf{dim}(E).$$

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