

GLOBAL STABILITY FOR THE N-SPECIES LOTKA-VOLTERRA TREE SYSTEMS OF INTERACTING POPULATION-DIFFUSION EQUATIONS

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ABSTRACT. Applying invariant region method and with the aid of quadratic form theory we obtain a set of sufficient conditions for the globally asymptotic stability of the solutions to Cauchy problem for the n -species Lotka-Volterra tree systems of interacting population-diffusion equations. The criteria in this paper are in explicit forms of the parameters, and thus, are easily verifiable. Moreover, this criteria is applicable to competition model, cooperation model, as well as to predator-prey model.

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1. INTRODUCTION

We consider the n -species Lotka-Volterra system of interacting population-diffusion equations as follows

$$(1) \quad (u_i)_t - D \Delta u_i = u_i \left(b_i - \sum_{j=1}^n a_{ij} u_j \right), \quad (t, x) \in [0, \infty) \times \mathbb{R}_+^m, \quad i \in \mathcal{N},$$

where $\mathcal{N} = \{1, \dots, n\}$ and n is the number of species. In general, D represents a diffusivity matrix with no cross-diffusion among the species, so that D simply becomes a diagonal matrix. We consider the case of equal diffusion coefficients,

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i.e., $D > 0$ is diffusion constant. In (1) the function $u_i(t, x)$ represents the density of species i at time t and at location x , the constant b_i is the carrying capacity of species i , and a_{ij} represents the effect of interspecific (if $i \neq j$) or intraspecific (if $i = j$) interaction. Depending on the sign of the interaction coefficients a_{ij} , system (1) is called a cooperative (if $a_{ij} \geq 0, i \neq j$), or a competitive (if $a_{ij} \leq 0, i \neq j$), or a prey-predator one (if $a_{ij}a_{ji} \leq 0, i \neq j$).

In vector form, system (1) is expressed as

$$(2) \quad u_t - D \Delta u = U(b - Au), \quad (t, x) \in [0, \infty) \times R_+^m$$

where $u = \text{col}(u_1, \dots, u_n)$ is an n -dimensional state vector, $U = \text{diag}(u_1, \dots, u_n)$, $b = \text{col}(b_1, \dots, b_n)$ is an n -dimensional real vector, and $A = (a_{ij})_{n \times n}$ is $n \times n$ community matrix. Furthermore, the following notation is adopted

$$R_+^m = \{x \in R^m : x_i > 0, i = 1, \dots, m\},$$

where m is any positive integer number and R^m represents m -dimensional Euclidean space.

In the following discussion, we assume that system (2) possesses a unique positive equilibrium solution $u^* = (u_1^*, \dots, u_n^*)$, $u_i^* > 0$, $i \in \mathcal{N}$, and $b - Au^* = 0$. Corresponding to system (2), the following Cauchy condition will be considered

$$(3) \quad u(0, x) = u^o(x), \quad x \in R^m$$

where $u^o(x)$ is smooth and the derivative of $u^o(x)$ is uniformly bounded and there are two constant C_1 and C_2 ($C_1 \geq C_2 > 0$) such that

$$C_2 \leq u^o(x) \leq C_1$$

The smoothness of $U(b - Au)$ and $u^o(x)$ ensures the existence and uniqueness of local solution for the problem (2) - (3). In fact, a proof of this conclusion for a more general system

$$(4) \quad \begin{aligned} u_t - D \Delta u &= f(u), & (t, x) \in [0, \infty) \times R_+^m \\ u(0, x) &= u^o(x), & x \in R^m \end{aligned}$$

follows from the fundamental solution argument in [9] (Theorem 14.2) under the assumptions that $f(u) = (f_1(u), \dots, f_n(u))$ with $f_i(u_1, \dots, u_n)$ smooth functions on R_+^n and that $D = \text{diag}(D_1, \dots, D_n)$, $D_i > 0$, $i \in \mathcal{N}$.

Biologically speaking, system (4) is called a system of interacting population-diffusion equations in agreement with Debnath in [2]. For a chemical problem, system (4) called a system of nonlinear diffusion-reaction equations. For treatment methods to such equations references [3, 4] have been consulted.

