POLYGONS AND CHAOS

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ABSTRACT. An infinite number of periodic trajectories are derived for the map giving rise to the Mandelbrot set at a value of the parameter corresponding to the extreme point on the real axis of the Mandelbrot set. Beginning with the edge of a family of star n-gons as the seed, the trajectory of the logistic map cycles through a sequence of edges of other star n-gons. Each n-gon for n odd is shown to have its own characteristic cycle length. The logistic map is shown to be the first of two infinite families of maps, all exhibiting periodic trajectories, derived from two families of polynomials, the Chebyshev polynomials and another related to the Lucas sequence. Using the family of Lucas polynomials, the Mandelbrot set is generalized to an infinite family of sets with similar properties.

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1. INTRODUCTION

This paper describes a connection between the edges of star polygons and dynamical systems in a state of chaos. A sequence of dynamical maps is derived from the Chebyshev polynomials of the first kind and another family of polynomials related to the Lucas sequence, and these maps exhibit periodic trajectories of all periods with each polygon having its own characteristic cycle length. The first polynomial of the family is the map giving rise to the Mandelbrot set at a value of its parameter corresponding to the extreme left-hand point on the real axis of the Mandelbrot set. This leads to a new family of Mandelbrot sets and to new connections between chaotic dynamics and both Euclidean geometry and the theory of numbers.

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2. p-CYCLES

Consider the complex transformation $z \to z^m$ restricted to the unit circle S^1 which can be denoted by $P_m : \exp(i\theta) \to \exp(im\theta)$. Observe that this map takes $\operatorname{Re} z = \cos\theta$ into $\operatorname{Re} z^m = \cos m\theta$.

This is the defining property of the *Chebyshev polynomials of the first kind* T_m, viz.,

$$T_m : \quad \operatorname{Re} z \to \operatorname{Re} z^m \quad \text{or} \quad T_m(\cos) = \cos(m\theta). \tag{1a}$$

In dynamical system terms, there is a *semi-conjugacy* $h_T : S^1 \to [-1,1]$ defined by,

$$h_T(\exp(i\theta)) = \cos\theta \tag{1b}$$

from the map $P_m : S^1 \to S^1$ to the map $T_m : [-1,1] \to [-1,1]$, $x \mapsto T_m(x)$. It should be noted that much of what we discuss in the sequel can be inferred from this semi-conjugacy.

Beginning with the seed $\operatorname{Re} z_0$ where $z_0 = \exp(i\theta)$,

$$\operatorname{Re} z_0 \to \operatorname{Re} z_1 \to \operatorname{Re} z_2 \to \ldots \to \operatorname{Re} z_q \to \ldots$$

where the sequence $\{\operatorname{Re} z_q\}$ for $z_q = \exp(im^q\theta)$ is the trajectory of T_m.

If $\theta_0 = \frac{2\pi}{n}$ for n an odd integer, then $z_q = \exp(\frac{2\pi i m^q}{n})$ where m is relatively prime to n. Therefore the points of the trajectory are roots of the equation $z^n = 1$ for z complex the solutions of which are,

$$z = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} = \exp(\frac{2\pi k i}{n}) \text{ for } k = 0,1,2,3,\ldots,n-1 \tag{2}$$

the n *roots of unity*. The points in a cartesian coordinate system,

$$(\cos\frac{2\pi k}{n}, \sin\frac{2\pi k}{n}) \text{ for } k = 0,1,2,3,\ldots,n-1 \tag{3}$$

lie at the vertices of a regular n-gon with unit radius that we shall refer to us as a *cyclotomic n-gon.* If the point $(1,0)$ is distinguished, the other points satisfy the equation,

$$\frac{z^n-1}{z-1} = z^{n-1} + z^{n-2} + \ldots + z^3 + z^2 + z + 1 = 0$$

referred to us as n-th cyclotomic polynomial when n is prime. We shall continue to use this terminology for n non-prime but odd. The vertices for the case $n = 7$ are shown in Figure 1. Notice that vertices in the upper half-circle and lower half-circle are symmetrically placed. Therefore, since the real parts of these vertices are equal, only the values of k corresponding to the upper half-circle need to be considered. This is a consequence of the semi-conjugacy.

with unit radius and vertices located at the roots of unity. Since $cos\frac{2\pi k}{n} = cos\frac{2\pi(n-k)}{n}$ or $cos\frac{2\pi k}{n} = cos\frac{2\pi(-k)}{n}$ it follows that $k \pmod n \equiv -k \pmod n$. Therefore, $\operatorname{Re} z_p = \operatorname{Re} z_0$ when p is the smallest positive integer such that,

$$k = m^p \equiv \pm 1 \pmod n \quad \text{where } m \text{ is relatively prime to } n. \tag{4}$$

In this case the trajectory forms a cycle of length p, or what we shall refer to us as a *p-cycle.*

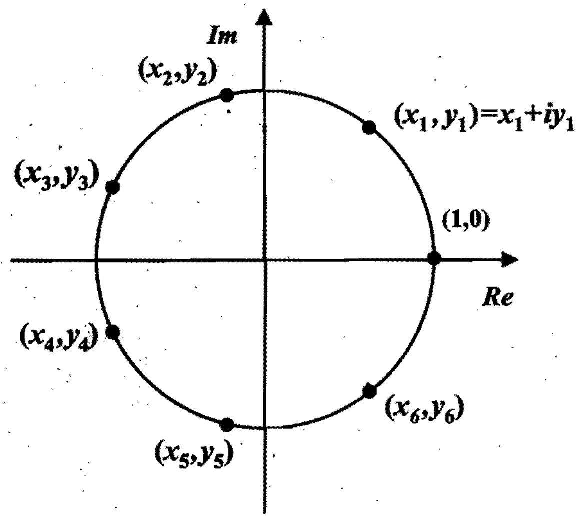

FIGURE 1. A cyclotomic 7-gon with unit radius and vertices located at the roots of unity.

