

ON THE RADIUS OF CONVEXITY OF FUNCTIONS WHOSE DERIVATIVE HAS A POSITIVE REAL PART

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ABSTRACT. In this paper: (i) we give a short proof of the MacGregor theorem for the radius of convexity of the class P' of the functions stated in the title and indicate all extremal functions; (ii) more generally, we find the radius of convexity of order α of the class P' with all extremal functions; (iii) we find the radius of convexity of the class P' when the second coefficient is fixed with all extremal functions; (iv) we compare our results with other results.

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1. INTRODUCTION

Let P' denote the class of all analytic functions

(1) $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $|z| < 1$,
of bounded turning, i.e. for which

(2) $\operatorname{Re} f'(z) > 0$, $|z| < 1$,

(see the notation and the name, for example, in Goodman [5, vol. I, p. 101, problem 7]). MacGregor [8, pp. 533-534, Theorem 2] has found the radius of convexity r_c of the class P' (see also Goodman [5, Vol. I, p. 129, problem 18; Vol. II, p. 96]), i.e. in the disc

(3) $|z| \leq r_c = \sqrt{2} - 1$

the inequality

(4) $\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \geq 0$

holds with an extremal function

(5) $f_0(z) = -z - 2\ln(1-z) = z + 2 \sum_{n=2}^{\infty} \frac{z^n}{n} \in P'$, $|z| < 1$,

at the "critical point" $z = -(\sqrt{2} - 1)$ for which in (4) the equality sign holds. Later the MacGregor theorem was rediscovered by Libera [6, p.149, Corollary (2.2)] in a different way and with the extremal functions

$$(6) f(z, \epsilon) = -z - 2\bar{\epsilon}\ln(1 - \epsilon z) = z + 2 \sum_{n=2}^{\infty} \epsilon^{n-1} \frac{z^n}{n} \in P', |z| < 1, |\epsilon| = 1,$$

at the "critical points" $z = -\bar{\epsilon}(\sqrt{2} - 1)$ for which in (4) the equality holds. The function (5) follows from (6) for $\epsilon = 1$. Evidently the function (6) is a rotation of the function (5), i.e. $f(z, \epsilon) = \bar{\epsilon}f_0(\epsilon z)$.

In this paper: (i) we give a short proof of the MacGregor theorem and show that all the extremal functions are those in (6); (ii) we find the radius of convexity of order $-\infty < \alpha < 1$ of the class P' ; (iii) we find the radius of convexity of the class P' when the second coefficient a_2 is fixed; (iv) we compare our results with other results.

2. ANOTHER PROOF OF THE MACGREGOR THEOREM

It follows from (1)-(2) that the derivative $f'(z)$ is subordinate to the function $(1+z)/(1-z)$ and hence

$$(7) f'(z) = \frac{1+g(z)}{1-g(z)}, |z| < 1,$$

where $g(z)$ is an analytic function in $|z| < 1$ with $g(0) = 0$ and $|g(z)| < 1$ (see, for example, Goodman [5, Vol. I, p. 85, Theorem 9]). Via the logarithmic derivative of (7) we obtain

$$(8) \frac{f''(z)}{f'(z)} = \frac{2g'(z)}{1-g^2(z)}, |z| < 1.$$

The modulus of the derivative $h'(z)$ of an analytic function $h(z)$ of bound one in $|z| < 1$ satisfies the sharp inequality (see, for example, Caratheodory [1, Vol. II, p. 18, Inequality (290.4), and Vol. I, p. 76, Equality (82.1)])

$$(9) |h'(z)| \leq \frac{1-|h(z)|^2}{1-|z|^2}, |z| < 1,$$

with the equality sign only for the function

$$(10) h(z) = \eta \frac{a-z}{1-\bar{a}z}, |z| < 1, |\eta| = 1, 0 \leq |a| < 1.$$

For the function $g(z)$ in (7), i.e. if $h(z) \equiv g(z)$ with

$h(0) \equiv g(0) = 0$ from (9)-(10) we obtain the sharp inequality

$$(11) |g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2}, |z| < 1,$$

with the equality sign only for the function

$$(12) g(z) = \epsilon z, |z| < 1, |\epsilon| = 1.$$

Now from (8) and (11) we obtain

$$(13) \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(1 + \frac{2zg'(z)}{1-g^2(z)}\right) \geq 1 - 2\left|\frac{zg'(z)}{1-g^2(z)}\right| \geq$$

$$1 - 2\frac{|z||g'(z)|}{1-|g(z)|^2} \geq 1 - \frac{2|z|}{1-|z|^2} = \frac{1-2|z|-|z|^2}{1-|z|^2} = \frac{(\sqrt{2}+1+|z|)(\sqrt{2}-1-|z|)}{1-|z|^2} \geq 0$$

for $0 \leq |z| \leq \sqrt{2} - 1$. The last equality sign in (13) holds only for the function (12) at the point $z = -\bar{\epsilon}(\sqrt{2} - 1)$.

