THE FINE SPECTRE OF SOME CESÀRO GENERALIZED OPERATORS DEFINED ON ℓ^p $(p > 1)$

AMELIA BUCUR
DEPARTMENT OF MATHEMATICS
UNIVERSITY "LUCIAN BLAGA" OF SIBIU
STR. DR. I. RA'TIU, NR. 7
550012 SIBIU, ROMANIA

(Received 2 October 2003)

ABSTRACT. The aim of the paper is the study of the fine spectre for a class of Cesàro generalized operators, Rhaly operators, when those operators are defined on the spaces ℓ^p, $p > 1$.

AMS Classification: 33C45
Keywords: Spectrum, Eigenvalue, Eigenvector.

Let s and ℓ^p, $p > 1$, be the linear spaces of sequences:

$$s = \{f = (f(n))_{n\in\mathbf{N}} : f(n) \in K\}$$

$$\ell^p = \left\{ f = (f(n))_{n\in\mathbf{N}} : \sum_{n=0}^{\infty} |f(n)|^p < \infty \right\},$$

where $K = \mathbf{R} \bigvee \mathbf{C}$.

For $a = (a_n) \in s$ we define $R_a : s \to s$ by

$$(R_a f)(n) = a_n \sum_{i=0}^{n} f(i) \quad , \quad n \in \mathbf{N},$$

for every $f = (f(n))_{n\in\mathbf{N}} \in s$.

The space s may be replaced with other spaces of sequences. In these case, Rhaly operator R_a determines and is determined by an infinite matrix, lower

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 2 (2004)

triangular, noted also with R_a:

$$R_a = \begin{bmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & \dots \\ a_2 & a_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The dual of an Rhaly operator $R_a : \ell^p \to \ell^p$ is the operator $R_a^* : \ell^q \to \ell^q$, to which is associated by the infinite matrix, triangular:

$$R_a^* = \begin{bmatrix} \overline{a_0} & \overline{a_1} & \overline{a_2} & \dots \\ 0 & \overline{a_1} & \overline{a_2} & \dots \\ 0 & 0 & \overline{a_2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For $a = \left(\dfrac{1}{n+1}\right)_{n\in\mathbf{N}} \in s$ one obtains the discrete Cesàro operator and for $a = \left(\dfrac{1}{(n+1)^z}\right)_n \in \mathbf{N}$, with $z \in \mathbf{C}$, one obtains the z–Cesàro operator.

G. Leibowitz [1], [2], studies the continuity, compactness of these operators defined on the space ℓ^p, $p > 1$. M.C.Jr. Rhaly [3] studies these Rhaly operators which as the discrete Cesàro operator C_1, are not compact, in ℓ^2. The spectra $\sigma_p(R_a, \ell^2)$ and $\sigma(R_a, \ell^2)$ are investigated.

In this paper $\mathcal{I}$ present some results concerning the spectra of Rhaly operators from $(\ell^p, \ell^p)^*$, where:

$\varphi(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a$ is bijective and $(\lambda I - R_a)^{-1}$ is continuous$\}$;
$\sigma(R_a, \ell^p) = K \setminus \varphi(R_a, \ell^p)$;
$\sigma_p(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a$ is not injective$\}$;
$\sigma_c(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a$ is injective, is not surjective and $\overline{(\lambda I - R_a)(\ell^p)} = \ell^p\}$;
$\sigma_r(R_a, \ell^p) = \{\lambda \in K : \lambda I - R_a$ is injective, and $\overline{(\lambda I - R_a)(\ell^p)} \neq \ell^p\}$.

A Rhaly operator $R_a : \ell^p \to \ell^p$ $(p > 1)$ is correctly defined if the sequence $((n+1)a_n)$ is bounded, and R_a is continuous.

Let $S = \{a_n : n \in \mathbf{N}\}$.

Lemma 1. *Let R_a be a Rhaly matrix ($a \in s$), $C = \lambda I - R_a$, such that $c_{jj} \neq 0$ $\forall j \in N$. Then C^{-1} has the entries:*

$$
(1)\qquad
\begin{aligned}
&c_{jj} = \frac{1}{\lambda - a_j} \quad , \quad j \in N; \\
&c_{ij} = 0 \quad , \quad i,j \in N \, , \; i < j \\
&c_{ij} = c_{j+r,j} = \lambda^{r-1} a_{j+r} \left[\prod_{k=j}^{j+r} (\lambda - a_k) \right]^{-1} , \\
&\qquad r \in N^*, \; i,j \in N, \; i = j + r.
\end{aligned}
$$

Proof. The method of the mathematical induction is used.