The components of u in (4) represent the densities of certain species, which are interacting with each other at each point (x, t) according to the vector source term $f(u)$ and also diffusing through the habitat. Lotka-Volterra system (2) is a special case of system (4).

One of the most important systems in mathematical ecology is n -species Lotka-Volterra system of ordinary differential equations (that is the corresponding ODE system of (1)), about which a well known theorem proved by Takeuchi and Adachi

and Tokumaru [10, 11] saying that if there exists a constant positive diagonal matrix W such that $WA + A^T W$ is positive definite, then the n -species Lotka-Volterra system has a nonnegative and stable equilibrium point. In general, it is difficult to apply this theorem directly to a system since such a diagonal matrix W cannot be easily found. It may be more interesting to seek those criteria which are connected directly with the parameters a_{ij} , by which one can easily ensure the existence of a stable equilibrium point. Papers [5, 6, 7] are efforts in this direction. In [6, 7] a set of criteria for the globally asymptotically stable equilibrium for n -species Lotka-Volterra tree systems were obtained with the aid of matrix analysis and quadratic form theory. The criteria in [5, 6, 7] are in explicit forms of the coefficients a_{ij} and thus easy to verify.

For Lotka-Volterra system of ordinary differential equations the density functions of species are assumed to be constants for every fixed time t , i.e., density is assumed to be independent on location. If we suppose the density functions $u_1, \dots, u_n$ of species are dependent on both time t and location x , then we obtain n -species Lotka-Volterra system of interacting population-diffusion equations (2).

The purpose of this paper is to extend the proving method of [6, 7] from Lotka-Volterra tree systems of ordinary differential equations into Lotka-Volterra tree systems of interacting population-diffusion equations (2) and to obtain a set of criteria which ensures the globally asymptotic stability of the solutions to the Cauchy problem (2) - (3). The criteria in this paper are in explicit forms of the parameters a_{ij} and thus are easily verifiable in real applications.

2. MAIN RESULT

In paper [8], Z.Y.Lu investigated the Cauchy problem (2) - (3), and proved global stability for n -species cooperative system under some conditions. In this paper, we will show for the Cauchy problem (2) - (3) that under some conditions the global stability is valid to n -species competitive system, to n -species cooperative system as well as to n -species prey-predator system. To study global stability for reaction diffusion systems (which is interacting population-diffusion systems for our paper) the invariant region method was introduced by Weighberger in [12] and has been considerably developed by Chueh, Conlay and Smoller in [1, 9]. In order to deal with our problem, we give an outline of previous notions and argument.

Definition 1. [9] A closed subset $\Sigma \subset R^n$ is called an invariant region for the local solution defined by (3)-(4), if any solution $u(t, x)$ having its initial value $u(0, x) = u^0(x) \in \Sigma$ for all $x \in R^m$, satisfies $u(t, x) \in \Sigma$ for all $t \in [0, \delta)$ ($\delta > 0$).

Now, we consider region Σ of the form

$$\Sigma = \bigcup_{i=1}^k \{u \in R_+^n \mid G_i(u) \leq 0\}$$

where $G_i(u)$, $i = 1, \dots, k$ are smooth real-valued functions defined on open subset of R_+^n , and

$$dG_i = \left(\frac{\partial G_i}{\partial u_1}, \dots, \frac{\partial G_i}{\partial u_n} \right) \neq 0$$

Definition 2. [9] The smooth function $G : R_+^n \rightarrow R$ is called quasi-convex at u , if whenever $dG(u) = 0$, then $d^2G(u) \geq 0$, where

$$d^2G(u) = \left(\frac{\partial^2 G(u)}{\partial u_i \partial u_j} \right)_{n \times n}$$

Obviously (see [9] for details), if system (4) admits a bounded invariant region Σ , and $u^o(x) \in \Sigma$ for all $x \in R_+^n$, then the solution of problem (3)-(4) exists for all $t > 0$ and this solution must be bounded.