Thus from (12) and (7) we obtain the functions (6) which are unique extremal functions with the corresponding "critical points" $z = -\bar{\epsilon}(\sqrt{2} - 1)$ for this problem: For the functions (6) we have

$$(14) 1 + \frac{zf''(z, \epsilon)}{f'(z, \epsilon)} = \frac{1+2\epsilon z - \epsilon^2 z^2}{1-\epsilon^2 z^2}, |z| < 1, |\epsilon| = 1.$$

The right-hand side of (14) vanishes for $z = -\bar{\epsilon}r_c$ where r_c is determined by (3).

3. RADIUS OF CONVEXITY OF ORDER ALPHA OF THE CLASS P'

Let α be real with $-\infty < \alpha < 1$. From the sharp estimate in (13) and (14) we deduce that: In the disc

$$(15) |z| \leq r_c(\alpha) = \frac{\sqrt{2-2\alpha+\alpha^2}-1}{1-\alpha}, \quad -\infty < \alpha < 1,$$

the sharp inequality

$$(16) \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \geq \alpha, \quad -\infty < \alpha < 1,$$

holds with the equality sign only for the functions (6) at the "critical points" $z = -\bar{e}r_c(\alpha)$. Hence the number $r_c(\alpha)$ in (15) is the radius of convexity of order α of the class P' which decreases from 1 to 0 as α in (16) increases from $-\infty$ to 1 where $r_c(0) \equiv r_c = \sqrt{2}-1$ is the radius of convexity (3) of the class P' .

The conception for the radius of convexity of order α with $0 \leq \alpha < 1$ was first introduced by Robertson in [10]. Its extension for the largest range $-\infty < \alpha < 1$ was made in [13, p. 160, Definitions].

4. RADIUS OF CONVEXITY OF THE CLASS P' WITH FIXED SECOND COEFFICIENT

Let P be the class of all analytic functions

$$(17) p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad |z| < 1, \quad b_1 \equiv b,$$

which have positive real part, i.e. $\operatorname{Re} p(z) > 0$ in $|z| < 1$. The class P was introduced by Carathéodory and the coefficients b_n in (17) satisfy $|b_n| \leq 2$, $n \geq 1$ (see, for example, Goodman [5, Vol. I, p. 80, Theorem 3]). If $b_1 \neq 0$ in (17), the function $p(z \exp(-i \arg b_1)) \in P$, and therefore, it can be assumed that the coefficient $b_1 \equiv b$ in (17) is nonnegative, i.e. $0 \leq b \leq 2$, where $b = 2$ only for the function $p(z) = (1+z)/(1-z) \in P$. For the class P normalized in this way Tepper [12, pp. 523-524, Lemma 3] proved that in the disc $|z| < 1$ the inequality

$$(18) \left| \frac{zp'(z)}{p(z)} \right| \leq \frac{|z|}{1-|z|^2} \frac{b|z|^2+4|z|+b}{|z|^2+b|z|+1}, \quad 0 \leq b \leq 2,$$

holds. Later Goel [4, pp. 34-35, Lemma] rediscovered the Tepper inequality (18) in a different way, and noted the extremal function

$$(19) p(z) = \frac{1-z^2}{1-bz+z^2} \in P$$

for (18) at the "critical point" $z = -r$ where $0 < r < 1$. With the help of the extremal function (10) for the inequality (9) and the Tepper method in [12, pp. 523-524, Lemma 3] we can prove that the extremal function (19) is unique for the inequality (18) with the unique "critical": $z = -r$, $0 < r < 1$. In their papers Tepper [12] and Goel [4] have not applied the inequality (18) to the convexity of the functions (1) of the class P' with fixed second coefficient. Now we will fill this white field as follows. If $a_2 \neq 0$ in (1), the function $\exp(i \arg a_2) f(z \exp(-i \arg a_2)) \in P'$, and therefore, we can assume that the coefficient $a_2 \equiv a$ in (1) is nonnegative, i.e. $0 \leq a \leq 1$. Since $f(z) \in P'$ implies $f'(z) \in P$ then from (18) - (19) we obtain the inequality

$$(20) \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{2|z|}{1-|z|^2} \frac{a|z|^2+2|z|+a}{|z|^2+2a|z|+1}, \quad |z| \leq 1, \quad 0 \leq a \leq 1,$$

where the equality sign holds only for the function

$$(21) f'(z) = \frac{1-z^2}{1-2az+z^2} \in P, \quad |z| \leq 1, \quad 0 \leq a \leq 1,$$

at the "critical point" $z = -r$, $0 < r < 1$.

