THE MOTION OF A SPINNING PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD AS A CONSTRAINED SYSTEM

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ABSTRACT. The motion of a spinning point-particle in an external electromagnetic field is investigated as a constrained system. The comparison with the Dirac equation is discussed, and it is shown that the primary constraint can be derived explicitly from this equation. Further, the equations of motion are determined using the canonical method. The quantization of such a system is achieved using the WKB approximation.

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1. Introduction

In a previous paper [11], we proposed a general method for the quantization of constrained systems using the WKB approximation. In this method, the Hamilton-Jacobi function S is determined in configuration space in the same manner as for regular systems in the semiclassical limit. This enables one to determine the equations of motion and then to construct a suitable wave function. The constraints then become operators of this wave function to be satisfied in the semiclassical limit. In this work, we wish to apply the method to the motion of a spinning point-particle in an external electromagnetic field.

The present paper is organized as follows. In Sec. 2, the spinning point-particles are introduced in an electromagnetic field. In Sec. 3, a comparison with the Dirac equation of spin-$\frac{1}{2}$ particles is made. In Sec. 4, the equations of motion are determined using the canonical method. In Sec. 5, the quantization of the spinning point -particle is carried out using the WKB approximation. The paper closes with some concluding remarks (Sec. 6).

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2. Spinning Point-Particles in an Electromagnetic Field

The motion of a spinning point-particle of mass m and charge e in a uniform external electromagnetic field is described by the following Lagrangian [2, 4]:

$$L = \left(m + \frac{1}{2} D_{\mu\nu} F^{\mu\nu}\right) \sqrt{\dot{x}_\lambda \dot{x}^\lambda} + e A_\mu \dot{x}^\mu; \quad \mu, \nu, \lambda = 0, 1, 2, 3. \tag{1}$$

Here the charged particle is described in four-dimensional spacetime by its position x^μ; its four-velocity $\dot{x}^\mu$, the overdot denoting a derivative with respect to the invariant proper time τ; and its polarization tensor $D^{\mu\nu}$, an antisymmetric four-tensor which combines the intrinsic magnetic dipole moment $\vec{\mu}$ with the intrinsic electric dipole moment $\vec{d}$:

$$D^{\mu\nu} = \begin{pmatrix} 0 & d^1 & d^2 & d^3 \\ -d^1 & 0 & \mu^3 & -\mu^2 \\ -d^2 & -\mu^3 & 0 & \mu^1 \\ -d^3 & \mu^2 & -\mu^2 & 0 \end{pmatrix} \tag{2}$$

The quantity A^μ is the four-vector potential of the electromagnetic field tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3}$$

which describes all external fields. The term $D_{\mu\nu} F^{\mu\nu}$ represents the interaction with the magnetic moment of the particle. Here and throughout the paper the Einstein summation rule for repeated indices is used. Besides, the velocity of light in free space, c, is taken as unity.

It is often useful to relate the polarization tensor to the angular momentum spin tensor; these are proportional:

$$D^{\mu\nu} = \frac{e}{m} S^{\mu\nu}.$$

Thus, the above Lagrangian can be rewritten in terms of the angular momentum spin tensor $S^{\mu\nu}$ as:

$$L = m\left(1 + \frac{e}{2m^2} S_{\mu\nu} F^{\mu\nu}\right) \sqrt{\dot{x}_\lambda \dot{x}^\lambda} + e A_\mu \dot{x}^\mu. \tag{4}$$

The generalized momenta corresponding to this Lagrangian are

$$p^\alpha = \frac{\partial L}{\partial \dot{x}_\alpha} = m\left(1 + \frac{e}{2m^2} S_{\mu\nu} F^{\mu\nu}\right) \frac{\dot{x}^\alpha}{\sqrt{\dot{x}_\lambda \dot{x}^\lambda}} + e A^\alpha; \quad \alpha = 0, 1, 2, 3. \tag{5}$$

Up to terms linear in the electromagnetic field strength (low-energy limit), we get the following primary first-class constraint:

$$H' = \pi^\mu \pi_\mu - m^2 - e S_{\mu\nu} F^{\mu\nu} = 0, \tag{6}$$

where

$$\pi^\mu = p^\mu - e A^\mu. \tag{7}$$

The above constraint, Eq.(6), was obtained from a different starting point in [3]; it plays a crucial role in some of the further developments of the theory.