3. CHARACTERIZATION OF p-CYCLES

p-Cycles can be determined for all odd values of n. However, if n is an odd prime number or a power of an odd prime number, with the help of number theory, p-cycles can be characterized in greater detail. The *Euler-Fermat theorem*, the theory of quadratic residues and the concept of a primitive root modulo n all play a role. If n is not a prime number the number of integers relatively prime to n, the *Euler phi function* also plays a role.

The period p can be characterized with the help of the Euler-Fermat theorem.

Theorem 1 (Euler-Fermat Theorem) If m and n are positive integers and m is relatively prime to n then $m^{\phi(n)} \equiv 1 \pmod{n}$ where $\phi(n)$ is the Euler phi function [1].

Remark: If n is prime then $\phi(n) = n - 1$ and Theorem 1 reduces to "Fermat's Little Theorem".

Corollary: $m^{\frac{\phi(n)}{2}} \equiv \pm 1 \pmod{n}$. (5)

As a result of the Corollary and Equation 4, the cycle length p is a factor of $\frac{\phi(n)}{2}$.

Definition: m is a *quadratic residue* modulo n if and only if there exists an integer a such that $a^2 = m \pmod{n}$.

The *generalized Euler criterion* characterizes m in the mod n system.

Theorem 2 (Generalized Euler criterion) If n is an odd prime or the power of an odd prime, then for all m,

$$(m|n) = m^{\frac{\phi(n)}{2}} \pmod{n}$$

where the *Legandre symbol* $(m|n)$ equals 1 when m is a quadratic residue mod n and equals -1 when m is not a quadratic residue mod n.

We claim that the number T of p-cycles is

$$T = \frac{\phi(n)}{2p} \tag{6}$$

and address the question of how to find these T, p-cycles. Consider $x_q = \cos\frac{2\pi k_q}{n}$, where k_q is relatively prime to n. Each x_q lies on one p-cycle. There are $\phi(n)$ indices relatively prime to n. However, since $k \pmod{n}$ and $-k \pmod{n}$ lead to the same trajectory point, only the $\frac{\phi(n)}{2}$ k-values where $0 \leq k < \frac{\phi(n)}{2}$ and k is relatively prime to n need be considered. This is a consequence of the semi-conjugacy and can be observed in Figure 1 for the case of $n = 7$, by the symmetric points in the upper half-circle with positive values of k having the same real parts as the negative k-values in the lower half-circle.

Let S be the set of indices relatively prime to n for $0 \leq k < \frac{\phi(n)}{2}$,

$$S = \{k_0, k_1, k_2, \ldots, k_{\frac{\phi(n)}{2}-1}\}.$$

Trajectory 1 has indices,

$$k_q^{(1)} = m^q : k_0^{(1)} = 1, m, m^2, \ldots, m^q, \ldots, m^p = \pm 1 \pmod{n}$$

corresponding to the p-vector

$$\overrightarrow{k^{(1)}} = (k_0^{(1)} = 1, k_1^{(1)}, \ldots, k_q^{(1)}, \ldots, k_{p-1}^{(1)})$$

Let $k_0^{(i)}$ be the smallest integer from S not used in trajectories $1, 2, \ldots, i-1$. Then trajectory i has indices,

$$k_q^{(i)} = m^q : k_0^{(i)}, k_0^{(i)}m, k_0^{(i)}m^2, \ldots, k_0^{(i)}m^q, \ldots, k_0^{(i)}m^p = \pm k_0^{(i)} \pmod{n}$$

corresponding to the p-vector $\overrightarrow{k^{(i)}} = (k_0^{(i)}, k_1^{(i)}, \ldots, k_q^{(i)}, \ldots, k_{p-1}^{(i)})$. The process continues until $i = T$ at which point all indices from S have been exhausted.

There are two cases in which we can conclude that $p = \frac{\phi(n)}{2}$ and consequently that $T = 1$:

1) when $\frac{\phi(n)}{2}$ is prime;
2) when m is a primitive root modulo n.

In all other cases we are able to conclude only that p divides $\frac{\phi(n)}{2}$. Case 1 is obvious since $\frac{\phi(n)}{2}$ has no divisors except 1 and itself. We consider case 2.

Definition: The integer m is a primitive root modulo n if and only if for arbitrary $k \pmod{n}$ relatively prime to n there exist a unique integer q such that $m^q = k \pmod{n}$ where q is denoted by $q = \text{ind}_m k$ [1].

Lemma: If m is a *primitive root modulus* n then $m^{\frac{\phi(n)}{2}} = -1 \pmod{n}$.

Proof: From Theorem 1 follows that $m^{\frac{\phi(n)}{2}} = \pm 1 \pmod{n}$. However, if m is a primitive root modulo n then it cannot be that $m^{\frac{\phi(n)}{2}} = 1 \pmod{n}$. Therefore the lemma follows.

Theorem 3: If n is odd, then it has primitive roots modulo n only if n is a prime number or the power of a prime [1].

Therefore, from the Euler criterion, case 2 occurs only when m is not a quadratic residue mod n.

If n is a composite number that is not the power of a prime, nothing can be concluded about the period p other then that it is a factor of $\frac{\phi(n)}{2}$.

Table 1 lists p-values for $m = 2, 3, 4$ and 5 and $n = 7, 9, 11, 13, 15$ and 17.

Table 1 Least Exponents p such thet $m^p = \pm 1 \pmod{n}$

m	$n=7$	$n=9$	$n=11$	$n=13$	$n=15$	$n=17$
2	3	3	5	6	4	4
3	3	...	5	3	...	8
4	3	3	5	3	2	2
5	3	3	5	2	...	8

The process for determining the T, p-cycles is illustrated for $m = 2$, $n = 7, 9, 11$ and 17 in Section 5 and $m = 4$, $n = 15$ in Section 6.