By (20) we obtain

$$(22) \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq 1 - \left|\frac{zf''(z)}{f'(z)}\right| \geq 1 - \frac{|2z|}{1-|z|^2} \frac{a|z|^2+2|z|+a}{|z|^2+2a|z|+1} \\ = \frac{1-4|z|^2-4a|z|^3-|z|^4}{(1-|z|^2)(|z|^2+2a|z|+1)} \geq 0$$

if $|z| \leq r_c(a)$ where $r = r_c(a)$ with $0 < r_c(a) < 1$ is the unique positive root of the equation

$$(23) A(r, a) \equiv 1 - 4r^2 - 4ar^3 - r^4 = 0, 0 \leq a \leq 1.$$

The last equality sign in (22) holds only for the function (21), i.e. for the function

$$(24) f(z, a) = -z - e^{i\psi} \ln(1 - ze^{-i\psi}) - e^{-i\psi} \ln(1 - ze^{i\psi}) = z + 2 \sum_{n=2}^{\infty} \cos(n - 1)\psi \frac{z^n}{n} \in P',$$

where

$$(25) \psi = \arccos a, 0 \leq \psi \leq \frac{\pi}{2},$$

at the "critical point" $z = -r_c(a)$. In fact, for (24)-(25) we have

$$(26) 1 + \frac{zf''(z, a)}{f'(z, a)} = \frac{1-4z^2+4az^3-z^4}{(1-z^2)(1-2az+z^2)}.$$

According to (23) the right-hand side of (26) vanishes for $z = -r_c(a)$. Hence the number $r_c(a)$ is the radius of convexity of the class P' of functions (1)-(2) with fixed second coefficient $a_2 \equiv a$ ($0 \leq a \leq 1$).

Further we will examine the solution of the equation (23) for $0 \leq a \leq 1$. Since $A(\sqrt{2}-1, a) = 4(5\sqrt{2}-7)(1-a) \geq 0$ with the equality sign only for $a = 1$, then $r_c(a) \geq \sqrt{2}-1$ with the equality sign only for $a = 1$. Now we will find the explicit expression of $r_c(a)$ with the help of the Ferrari method (see, for example, Dickson [2, Chapter IV, pp. 38-46] or [3, pp. 50-51]). The equation (23) can be written in the form

$$(27) -A(r, a) = (r^2 + 2ar + t)^2 - [2(2a^2 - 2 + t)r^2 + 4atr + 1 + t^2] = 0,$$

where t is an arbitrary parameter. If t is any root of the equation

$$(28) 4a^2t^2 - 2(2a^2 - 2 + t)(1 + t^2) = 0,$$

then the expression in the square brackets of (27) is a perfect square of a linear function of r . The normal form of (28) is

$$(29) B(t, a) \equiv t(t-1)^2 - 2(1-a^2) = t^3 - 2t^2 + t - 2(1-a^2) = 0.$$

In (29) the real roots $t \geq 0$ with the equality sign only for $a = 1$ ($0 \leq a \leq 1$).

The Horner transformation

$$(30) t = u + \frac{2}{3},$$

reduces (29) to the equation

$$(31) u^3 - \frac{1}{3}u + 2\left(a^2 - \frac{26}{27}\right) = 0, 0 \leq a \leq 1.$$

By the Cardano formula (see, for example, Dickson [2, Chapter III, pp. 31-37] or [3, pp. 45-49]) from (31) we obtain that

$$(32) u = \frac{1}{3} \sum_{v=0}^2 \sqrt[3]{26 - 27a^2 + (-1)^v \sqrt{(25 - 27a^2)(27 - 27a^2)}}$$

for $0 \leq a \leq 1$ where the values of the two cubic radicals are taken so that their product is equal to 1. Thus the formulas (32) and (30) give the roots of the equation (29) (or (28)) in t . In particular, for $a = 1$ from (32), (30) and (27) we obtain, for example, $u = -2/3$, $t = 0$, and hence, $r_c(1) = \sqrt{2} - 1$, respectively.