Next, we are particularly interested in particles which have no intrinsic electric dipole moment in the rest frame of the particle, as in the case for real Dirac fermions. Thus, in the rest frame of the particle ($\dot{x}^i = 0$), the spin tensor $S^{\mu\nu}$ has only three non-zero components:

$$S^{ij} = \epsilon^{ijk} S^k; \quad S^{i0} = 0; \quad i, j, k = 1, 2, 3. \tag{8}$$

3. Comparison with the Dirac Equation

At this point it is instructive to compare our constraint, Eq.(6) , with the relativistic quantum motion of spin-$\frac{1}{2}$ particles defined by the Dirac equation,

$$[\gamma^{\mu}(i\hbar\partial_{\mu} - eA_{\mu}) - m]\psi = 0. \tag{9}$$

Multiplying Eq.(9) by

$$[\gamma^{\nu}(i\hbar\partial_{\nu} - eA_{\nu}) + m]$$

leads to the equation

$$\left[(i\hbar\partial_{\mu} - eA_{\mu})^2 - m^2 - eS_{\mu\nu}F^{\mu\nu}\right]\psi = 0. \tag{10}$$

Of course, this is achieved by the standard operator identification:

$$p_{\mu} = i\hbar\partial_{\mu}; \quad S^{\mu\nu} = \frac{i\hbar}{4}[\gamma^{\mu}, \gamma^{\nu}]; \quad [\pi^{\mu}, \pi^{\nu}] = -i\hbar eF^{\mu\nu},$$

together with the anticommutation relations

$$\{\gamma^{\mu}, \gamma^{\nu}\}_{+} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}.$$

where the metric tensor $g^{\mu\nu}$ is given by

$$g_{\mu\nu} = diag(1, -1, -1, -1),$$

and the Dirac matrices are defined as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

We then obtain the direct quantum analogue of the primary constraint (6). Therefore this constraint can be reinterpreted as a statement about quantum-mechanical expectation values.

4. Equations of Motion Using the Canonical Method

Another interesting point is to find the general equations of motion by the canonical method [6,7,9,10,12]:

$$dp^{\alpha} = -\frac{\partial H_0'}{\partial x_{\alpha}}d\tau - \frac{\partial H'}{\partial x_{\alpha}}dx_0;$$
$$dx^{\alpha} = \frac{\partial H_0'}{\partial p_{\alpha}}d\tau + \frac{\partial H'}{\partial p_{\alpha}}dx_0.$$

Since the canonical Hamiltonian H_0 is identically zero:

$$H_0 = p_{\alpha}\dot{x}^{\alpha} - L \equiv 0,$$

we get

$$dp^{\alpha} = (2e\partial^{\alpha}A^{\mu}\pi_{\mu} + eS_{\mu\nu}\partial^{\alpha}F^{\mu\nu})\,dx_0; \tag{11}$$
$$dx^{\alpha} = 2\pi^{\alpha}dx_0 = 2\,(p^{\alpha} - eA^{\alpha})\,dx_0. \tag{12}$$

Choosing x_0, which is just an arbitrary parameter here, such that $dx_0 = \frac{1}{2m}dt$, then taking the second time derivative of Eq.(12), we get

$$\ddot{x}^{\alpha} = \frac{1}{m}\left(\dot{p}^{\alpha} - e\dot{A}^{\alpha}\right); \tag{13}$$

or

$$\ddot{x^{\alpha}} = \frac{e}{m}\left(F^{\alpha\mu}\dot{x_{\mu}} + \frac{1}{2m}\partial^{\alpha}S_{\mu\nu}F^{\mu\nu}\right), \tag{14}$$

which is the modified Lorentz force law.