4. Polygons and Chaos

For k and n relatively prime integers and n odd, given a value of $k = 0, 1, 2, \ldots, \frac{n-1}{2}$, it can be shown that there exists a value of $j = 0, 1, 2, \ldots, \lfloor \frac{2n-1}{2} \rfloor$ such that,

$$2 \cos \frac{2\pi k}{n} = 2 \sin \frac{2\pi j}{2n} \text{ where } 4k + j = n \qquad (7a \text{ and } b)$$

for j and $2n$ relatively prime. A regular n-gon $\{n/e\}$ where n and e are relatively prime integers, is a polygon with n vertices and a cycle of directed edges of equal length for which the k-th vertex connects to $(k+e)$-th vertex. Figure 2 illustrates the three species of positively directed (clockwise) star 7-gons. There are also three negatively directed (counterclockwise) star 7-gons. The values $2 \sin \frac{2\pi j}{2n}$, can be shown to be the edge lengths of star $2n$-gons $\{2n/j\}$ with unit radii when $j > 0$ with clockwise oriented edges, and $\{2n/n+j\}$ when $j < 0$ with counterclockwise oriented edges, where j is relatively prime to $2n$ [2], i.e., $e = j$ when $j > 0$ and $e = 2n - j$ when $j < 0$. Therefore we can view the p-cycles of the previous section corresponding to an n-cyclotomic polynomial as being sequences of edges of star $2n$-gons with unit radii for odd n. The k-values for the T, p-cycles were represented in Section 3 by a p-vector. We also denote the cycles of j-values of T, p-cycles by the vector $\overrightarrow{j^{(i)}} = (j_0^{(i)}, j_1^{(i)}, \ldots, j_q^{(i)}, \ldots, j_{p-1}^{(i)})$ for $i = 1, 2, \ldots, T$ and the e-indices for the cycle of edges by the p-vector $\overrightarrow{e^{(i)}} = (e_0^{(i)}, e_1^{(i)}, \ldots, e_q^{(i)}, \ldots, e_{p-1}^{(i)})$.

oriented star 7-gons.

This encourages us to consider the transformations of the circle of radius 2, S_2^1, defined by

$$P_{m_2}: \quad 2\exp(i\theta) \rightarrow 2\exp(im\theta).$$

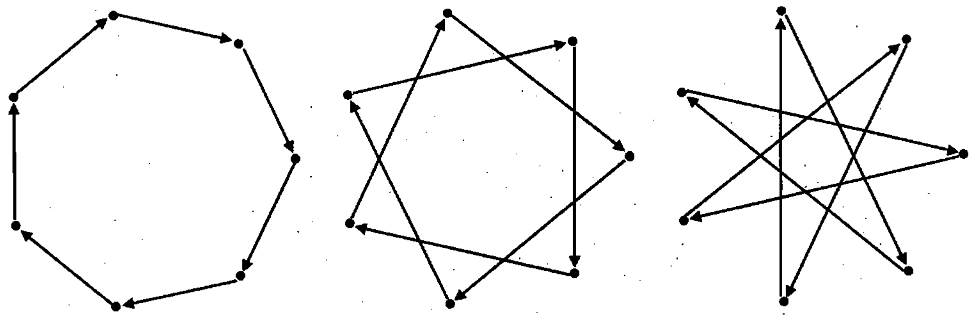

FIGURE 2. The three positively oriented star 7-gons.

This map takes $2\mathrm{Re}\, z = 2\cos\theta$ into $2\mathrm{Re}\, z^m = 2\cos m\theta$, which is also the defining property of the *Lucas polynomials* [2,3],

$$L_m(2\cos\theta) = 2\cos m\theta. \tag{8a}$$

In other words, the maps $2z \to 2z^m$ restricted to S_2^1 are semi-conjugate to the Lucas polynomials $\{L_m\}$ for $m \geq 1$ via the map,

$$h_L(2\exp(i\theta)) = 2\cos\theta. \tag{8b}$$

This has some interesting dynamical consequences.

Letting $x = \cos\theta$, as a result of Equations 1a and 8a, $\frac{1}{2}L_m(2x) = T_m(x)$ and transformations 1a and 8a can be rewritten as,

$$x \mapsto T_m(x) \quad \text{and} \quad 2x \mapsto L_m(2x)$$

where the first Chebyshev and Lucas polynomials are listed in Table 2.

Table 2. The Lucas and Chebyshev Polynomials

L_1	x	1	T_1	x	1
L_2	$x^2 - 2$	3	T_2	$2x^2 - 1$	3
L_3	$x^3 - 3x$	4	T_3	$4x^3 - 3x$	7
L_4	$x^4 - 4x^2 + 2$	7	T_4	$8x^4 - 8x^2 + 1$	17
L_5	$x^5 - 5x^3 + 5x$	11	T_5	$16x^5 - 20x^3 + 5x$	41
L_6	$x^6 - 6x^4 - 9x^2 - 2$	18	T_6	$32x^6 - 48x^4 + 18x^2 - 1$	99
L_7	$x^7 - 7x^5 + 14x^3 - 7x$	29	T_7	$64x^7 - 112x^5 + 56x^3 - 7x$	239
...	...	...	...	...	...

Notice that, ignoring signs, the sum of the coefficients of the Lucas polynomials form the Lucas sequence or L-sequence $\{l_k\}$: 1 3 4 7 11 18 29 ... which satisfies the recursion relation $a_{n+2} = a_{n+1} + a_n$, i.e., it is a Fibonacci sequence beginning with 1, 3. On the other hand the Chebyshev polynomials form the Chebyshev sequence (T-sequence) $\{t_k\}$: 1 3 7 17 41 99 239 ... which satisfies the recursion relation $a_{n+2} = 2a_{n+1} + a_n$, i.e., a Pell sequence beginning with 1, 3. The standard Fibonacci sequence (F-sequence) $\{f_k\}$ is: 0 1 1 2 3 5 8 13 21 ... and $l_k = f_{k-1} + f_{k+1}$ while the standard Pell sequence (P-sequence) $\{p_k\}$ is: 0 1 2 5 12 29 70 ... and $t_k = p_{k-1} + p_k$ [2,3]. Also,

$$\lim_{k\to\infty} \frac{l_{k+1}}{l_k} = \tau_1 \quad \text{and} \quad \lim_{k\to\infty} \frac{t_{k+1}}{t_k} = \tau_2$$

where $\tau_1 = \frac{1+\sqrt{5}}{2}$ and $\tau_2 = 1 + \sqrt{2}$. Also $\tau_1 - \frac{1}{\tau_1} = 1$ and $\tau_2 - \frac{1}{\tau_2} = 2$. For this reason τ_1 and τ_2 are referred to as the golden and silver means respectively. The Lucas and Chebyshev polynomials can also be generated from initial polynomials, 2 and z for the Lucas and 1 and z for the Chebyshev, by the recursive formulas:

$$L_{n+1} = zL_n - L_{n-1} \quad \text{and} \quad T_{n+1} = 2z_n - T_{n-1} \tag{9}$$

where $z \in C$.