In particular, for

$$(33) a^2 = \frac{25}{27} = 0.9259259 \dots, a = \frac{5}{3\sqrt{3}} = 0.9622504 \dots,$$

from (32) and (30) we obtain for example,

$$(34) u = \frac{2}{3}, t = \frac{4}{3},$$

respectively. The values (33)-(34) reduce (27) to the equation

$$(35) -A\left(r, \frac{5}{3\sqrt{3}}\right) = \left(r^2 + \frac{10}{3\sqrt{3}}r + \frac{4}{3}\right)^2 - \left(\frac{8}{3\sqrt{3}}r + \frac{5}{3}\right)^2 = 0.$$

from (35) we obtain the two equations

$$(36) (r + \sqrt{3})^2 = 0, r^2 + \frac{2}{3\sqrt{3}}r - \frac{1}{3} = 0,$$

which supply the roots of the equation (23). Hence the second equation in (36) determines the unique positive root of the equation (23), i. e.

$$(37) r_c\left(\frac{5}{3\sqrt{3}}\right) = \frac{1}{3} \left(\sqrt{\frac{10}{3}} - \sqrt{\frac{1}{3}}\right) = 0.4161305 \dots$$

In particular, for $a = 0$ from (32) we obtain, for example ,

$$(38) u = \frac{1}{3} \sum_{v=0}^1 \sqrt[3]{26 + (-1)^v 15\sqrt{3}} = \frac{1}{3} \sum_{v=0}^1 \sqrt[3]{2 + (-1)^v \sqrt{3}} = \frac{4}{3}.$$

It follows from (38) and (30) that $t=2$. Thus (27) is reduced to

$$(39) (r^2 + 2)^2 - 5 = 0, r_c(0) = \sqrt{\sqrt{5} - 2} = 0.4858681 \dots$$

Remark. The cases $a=0$ and $a=1$ can be examined directly from the equation (23). Now for

$$(40) 0 < a^2 < \frac{25}{27} = 0.9259259 \dots, 0 < a < \frac{5}{3\sqrt{3}} = 0.9622504 \dots,$$

the equation (31) has one real (positive) root and two complex conjugate roots, i.e. the equation (29)(or (28)) has a unique positive root. This root is in the interval

$$(41) 2(1 - a^2) < t < 2$$

since

$$B(2(1 - a^2), a) = 4(1 - a^2)^2(-2a^2) < 0$$

and

$$B(2, a) = 2a^2 > 0$$

as it follows from (29). Under the condition (40) if t is determined by (30) and (32), for example , by the positive values of the cubic radicals, then by means of (28) the equation (27) takes the form

$$(42) (r^2 + 2ar + t)^2 - [r\sqrt{2(2a^2 - 2 + t)} + \sqrt{1 + t^2}]^2 = 0,$$

where the radicals are arithmetic, having in mind (41). From (42) we obtain the equations

$$(43) r^2 + 2\left(a + \sqrt{a^2 - 1 + \frac{t}{2}}\right)r + t + \sqrt{1 + t^2} = 0$$

and

$$(44) r^2 + 2\left(a - \sqrt{a^2 - 1 + \frac{t}{2}}\right)r + t - \sqrt{1 + t^2} = 0$$

which supply the roots of the equation (23). The equation (43) has no positive roots. thus the equation (44) supplies the unique positive root $r = r_c(a)$ of the equation (23) under the condition (40). The other root of the equation (44) is negative. Hence

$$(45) r_c(a) = \sqrt{a^2 - 1 + \frac{t}{2}} - a + \sqrt{(\sqrt{a^2 - 1 + \frac{t}{2}} - a)^2 - t + \sqrt{1 + t^2}},$$

where t is the unique positive root of the equation (29) under the condition (40) and the radicals are arithmetic. Certainly the formula (45) is also valid for the cases $a = 0$ and $a = 5/3\sqrt{3}$ and it gives again the values (39) and (37), respectively.

Finally, for

$$(46) \frac{25}{27} < a^2 < 1, \frac{5}{3\sqrt{3}} < a < 1,$$

it follows from the formula (32) that the equation (31) has three real and distinct roots and the middle of them is

$$(47) \quad u = \frac{2}{3} \cos \frac{\varphi+4\pi}{3}, \quad \frac{4\pi}{3} < \frac{\varphi+4\pi}{3} < \frac{5\pi}{3}, \quad -\frac{1}{3} < u < \frac{1}{3},$$

where φ is determined by the equation

$$(48) \quad \cos \varphi = 26 - 27a^2, \quad 0 < \varphi < \pi.$$

From (47) and (30) we find the middle root

$$(49) \quad t = \frac{4}{3} \cos^2 \frac{\varphi+4\pi}{6}, \quad \frac{2\pi}{3} < \frac{\varphi+4\pi}{6} < \frac{5\pi}{6}, \quad \frac{1}{3} < t < 1,$$