5. Quantization Using the WKB Method

Now the primary constraint, Eq.(6), can be written more explicitly as

$$p^0 = eA_0 + \left[\left(\vec{p} - e\vec{A}\right)^2 + m^2 + eS_{\mu\nu}F^{\mu\nu}\right]^{\frac{1}{2}}. \tag{15}$$

It is interesting to quantize this constraint using the WKB approximation [8, 11]. As a special case, assume the field $\vec{B}$ to be along the z-axis. The four-vector potential A may be chosen such that

$$A_0 = A_x = A_z = 0; \quad A_y = Bx.$$

With these assumptions, together with Eq.(8), Eq.(15) becomes

$$p_0 - \left[p_x^2 + p_z^2 + (p_y - eBx)^2 + m^2 - \hbar eB\sigma_z\right]^{\frac{1}{2}} = 0. \tag{16}$$

Thus, the coordinates y and z have become cyclic in the original Lagrangian; i.e., their generalized momenta are constants of motion:

$$p_y = \alpha_y; \quad p_z = \alpha_z.$$

The corresponding Hamilton-Jacobi partial differential equation (HJPDE) is [8, 11]:

$$\frac{\partial S}{\partial x_0} = \left[\left(\frac{\partial S}{\partial x}\right)^2 + \alpha_z^2 + (\alpha_y - eBx)^2 + m^2 - \hbar eB\sigma_z\right]^{\frac{1}{2}}.$$

The solution for this equation can be obtained as

$$S(x, x_0, E) = f(x_0) + \alpha_y y + \alpha_z z + W_x(x, E). \tag{17}$$

Here $f(x_0) = -Ex_0$; then

$$W_x(x, E) = \int \left[E^2 - \alpha_z^2 - (\alpha_y - eBx)^2 - m^2 + \hbar eB\sigma_z\right]^{\frac{1}{2}} dx.$$

Finally, then, we have

$$S(x, x_0, E) = -Ex_0 + \alpha_y y + \alpha_z z + \int \left[E^2 - \alpha_z^2 - (\alpha_y - eBx)^2 - m^2 + \hbar eB\sigma_z\right]^{\frac{1}{2}} dx. \tag{18}$$

Following [1,5,10,11,12], the equations of motion arising from the above Hamilton-Jacobi function can be determined as follows:

$$\lambda_x = \frac{\partial S}{\partial E} = -x_0 + \int \frac{E dx}{\left[E'^2 - (\alpha_y - eBx)^2\right]^{\frac{1}{2}}}.$$

After integration, we obtain

$$x = \frac{1}{eB}\left[E' sin \frac{eB}{E}(\lambda_x + x_0) + \alpha_y\right], \tag{19}$$

where

$$E' = \left[E^2 - \alpha_z^2 - m^2 + \hbar eB\sigma_z\right]^{\frac{1}{2}}.$$

Further,

$$p_x = \frac{\partial S}{\partial x} = \left[E'^2 - (\alpha_y - eBx)^2\right]^{\frac{1}{2}}.$$

Substituting for x, we get

$$p_x = E' cos\left[\frac{eB}{E}(\lambda_x + x_0)\right]. \tag{20}$$

In a similar way, we can determine the equations of motion for the remaining coordinates, y and z, as

$$\lambda_y = \frac{\partial S}{\partial \alpha_y} = y - \int \frac{(\alpha_y - eBx)dx}{\left[E'^2 - (\alpha_y - eBx)^2\right]^{\frac{1}{2}}};$$

$$\lambda_z = \frac{\partial S}{\partial \alpha_z} = z - \int \frac{\alpha_z dx}{\left[E'^2 - (\alpha_y - eBx)^2\right]^{\frac{1}{2}}}.$$

These can be integrated to give

$$y = \lambda_y + \frac{E'}{eB} cos\left[\frac{eB}{E}(\lambda_x + x_0)\right]. \tag{21}$$

$$z = \lambda_z + \frac{\alpha_z}{E}(\lambda_x + x_0). \tag{22}$$

It is straightforward to check that these equations of motion are equivalent to those obtained by the canonical method.