Consider the second Lucas polynomial L_2, $z^2 - 2$ and its iterative map,

$$z \mapsto z^2 - 2, \tag{10a}$$

It represents the extreme left-hand point on the real axis, i.e., $c = -2$, of the *Mandelbrot set* (see Figure 3) for the map,

$$z \mapsto z^2 + c, \tag{10b}$$

where c is a complex number. The values, c, in the Mandelbrot set, M, are defined as the bounded iterates of the complex polynomial map P_c in Map 10b evaluated as its critical point $z = 0$ where $P_c(0) = 0$, i.e.,

$$M = \{c \in C : P_c^n(0) \text{ does not approach } \infty\}. \tag{10c}$$

In what follows, we restrict z to the real line where Map 10a is written as

$$x \mapsto x^2 - 2, \tag{11a}$$

for $x \in \Re$. Beginning with a seed x_0, Equation 11a generates the sequence: $x_0, x_1, x_2, \ldots, x_q, \ldots$ the *trajectory* of the map.

The map given by Equation 11a is a transformed version of the logistic map,

$$x \mapsto \lambda x(1-x) \quad \text{for} \quad \lambda = 4 \tag{11b}$$

which has been studied in great detail [4,5]. The fact that $c = -2$ in Equation 10b or $\lambda = 4$ in Equation 11b, means that this map is in a state of *chaos*, and the portion of the interval $[0,1]$ that does not lead to escape is a Cantor set. According to a theorem of Sharkovskii [6], after a trajectory of length 3 appears, trajectories with periods of every length are present. As a result of our analysis when $\lambda = 4$ or $c = -2$, these cycles can be characterized as edges of star $2n$-gons of unit radii in which each value of n has its own characteristic cycle length.

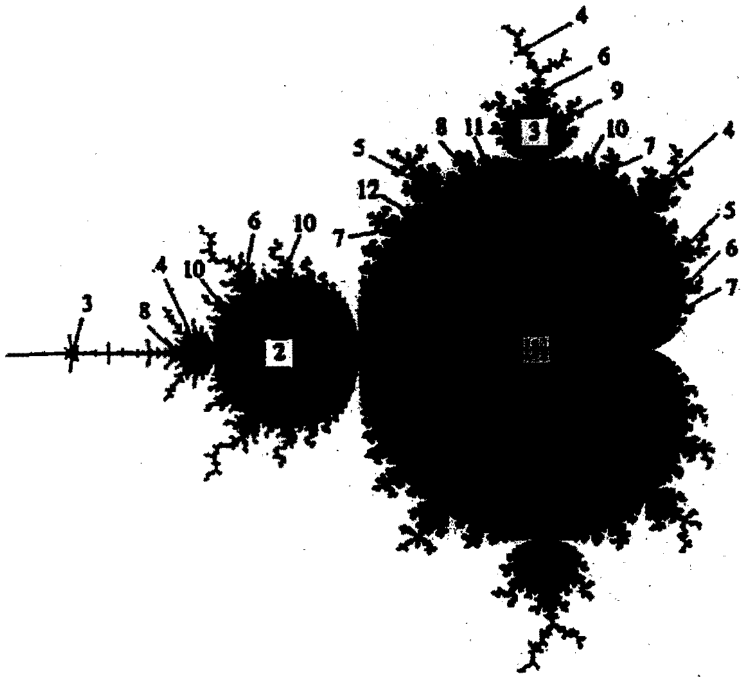

FIGURE 3. The Mandelbrot set.

5. RESULTS

For any value of m relatively prime to n the L_m transformations defined by Equation 8a have p-cycles, some p-values of which are listed in Table 1. In this section we shall study the dynamics of the map given by Map 11a corresponding to $m = 2$, in some detail giving the results of the p-cycles for cyclotomic n-gons where $n = 7, 9, 11$ and 17. The trajectory of the map will be shown to correspond to a sequence of edge lengths of star $2n$-gons with unit radii. The value of p is determined from Equation 4 while the number of trajectories T is determined from Equation 6. The k-values from the set S of integers $1 \leq k < \frac{\phi(n)}{2}$, relatively prime to n are determined by the procedure described in Section 3 for each of the T, p-cycles, remembering that values of k and $-k$ lead to the same trajectory point because of semi-conjugacy. The j-values are determined from k using Equation 7b. For the e-values, $e = j$ if $j > 0$ and $e = 2n + j$ if $j < 0$. The index q records the order of the elements of these vectors in the p-cycle. The results are recorded in Table 3.

Example 1: $m = 2$, $n = 7$ where 2 is not a primitive root modulo 7 but 2 is a quadratic residue mod 7. $S = \{1, 2, 3\}$.

$$k_q^{(1)} = 2^q : k_0^{(1)} = 1, 2, 4, 8 = 1 \pmod 7.$$

Therefore $p = 3$ and $T = 1$ and

$$\overrightarrow{k^{(1)}} = (1,2,3),\ \overrightarrow{j^{(1)}} = (3,-1,-5),\ \overrightarrow{e^{(1)}} = (3,13,9)$$

For the cyclotomic 7-gon, values of $2\cos\frac{2\pi k}{7}$ for $k = 1,2,3$ are listed in Table 3 corresponding to the trajectory of Map 11a: $x_0 \to x_1 \to x_2 \to x_0$, or

$$2\cos\frac{2\pi}{7} = 1.2469\ldots \to 2\cos\frac{4\pi}{7} = -0.44509\ldots \to 2\cos\frac{6\pi}{7} = -1.80189\ldots \to 1.2469\ldots$$