For the calculation of the elements c_{jj} one uses determinants of some lower triangular matrices with the entries on the diagonal $\lambda - a_k$, $k \neq j$. Obvious $c_{00} = \dfrac{1}{\lambda - a_0}$.

Supposing that, for an $i \in \mathbf{N}^*$, $c_{i0}, c_{i1}, \ldots, c_{i,i-1}$ have the given form, it can be proved the entries $c_{i+1,0}, c_{i+1,1}, \ldots, c_{i+1,i}$ are also given by the above relations.

The condition $c_{i+1,i}(\lambda - a_i) + c_{i+1,i+1}(-a_{i+1}) = 0$ implies:

$$c_{i+1,i} = \frac{a_{i+1}}{(\lambda - a_i)(\lambda - a_{i+1})}.$$

Since $c_{i+1,i-1}(\lambda - a_{i-1}) + c_{i+1,i}(-a_i) + c_{i+1,i+1}(-a_{i+1}) = 0$ is equivalent with

$$c_{i+1,i-1} = \frac{a_i}{\lambda - a_{i-1}} \cdot \frac{a_{i+1}}{\prod_{k=i}^{i+1}(\lambda - a_k)} + \frac{a_{i+1}}{\lambda - a_{i-1}} \cdot \frac{1}{\lambda - a_{i+1}},$$

we have

$$c_{i+1,i-1} = \frac{\lambda \cdot a_{i+1}}{(\lambda - a_{i-1})(\lambda - a_i)(\lambda - a_{i+1})} .$$

Also, from the equality

$$c_{i+1,i-2}(\lambda - a_{i-2}) + c_{i+1,i-2}(-a_{i-1}) + c_{i+1,i}(-a_i) + c_{i+1,i+1}(-a_{i+1}) = 0,$$

it results that

$$
\begin{aligned}
c_{i+1,i-2} = {} & \frac{\lambda a_{i-1} a_{i+1}}{(\lambda - a_{i-2})(\lambda - a_{i-1})(\lambda - a_i)(\lambda - a_{i+1})} + \\
& + \frac{a_i a_{i+1}(\lambda - a_{i-1})}{(\lambda - a_{i-2})(\lambda - a_{i-1})(\lambda - a_i)(\lambda - a_{i+1})} + \\
& + \frac{a_{i+1}(\lambda - a_{i-1})(\lambda - a_i)}{(\lambda - a_{i-2})(\lambda - a_{i-1})(\lambda - a_i)(\lambda - a_{i+1})} = \\
& = \frac{\lambda^2 a_{i+1}}{(\lambda - a_{i-2})(\lambda - a_{i-1})(\lambda - a_i)(\lambda - a_{i+1})}
\end{aligned}
$$

Ressoning analogously we can find expression of

$$c_{i+1,i-3}, c_{i+1,i-4}, \ldots, c_{i+1,0}.$$

Let $a = (a_n)_{n \in \mathbf{N}}$ a sequence convergent of real numbers.

Theorem 1. *If R_a is a Rhaly operator with $a_n \neq 0$, $n \in N$ and $((n+1)a_n)$ bounded, then $0 \in \varphi(R_a, \ell^p)$.*

Proof. The operator $0 \cdot I - R_a$ is bijective. The sequence $((n+1)a_n)$ being bounded, the Rhaly operator R_a is continuous on ℓ^p for any $p > 1$.

It remains to prove that the operator $(0 \cdot I - R_a)^{-1}$ is not continuous. It is equivalent to the fact that the operator $(0 \cdot I - R_a^*)$ is not surjective.

Be $g \in \ell^2$. It can be shown that the equation $(0 \cdot I - R_a^*)f = g$ is equivalent with $f(n) = \dfrac{g(n-1) - g(n)}{a_n}$, $n \in \mathbf{N}^*$. $g \in \ell^q$ is equivalent with $\sum\limits_{n=0}^{\infty} |g(n)|^2 < \infty$. The sequence (a_n) is bounded (thus, the sequence $((n+1)a_n)$ is unlimited, which contradicts the hypothesis). Hence $\exists M > 0$ one of its superior limits. It follows that

$$\sum_{n=1}^{\infty} |f(n)|^q = \sum_{n=1}^{\infty} \frac{|g(n-1) - g(n)|^q}{|a_n|^q} \geq$$

$$\geq \frac{1}{M^2} \sum_{h=1}^{\infty} |g(n-1) - g(n)|^q \geq \frac{1}{M^2} \sum_{h=1}^{\infty} \frac{1}{n} = \infty,$$

for g chosen so that

$$g(n) - g(n-1) = \frac{1}{n^{1/q}} \quad , \quad n > 1,$$

and

$$g(n-1) + \frac{1}{n^{1/q}} \leq \frac{1}{n-1} \quad , \quad n \geq 1.$$

Then,

$$\sum_{h=1}^{\infty} |g(n)|^q = \sum_{h=1}^{\infty} \left| g(n-1) + \frac{1}{n^{1/q}} \right|^q \leq \sum_{h=1}^{\infty} \frac{1}{(n-1)^q} < \infty.$$

Therefore $f \notin \ell^q$, that is the equation $(0 \cdot I - R_a^*)f = g$ has no solution, and the operator $0 \cdot I - R_a^*$ is not surjective, which remained to be demonstrated.