At the quantum level, the primary constraint, Eq.(16), implies the following condition in the semiclassical limit (the WKB approximation)[8, 11]:

$$\left[p_0 - \left[p_x^2 + p_z^2 + (p_y - eBx)^2 + m^2 - e\hbar B\sigma_z\right]^{\frac{1}{2}}\right]\psi = 0, \tag{23}$$

where the wave function ψ can be determined as [8, 11]

$$\psi = \frac{1}{\sqrt{p_x}} exp\left[\left(\frac{i}{\hbar}\right)\left(-Ex_0 + \alpha_y y + \alpha_z z + \int \left[E'^2 - (\alpha_y - eBx)^2\right]^{\frac{1}{2}} dx\right)\right].$$

If ψ is an eigenvector of σ_z with eigenvalues ± 1, then we can write

$$\psi = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}; \quad \psi = \begin{pmatrix} 0 \\ \psi^-, \end{pmatrix}$$

with eigenvalues $+1$, -1, respectively.

6. Conclusion

In this paper we have investigated the motion of a spinning point -particle in an external uniform electromagnetic field as a constrained system. We have shown that the equations of motion can be derived using the canonical method. A modified Lorentz law has been deduced. We have also shown that the primary constraint can be derived from the Dirac equation of spin- $\frac{1}{2}$ particles using the standard commutation relations.

In addition, the primary first-class constraint, Eq.(6), which is derived in the low-energy limit thanks to the singularity of the original Lagrangian, has been imposed as an operator on the physical quantum states:

$$\left(\pi^\mu \pi_\mu - m^2 - eS_{\mu\nu}F^{\mu\nu}\right)\psi = 0.$$

It has been shown how the above wave function can be obtained in the semiclassical limit (the WKB approximation) in a special case when the magnetic field is directed along the z-axis. Thus, this constraint can be reinterpreted as a statement about quantum-mechanical expectation values.

References

[1] Arnold, I.V., Mathematical Methods of Classical Mechanics, 2nd edition, *Springer-Verlag*, Berlin, 1989.

[2] Barut A.O., Electrodynamics and Classical Theory of Fields and Particles, *Dover, New York*, 1980.

[3] Chichian M., Gonzales Felipe R., and Louis Martinez D., Anyons in an External electromagnetic Field: Hamiltonian and Lagrangian Formulations, *Phys. Rev. Lett.* **71** (1993) 3405-3408.

[4] Corben H.C., Classical and Quantum Theories of Spinning Particles, *Holden-Day, San Francisco*, 1968.

[5] Goldstein, H., Classical Mechanics, 2nd edition, ***Addison-Wesley***, Reading-Mass, 1980.

[6] Guler Y., Integration of Singular Systems, *Il Nuovo Cimento*, **107B** (1992) 1143-1149.

[7] Nawafleh Khaled I., Constrained Hamiltonian Systems: A Preliminary Study, M.Sc. thesis, University of Jordan, Amman, Jordan, 1998.

[8] Nawafleh Khaled I., Rabei Eqab M. and Ghassib Humam B., Quantization of the Motion of a Relativistic Particle in an External Electromagnetic Field Using the WKB Approximation, *Submitted for publication.*

[9] Rabei Eqab M. and Guler Y., Hamilton-Jacobi Treatment of Second-Class Constraints, *Phys. Rev. A.* **46**(1992) 3513-3515.

[10] Rabei Equab M., On Hamiltonian Systems with Constraints, *Hadronic Journal*, **19**, (1996) 597-605.

[11] Rabei Eqab M., Nawafleh Khaled I. and Ghassib Humam B., Quantization of Constrained Systems Using the WKB Approximation, *Phys. Rev. A.* ***66***, 024101 (2002).

[12] Rabei Eqab M., Nawafleh Khaled I. and Ghassib Humam B., Some Physical Applications of the Canonical Method, *Hadronic Journal*, **22** (1999) 241-255.