These are edge lengths of star 14-gons. If the vertices of the 14-gon are numbered from 0 to 13 then a sequence of edges can be associated with these star $2n$-gons as shown in Figure 4. The cycle of edges extend from vertex number 0 of the 14-gon to the darkened vertices $3 \to 13 \to 9 \to 3$. The *orders* of the edges in the cycles are also indicated in Table 3, beginning with the seed $q = 0$ corresponding to $k = 1$ and in Figure 4 by the boxed integers. Notice the regular skip pattern of highlighted vertices: 4, 4, 6. We shall find this pattern to hold for all cycles when n is prime. The proof is found in Appendix A. If n is not prime then there exists a modified pattern depending on the number of integers missing from the set S. Also note that for each edge cycle within the $2n$-gon its mirror image having the same edge lengths but opposite signs, illustrated by open vertices, is also a 3-cycle, i.e., $11 \to 1 \to 5 \to 11\ldots$ for the 14-gon. Each edge cycle of a $2n$-gon will have a corresponding mirror image cycle. Finally, Adamson discovered that the product of the cycle values equals -1, e.g.,

$$(1.2469\ldots)(-0.44509\ldots)(1.80189\ldots) = -1,$$

and that the product of edges within any cycle will always equal ± 1 for $m = 2$, a result that can be proven from a number theoretic description. The 3-cycle for $m = 2$, $n = 7$ assures chaos by Sharkovskii's theorem.

from vertex 0 to vertex k (closed circles) of the star 14-gon, $\{14/k\}$, are elements of the 3-cycle of the Mandelbrot map related to the cyclotomic 7-gon. The number in the square boxes are the order of the points in the 3-cycle. The open circles represent the edges of a mirror image 3-cycle.

Example 2: $m = 2$, $n = 9$ where 2 is a primitive root modulo 9 and therefore not a quadratic residue mod 9. $S = \{1,2,4\}$.

$$k_q^{(1)} = 2^q : k_0^{(1)} = 1,2,4,8 = -1 \pmod 9.$$

Therefore $p = 3$ and $T = 1$ and

$$\overrightarrow{k^{(1)}} = (1,2,4),\ \overrightarrow{j^{(1)}} = (5,1,-7),\ \overrightarrow{e^{(1)}} = (5,1,11)$$

The skip pattern is 4,6,8 deviates from the 4,4,...,4,6 pattern due to the absence of 3 from S. Values of the sequence of edge lengths $2\cos\frac{2\pi k}{9}$ for $k = 1,2,3$ are listed in Table 3.

Example 3: $m = 2$, $n = 11$ where 2 is a primitive root modulo 11 and therefore not a quadratic residue mod 11. $S = \{1,2,3,4,5\}$.

$$k_q^{(1)} = 2^q : k_0^{(1)} = 1,2,4,8 = -3, 16 = 5, 32 = -1 \pmod{11}.$$

Therefore $p = 5$ and $T = 1$ and

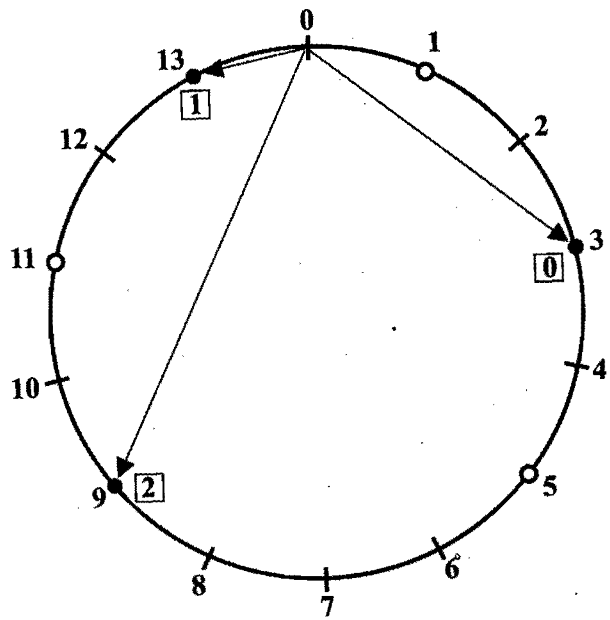

FIGURE 4. The edge lengths from vertex 0 to vertex k (closed circles) of the star 14-gon, $\{14/k\}$, are elements of the 3-cycle of the Mandelbrot map related to the cyclotomic 7-gon. The number in the square boxes are the order of the points in the 3-cycle. The open circles represent the edges of a mirror image 3-cycle.

$$\overrightarrow{k^{(1)}} = (1, 2, 3, 4, 5),\ \overrightarrow{j^{(1)}} = (7, 3, -5, -1, -9),\ \overrightarrow{e^{(1)}} = (7, 3, 17, 21, 13)$$

The skip pattern is 4,6,4,4,4. Values of the sequence of edge lengths $2\cos\frac{2\pi k}{11}$ for $k = 1, 2, 3, 4, 5$ are listed in Table 3 and shown on the 22-gon of Figure 5.

Example 4: $m = 2$, $n = 17$ where 2 is not a primitive root modulo 17 while 2 is a quadratic residue mod 17. $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$k_q^{(1)} = 2^q : k_0^{(1)} = 1, 2, 4, 8, 16 = -1 \pmod{17}.$$

Therefore $p = 4$ and $T = 2$ and

$$\overrightarrow{k^{(1)}} = (1, 2, 4, 8),\ \overrightarrow{j^{(1)}} = (13, 9, 1, -15),\ \overrightarrow{e^{(1)}} = (13, 9, 1, 19)$$

where $k_0^{(2)} = 3$.

$$k_q^{(2)} = 3 \cdot 2^q : k_0^{(2)} = 3, 6, 12 = -5, 24 = 7, 48 = -3 \pmod{17}.$$