Theorem 2. *Be R_a a Rhaly operator so that $((n+1)a_n)$ is bounded. If $\lambda \in \sigma(R_a, \ell^p)$ and $\lambda \notin S$, then $\lambda \in \sigma_c(R_a, \ell^p)$.*

Proof. We notice that $\lambda I - R_a$ is a triangular matrix. $\lambda \notin S$ is equivalent with $\lambda - a_n \neq 0$, $n \in \mathbf{N}$; $det\,(\lambda I - R_a) \neq 0$, and $\lambda I - R_a$ is an injective operator.

To study the following equality, $\overline{(\lambda I - R_a)(\ell^p)} = \ell^p$, it can be shown that the equation $(\lambda I - R_a^*)f = 0$ has no solution $f \in \ell^q$, f non–null. We have

$$f(n+1) = \frac{\lambda}{\lambda - a_n} f(n) \; ; \tag{2}$$

$$f(n+1) = \frac{\lambda^{n+1}}{(\lambda - a_n)(\lambda - a_{n-1}) \ldots (\lambda - a_0)} f(0) \; , \; n \geq 0. \tag{3}$$

We assume that $f(0) \neq 0$ (for $f(0) = 0$, $f = 0$). Then

$$\left| \frac{f(n+1)}{f(n)} \right|^q = \frac{|\lambda|^q}{|\lambda - a_n|^q} \geq 1.$$

Thus $\sum |f(n)|^q$ is divergent, that is the equation $(\lambda I - R_a^*)f = 0$ has no solution $f \in \ell^q$, f non–null.
Therefore, $\lambda \in \sigma_c(R_a, \ell^p)$.

Theorem 3. *Be R_a a Rhaly operator with the quality that the elements of the main diagonal of the matrix R_a are distinct, non–null and the sequence $((n+1)a_n)$ is bounded. If $\lambda = a_n$, $n \in \boldsymbol{N}$, then $\lambda \in \sigma_p(R_a, \ell^p)$.*

Proof. It must be shown that the operator $\lambda I - R_a$ is not injective. It can be shown that $(\lambda I - R_a)f = 0$, $f \in \ell^p$ and $f \neq 0$.

The equation $(\lambda I - R_a)f = 0$ implies

$$\sum_{k=0}^{n-1} a_n f(k) = (\lambda - a_n) f(k) \quad , \quad n \geq j. \tag{4}$$

If $\lambda = a_j$, $j \geq 1$, we get

$$a_j - a_n = a_n \sum_{k=0}^{n-1} f(k) \quad , \quad n \geq j\ ; \tag{5}$$

$$f(n) = a_j \left(\frac{1}{a_{n+1}} - \frac{1}{a_n} \right). \tag{6}$$

It follows that

$$\sum_{n=0}^{\infty} |f(n)|^p < \infty$$

and

$$|a_j| \sum_{n=j}^{\infty} \left| \frac{1}{a_{n+1}} - \frac{1}{a_n} \right|^p < \infty\ .$$

Be $m < -1$ an inferior limit of the sequence (a_n). Then

$$|a_j|^p \sum_{n=j}^{\infty} \left| \frac{1}{a_{n+1}} - \frac{1}{a_n} \right|^p \leq 2|a_j|^p \sum_{n=j}^{\infty} \left| \frac{1}{a_n} \right|^p <$$

$$< 2|a_j|^p \sum_{n=j}^{\infty} \left| \frac{1}{m} \right|^p = 2 \left| \frac{a_j}{m} \right|^p \frac{|m|}{|m|-1} < \infty.$$

If $\lambda = a_0$, $(\lambda I - R_a)e_0 = 0$.
So the operator $\lambda I - R_a$ is not injective; $\lambda \in \sigma_p(R_a, \ell^p)$.

The following theorems contain the results with reference to Rhaly operators R_a with the quality to

$$0 < \lim_{n\to\infty} (n+1)a_n < \infty.$$

Theorem 4. *Be R_a a Rhaly operator with the quality that a is a sequence of positive real numbers. If $\lambda \in K$ has the qualities:*

$$\left| \lambda - \frac{\alpha p}{2(p-1)} \right| < \frac{\alpha p}{2(p-1)}$$

and $\lambda \in S$, then $\lambda \in \sigma_c(R_a, \ell^p)$.