of the equation (29) (or (28)). With the help of the value (49) from (28) and (27) we obtain again the equations (42)-(44) and the formula (45) which gives the unique positive root $r_c(a)$ of the equation (23) in the case (46), having in mind that t is determined by (48)-(49). For example, in the simplest case

$$(50) \quad a^2 = \frac{26}{27} = 0.9629629 \dots, \quad a = \frac{1}{3} \sqrt{\frac{26}{3}} = 0.9813067 \dots,$$

in (48) we obtain $\varphi = \pi/2$ and (49) (and (47)) gives $t = 2/3$ (and $u = 0$, respectively). These values follow also from (30) and (31) according to (50). Thus by formula (45) we obtain

$$(51) \quad r_c \left(\frac{1}{3} \sqrt{\frac{26}{3}} \right) = \frac{1}{3} \left(\sqrt{\frac{16+\sqrt{13}}{3}} + \sqrt{\frac{8}{3}} - \sqrt{\frac{26}{3}} \right) = 0.4151578 \dots$$

5. A COMPARISON WITH OTHER RESULTS

The functions (1) of the class P' satisfy the inequality

$$(52) \quad \operatorname{Re} \frac{f(z)}{z} > 0, \quad |z| < 1,$$

according to the Sakaguchi [11] and Libera [7] results (see also Goodman [5, Vol. I, p. 102, Problem 19: on the left-hand side of the equation the letter "f" must be replaced by another letter, for example, by "F"; Vol. II, pp. 154-155, Lemma 5]). Hence the functions $f(z) \in P'$ are close-to-star as well as close-to-convex (see, for example, Goodman [5, Vol. II, p. 2; p. 29, problem 16] and Reade, Ogawa and Sakaguchi [9, p. 1]). In general let H denotes the set of all analytic functions $f(z)$ in $|z| < 1$ normalized by $f(0) = f'(0) - 1 = 0$ satisfying the inequality (52). MacGergor [8, p. 534, Theorem 3], Reade, Ogawa and Sakaguchi [9, p. 2, Theorem] and Goel [4, pp. 37-38, Theorem 2] (see also Goodman [5, Vol. II, p. 98]) have found the radii of starlikeness (and univalence) r_s , convexity r_c and starlikeness $r_s(a)$ depending on the second coefficient $a_2 = 2a$, $0 \leq a \leq 1$, of the class H as follows:

$$(53) \quad r_s = \sqrt{2} - 1$$

with an extremal function

$$(54) \quad f(z) = \frac{z(1+z)}{1-z} \in H \text{ at the "critical point" } z = -r_s; \quad r = r_c = 0.179 \dots \text{ is the unique positive root of the equation}$$

$$(55) \quad 1 - 5r - 3r^2 - r^3 = 0$$

with the extremal functions (54) at the "critical point" $z = -r_c$; and $r = r_s(a)$ is the unique positive root of the equation

$$(56) \quad 1 - 4r^2 - 4ar^3 - r^4 = 0, \quad 0 \leq a \leq 1,$$

with the extremal function

$$(57) \quad f(z) = \frac{z(1-z^2)}{1-2az+z^2} \in H$$

at the "critical point" $z = -r_s(a)$, respectively. The extremal function (54) and (57) of the class H do not belong to the class P' although $P' \subset H$. The radius of starlikeness (and univalence) r_s in (53) of the class H is equal to the radius of

convexity r_c in (3) of the class P' . Now we note that the explicit value of the root $r = r_c$ of (55) is

$$(55') \quad r_c = \frac{\sqrt[3]{2}}{\sqrt[3]{3}} (\sqrt[3]{\sqrt{29} + \sqrt{27}} - \sqrt[3]{\sqrt{29} - \sqrt{27}}) - 1 = 0.1795 \dots$$

The radius of starlikeness $r_s(a)$, determined by (56), of the class H with fixed second coefficient is equal to the radius of convexity $r_c(a)$, determined by (23), of the class P' with fixed second coefficient. The theory of the equation (23) in the section 4 for the class P' can be used as a theory of the equation (56) for the class H . For example, the cases (33), (37), (50) and (51) give new corresponding results for the starlikeness of the class H . In conclusion the radius of starlikeness of the class P' with fixed second coefficient remains unknown as well as the radius of starlikeness of the full class P' (see, for example, Goodman [5, Vol. II, p. 95, section 2. 13]).

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