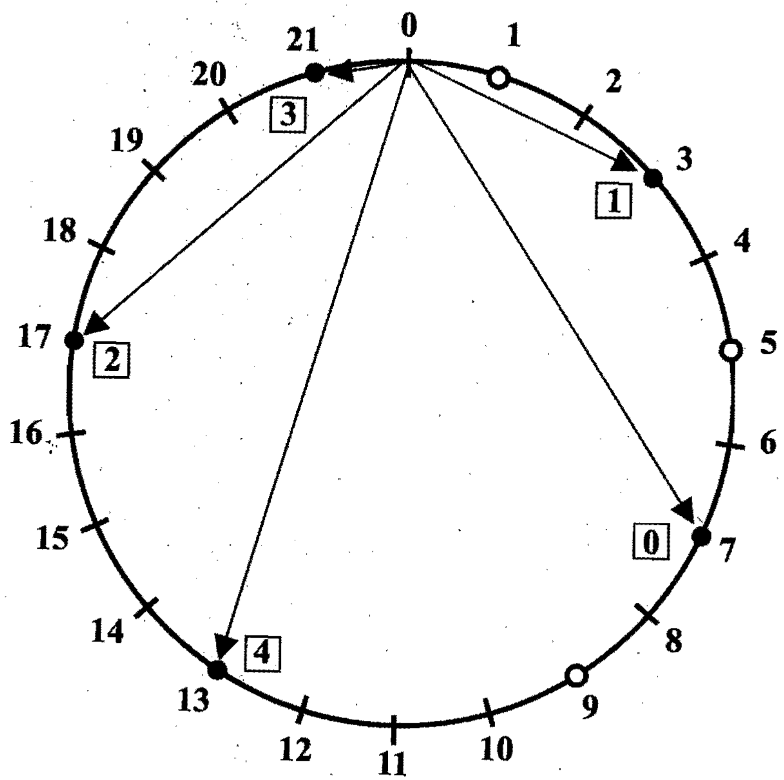

FIGURE 5. The edge lengths from vertex 0 to vertex k of the star 22-gon, $\{22/k\}$, are elements of the 5-cycle of the Mandelbrot map related to the cyclotomic 11-gon. The number in the square boxes are the order of the points in the 5-cycle.

$$\overrightarrow{k^{(2)}} = (3,6,5,7),\ \overrightarrow{j^{(2)}} = (5,-7,-3,-11),\ \text{and}\ \overrightarrow{e^{(2)}} = (5,27,31,23).$$

Trajectories 1 and 2 together form a skip pattern of 4,4,4,6,4,4,4. Values of the sequence of edge lengths for trajectories 1 and 2 are listed in Table 3. Order numbers for trajectory 2 are indicated by primes.

Table 3. Cycles for the Logistic Equation Corresponding to Cyclotomic Polygons

	Cyclotomic 7-gon				Cyclotomic 9-gon			
k	$2\cos\frac{2\pi k}{7}$	$2\sin\frac{\pi j}{14}$ j	$\{14/j\}$ or $\{14/14+j\}$	Order q	$2\cos\frac{2\pi k}{9}$	$2\sin\frac{\pi j}{18}$ j	$\{18/j\}$ or $\{18/18+j\}$	Order q
1	1.24696 ...	3	3	0	1.53208 ...	5	5	0
2	−0.44509 ...	−1	13	1	0.34729 ...	1	1	1
3	−1.80189 ...	−5	9	2	−0.500 ...	...	...	...
4					−1.87938 ...	−7	11	2

	Cyclotomic 11-gon				Cyclotomic 17-gon			
k	$2\cos\frac{2\pi k}{11}$	$2\sin\frac{\pi j}{22}$ j	$\{22/j\}$ or $\{22/22+j\}$	Order q	$2\cos\frac{2\pi k}{17}$	$2\sin\frac{\pi j}{34}$ j	$\{34/j\}$ or $\{34/34+j\}$	Order q
1	1.68250 . . .	7	7	0	1.8649 . . .	13	13	0
2	0.83082 . . .	3	3	1	1.4780 . . .	9	9	1
3	−0.28462 . . .	−1	21	3	0.89147 . . .	5	5	0'
4	−1.30972 . . .	−5	17	2	0.1845. . .	1	1	2
5	−1.91898 . . .	−9	13	4	−0.5473 . . .	−3	31	2'
6					−1.205 . . .	−7	27	1'
7					−1.70043 . . .	−11	23	3'
8					−1.9659 . . .	−15	19	3

6. Other Lucas Polynomials

What we have discovered for the second Lucas polynomial, holds true for all the other Lucas polynomials from Table 2. For example, the third Lucas polynomial L_3 with alternating signs leads to the recursive map,

$$x \mapsto x^3 - 3x \tag{12}$$

With $m = 3$, taking the seed $x_0 = 2\cos\frac{2\pi}{n}$ for odd n, results in the sequence $\{2\cos\frac{2\pi k}{n}\}$ where now $k = 1, 3, 9, 27, 81\ldots \pmod{n}$, i.e., $3^q \pmod{n}$ for $q = 0, 1, 2, 3, \ldots, p$, a p-cycle as predicted by Table 1. The sequence is based on powers of 3 since we are using the 3^{rd} Lucas polynomial. The cycle lengths can be determined by the procedure given in Section 2. This generalizes the m-th Lucas polynomial L_m with alternating signs in which case the iterates correspond to k-values that are powers of m. These polynomial maps also represent dynamical systems in a state of chaos. However, the products of the edges in a cycle are not always equal to ± 1. We give one example:

Example: $m = 2$, $n = 15$. 4 is not a primitive root modulo 15. $S = \{1, 2, 4, 7\}$.

$$k_q^{(1)} = 4^q : k_0^{(1)} = 1, 4, 16 = 1 \pmod{15}.$$

Therefore $p = 2$ and $T = 2$ and

$$\overrightarrow{k^{(1)}} = (1, 4),\ \overrightarrow{j^{(1)}} = (11, -1),\ \overrightarrow{e^{(1)}} = (11, 29) \text{ where } k_0^{(2)} = 2.$$

$$k_q^{(2)} = 2 \cdot 4^q : k_0^{(2)} = 2, 8 = -7, 32 = 2 \pmod{15}.$$

$$\overrightarrow{k^{(2)}} = (2, 7),\ \overrightarrow{j^{(2)}} = (7, -13),\ \text{ and } \overrightarrow{e^{(2)}} = (7, 17).$$

The skip pattern is 4,6,12,8 reflecting the three missing integers from S.