Proof. $\lambda I - R_a$ is a triangular matrix. $\lambda \notin S$, then all its elements from the main diagonal are non–null, the operator $\lambda I - R_a$ being then injective.
$\overline{(\lambda I - R_a)(\ell^p)} = \ell^p$ is a quality that can be studied using the injectivity of the operator $\lambda I - R_a^*$. For this it can be shown that the equation $(\lambda I - R_a^*)f = 0$ has

no solution $f \in \ell^q$, non–null.

It results the relation of recurrence:

$$f(n+1) = \frac{\lambda}{\lambda - a_n} f(n) \quad , \quad n \in \mathbf{N}.$$

From this results the equalities:

$$f(n+1) = \frac{\lambda^{n+1}}{(\lambda - a_n)(\lambda - a_{n-1}) \ldots (\lambda - a_0)} f(0), \ n \geq 1. \tag{7}$$

We assume that $f(0) \neq 0$, ($f(0) = 0$ implies $f = 0$). Then,

$$\left| \frac{f(n+1)}{f(n)} \right|^q = \frac{|\lambda|^q}{|\lambda - a_n|^q} \geq 1. \tag{8}$$

Hence $\sum |f(n)|^q$ is divergence, that is the equation $(\lambda I - R_a^*) f = 0$ has no solution $f \in \ell^q$, f non–null.

Theorem 5. *Be R_a a Rhaly operator with $a = (a_n)$ a sequence of positive real numbers. If*

$$\left| \lambda - \frac{\alpha p}{2(p-1)} \right| < \frac{\alpha p}{2(p-1)},$$

$\alpha \neq 0$, $\lambda \neq 0$, then $\lambda \in \sigma_c(R_a, \ell^p)$.

Proof. Be $\lambda \in K$ with the qualities from the non- equality, $det\,(\lambda I - R_a) \neq 0$ so the operator $\lambda I - R_a$ is injective ($\lambda \notin S$). Thus, $\lambda \in S$ and $\lambda = a_n$ with $n \in \mathbf{N}$.

The property

$$\left| \lambda - \frac{\alpha p}{2(p-1)} \right| < \frac{\alpha p}{2(p-1)}$$

is equivalent with:

i) $\lambda = \dfrac{\alpha p}{p-1} = \alpha q$, and $a_n = q \lim\limits_{n \to \infty} (n+1) a_n$, implies the following equality:

$$\alpha - \alpha = q \lim_{n \to \infty} \left\{ (n+1) \lim_{n \to \infty} [(n+1)a_n - (n+1)a_{n+1}] \right\} -$$

$$-q \lim_{n \to \infty} (n+2) a_{n+1}.$$

Then $\lambda = 0$, which contradicts the hypothesis.

ii) $\lambda + \dfrac{\alpha p}{2(p-1)} = \dfrac{\alpha p}{2(p-1)}$ implies $\lambda = 0$, which is false.

We consider the equation $(\lambda I - R_a^*) f = 0$, $f \in 0$, $f \neq 0$. We get the relations

$$f(n+1) = \frac{\lambda^{n+1}}{(\lambda - a_n)(\lambda - a_{n-1}) \ldots (\lambda - a_0)} f(0), \ n \in \mathbf{N}. \tag{9}$$

We assume that $f(0) \neq 0$ (if not, $f = 0$). Then

$$\left| \frac{f(n+1)}{f(n)} \right|^q = \frac{|\lambda|^q}{|\lambda - a_n|^q} \geq 1.$$

so $\sum |f(n)|^q$ is divergence, that is the equation $(\lambda I - R_a^*) f = 0$ has no solution $f \in \ell^q$, f non–null. Then $\lambda \in \sigma_c(R_a, \ell^p)$.

Observation 1. a) *The results above can be particularized for z–Cesàro operators.*

b) *The spectrum of a Rhaly operator $R_a : \ell^p \to \ell^p$ with the sequence $((n+1)a_n)$ bounded, where $a = (a_n)$ is a sequence of positive real numbers verifies the equality:*

$$\sigma(R_a, \ell^p) = \left\{ \left| \lambda - \frac{\alpha p}{2(p-1)} \right| \leq \frac{\alpha p}{2(p-1)} \bigcup S \right\}.$$

References

[1] Leibowitz G., The imaginary part of a Rhaly matrix, *Bull. London Math. Soc.*, 19 (1987) 167–168.

[2] Leibowitz G., Rhaly matrices, *J. Math. Analysis Appl.*, 128 (1987) 272–286.

[3] Rhaly, M., Terraced matrices, *Bull. London Math. Soc.* 21 (1989) 399–406.