7. New Mandelbrot and Julia Sets

Each of the Lucas polynomials has new Mandelbrot and Julia sets associated with it defined by the dynamical map:

$$z \mapsto L_m(c, z) \text{ for } c, z \in C$$

where $L_m(c, z)$ is generated by a similar recursion relation as Equation 9 that generated the Lucas polynomials with alternating signs in Table 2, i.e.,

$$-c, z,\ z(z) + c = z^2 + c,\ z(z^2 + c) - z = z^3 + (c - 1)z,$$

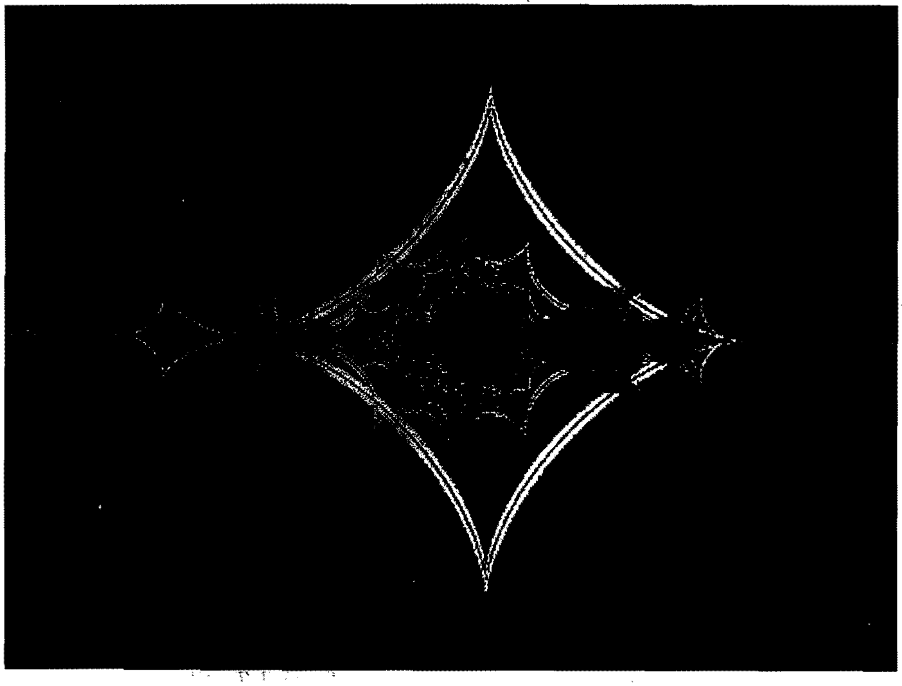

FIGURE 6. The generalized Mandelbrot set for $m = 4$.

$$z(z^3 + (c-1)z) - (z^2 + c) = z^4 + (c-2)z^2 - c, \ etc. \tag{13}$$

When $c = -2$ and z is restricted to the real line, these polynomials result in the Lucas polynomials with alternating signs. By *new Mandelbrot sets* we mean that, just as for the standard Mandelbrot set, $c \in C$ is in the Mandelbrot set of $L_m(c,\cdot)$ when the (iterative) orbits of the critical points of $L_m(c,\cdot)$ are bounded. For polynomials of even degree these new Mandelbrot sets are defined by Equation 10c where polynomials P_c are given by Equations 13. Mandelbrot sets for the odd degree polynomials have not yet been determined.

Figure 6 illustrates the new Mandelbrot set for $m = 4$ or $z \mapsto z^4 + (c-2)z^2 - c$ while Figure 7 shows one of the new Julia sets for L_3 or $z \mapsto z^3 + (c-1)z$ and a value of $c = 0.1683 + 0.7543i$. These were created by Javier Barrallo [2001]. Recall that the Julia set may be defined as the closure of the set of repelling periodic points for the function that generates Mandelbrot set.

8. ANOTHER DOUBLING SEQUENCE

The function $R_2(x) = 4x(1-x)$ has the property that $R_2(\sin^2\theta) = \sin^2 2\theta$. $R_2(x)$ is recognized as the logistic Equation 11b for the value $\lambda = 4$, the value of the parameter at which the non-repelling periodic points lie on a Cantor set [5]. Because

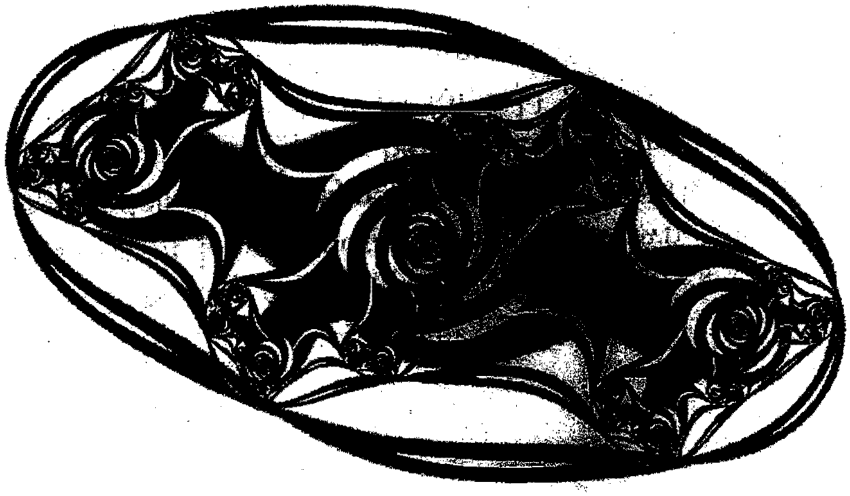

FIGURE 7. A new Julia set for $m = 3$ at a value of $c = 0.1683 + 0.7543i$.

of the doubling property, R_2 shares with L_2 and T_2 that, starting with an initial value of $\sin^2 \frac{2\pi}{n}$, R_2 generates the same p-cycles as L_2 and T_2. There also exists a sequence of polynomials with the property that $R_m(\sin^2\theta) = \sin^2 m\theta$.

9. Chaos and Number

There is an intimate relationship between chaos theory and number. We have shown that properties of number also lie at the basis of the polygon cycles. There is another relationship between the Lucas maps and pure number. The inverses of certain integers n when expanded in base b yield repeating decimals. For example in base 10, $\frac{1}{3} = 0.333\ldots$, an expansion with period 1 while $\frac{1}{11} = 0.0909\ldots$ has a period 2 expansion. It is easy to see that the period of the repeating decimal corresponds to the least integer p such that $b^p \equiv 1 \pmod{n}$. The first digit of the expansion of $\frac{1}{n}$ in base b is q where $b = qn + r$. When p is the least integer such that $b^p \equiv 1 \pmod{n}$ the remainder r is 1 after p decimal places. Consequently, the next quotient is again q and the process begins again leading to a period p expansion.

In order to apply this theory to p-cycles, Equation 4 must be squared to get,

$$(m^p)^2 \equiv 1 \pmod{n} \tag{14}$$

Equation 14 can be rewritten as,

$$(m^p - 1)(m^p + 1) \equiv 0 \pmod{n}.$$

Therefore, when the integer n is prime or the power of a prime, Equation 14 is equivalent to Equation 4. Therefore if n is taken to be a prime or the power of a prime, setting $b = m^2$ guarantees that the period of expansion of $\frac{1}{n}$ in base b yields the identical periods as the trajectories described in this paper for the

Lucas polynomial maps. For example, expanding $\frac{1}{7}$ in the base 4 (the square of 2), $\frac{1}{7} = 0.021021021\ldots = 0.\overline{021}$, a repeating decimal with a 3-cycle. Likewise $\frac{1}{11} = 0.\ldots = 0.\overline{01131}$ expanded in base 4, a 5-cycle, while $\frac{1}{17} = 0.\ldots = 0.\overline{0033}$, a 4-cycle. An algorithm for converting the decimal expansion of $\frac{1}{n}$ in base 10 to a decimal in base b is given in Appendix B.

10. Conclusion

Many things have come together in this study. We have shown the relationship between chaos and number theory. The well known theorem of Sharkovskii predicts that once a period 3-cycle appears in a dynamical system, periods of all lengths occur. We have shown that at an extreme point of the Mandelbrot set beyond which orbits of the logistic equation begin to escape, each of these periods can be characterized by a sequence of edge lengths of a family of star $2n$-gons, each n having a characteristic cycle length. The theory has been applied to all odd values of n, although when n is a prime or the power of a prime we are able to characterize the p-cycles with greater specificity. The value of $n = 7$, the heptagon, results in a 3-cycle assuring chaos. Corresponding to the family of Lucas polynomials we have been able to extend the notion of the Mandelbrot set for the quadratic polynomial map to an infinite class of polynomials with similar properties. Coxeter has shown star polygons to be related to the two-dimensional projections of higher-dimensional polyhedra or polytopes [7]. Geometry has shown itself once again to be the rich well-spring of mathematics. Rather than jettisoning these roots, the theory of dynamical systems and chaos has strongly embraced them.

11. Appendix A

Theorem A1 The trajectory points of the map $x \mapsto L_m(x)$ (see Table 2) are lengths of diagonals of a $2n$-gon stretching from vertex 0 to vertex k where the vertices are numbered from $k = 0$ to $2n - 1$. If n is an odd prime number and m is a primitive root modulo n, then the trace of the trajectory of this map intercepts every fourth vertex of the $2n$-gon and one spacing of 6 vertices.

Proof If n is an odd prime number and m is a primitive root modulo n, then according to Theorem 2, the trajectory $x \mapsto L_m(x)$ has period $\frac{n-1}{2}$ corresponding to the k-values: $1, 2, 3, \ldots, \frac{n-1}{2}, 1, 2, \ldots$ Values of j follow from Equation 7b,

$$j = (-4k + n) \pmod{2n}.$$

The difference $\Delta j = j_{k+1} - j_k$ of successive j-values corresponding to successive k-values is determined from,

$$\Delta j = -4\Delta k \pmod{2n} \tag{A1}$$

where Δj refers to the principal value. From Equation A1, $\Delta j = 4$ for $\Delta k = -1$ and $\Delta j = (6 - 2n) \pmod{2n} = 6$ for $\Delta k = \frac{n-1}{2} - 1 = \frac{n-3}{2}$. For example, referring to Example 3 of Section 5, for an 11-cyclotomic polynomial the k-values and their corresponding j-values are,

$$k: \ 1, 2, 3, 4, 5, 1, 2, \ldots$$

$$j:\ 7,3,-1,-5,-9,7$$

which has the following trace on the 22-gon,

$$7,3,21,17,13,7.$$

The spacings between these values are seen to be the sequence: 4,4,4,4,6.

12. Appendix B

A decimal in base 10 can be written in any other base by the following procedure illustrated for converting $\frac{1}{7} = 0.\overline{142857}$ in base 10 to base 4.

1. Multiply the decimal in base 10 by 4 and record a 0 if the result is less than 1, otherwise record the integer part. For example, $0.\overline{142857} \times 4 = 0.57148\ldots$ so record a 0 in the 1^{st} decimal place.

2. Multiply the result again by 4 to get $2.2857142\ldots$ and record a 2 as the 2^{nd} decimal place.

3. Multiply the decimal part of the preceding number by 4 to get: $1.1428568\ldots$ and record a 1 as the 3^{rd} decimal place.

4. Again multiply the decimal part of the preceding number by 4, but since the decimal part repeats we have the repeating decimal in base 4: $0.\overline{021}$.

In general, consider the decimal expansion of a real number $0 < a_0 < 1$ in base 10. Its decimal expansion in base b is then: $0.b_1b_2b_3\ldots$ where $b_n = [(a_{n-1} - [a_{n-1}])b]$ for $n = 1,2,3,\ldots$ where $[a]$ denotes the integer part of a.

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