We consider, without loss of generality, the invariant region in the case of $k = 1$ as follows

$$\Sigma_\alpha = \{u \in R_+^n \mid G(u) - \alpha \leq 0\} \quad (\alpha \geq 0)$$

Definition 3. [8] $G(u)$ is called a global Liapunov function for system (4), if it satisfies

- i) $G(u) \geq 0$, $G(u) = 0$ iff $u = u^*$,
- ii) $G(u) \rightarrow +\infty$, as $u \rightarrow \partial R_+^n$,
- iii) $dG(u) \cdot f(u) = dG(f) \leq 0$, the equality holds iff $u = u^*$,
- iv) $dG(u)$ is a left eigenvector of D ,
- v) $G(u)$ is quasi-convex in R_+^n .

The conditions i), ii) and iii) in above definition guarantee the global stability of u^* for the corresponding ODE system of (4). And iii), iv) and v) ensure the global existence of solutions of problem (3)-(4).

To obtain the global stability for the Cauchy problem (2) - (3) we need more notations in biological mathematics, and then we need to introduce a concept called degree of species.

Definition 4. System (1) (or (2)) is said to be a cycle system if there are distinct indices $i_1, i_2, \dots, i_m (m > 3)$ such that none of the elements $a_{i_1 i_2}, a_{i_2 i_3}, \dots, a_{i_m i_1}$ is vanishing. A system is said to be a chain system if it is not a cycle system.

Definition 5. For fixed $i, j \in N$ ($i \neq j$), species i and species j are said to be in interaction if $a_{ij}a_{ji} \neq 0$. And we denote

- (a) $\bullet_i \longleftrightarrow \bullet_j \iff a_{ij} > 0$ for cooperation interaction between species i and j ;
- (b) $\bullet_i \rightarrow \leftarrow \bullet_j \iff a_{ij} < 0$ for competition interaction between species i and j ;
- (c) $\bullet_i \longrightarrow \bullet_j \iff a_{ij}a_{ji} < 0$ for prey-predator interaction from species i to j .

The graph $G(A)$ of an $n \times n$ matrix A consists of n vertices $\bullet$ representing n species together with edges $\longleftrightarrow$ and/or edges $\rightarrow \leftarrow$ and/or edges $\longrightarrow$, representing the interrelations between species.

Definition 6. A system is said to be a tree system if the graph $G(A)$ is a connected graph without any cycle.

The concept of the degree of a species is introduced as follows.

Definition 7. The degree of i^{th} species is denoted by the positive integer m_i that is the number of species connected with the i^{th} species.

The main result of this paper is expressed by the following theorem.

Theorem 8. Assume $a_{ii} > 0$, $i \in \mathcal{N}$. And assume that m_i and m_j are the degrees of the i^{th} species and of the j^{th} species, respectively. If the inequalities

$$(5) \quad a_{ii}a_{jj} > m_i m_j a_{ij} a_{ji}, \quad i, j \in \mathcal{N}$$

are fulfilled, then, for any given initial function $u^0(x)$, the Cauchy problem (2)-(3) of Lotka-Volterra tree system has a unique global positive solution $u(t, x)$ such that

$$\lim_{t \rightarrow \infty} u(t, x) = u^*$$

uniformly for $x \in E$ (where E is any bounded subset of R^m).

Obviously, any tree system is composed with a number of chain systems and/or a number systems with models between one and multi-species. To explain the meanings of the theorem more vivid, we give three corollaries. The first concerns with Lotka-Volterra chain systems for cooperation models and for competition models. The second is about Lotka-Volterra models between one and multi-species for cooperation systems and for competition systems. The third one is about Lotka-Volterra prey-predator tree systems.

Lotka-Volterra Chain Systems

For competition chain system (if $a_{ii} > 0$, $i \in \mathcal{N}$ and $a_{i,i+1} > 0$, $a_{i+1,i} > 0$, $i \in \mathcal{N} - \{n\}$), for cooperation chain system (if $a_{ii} > 0$, $i \in \mathcal{N}$ and $a_{i,i+1} < 0$, $a_{i+1,i} < 0$, $i \in \mathcal{N} - \{n\}$), and for prey-predator chain system (if $a_{ii} > 0$, $i \in \mathcal{N}$ and $a_{i,i+1}a_{i+1,i} < 0$, $i \in \mathcal{N} - \{n\}$) the community matrix is

$$(6) \quad A = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

Corollary 9. Assume that $a_{ii} > 0$, $i \in \mathcal{N}$. If the inequalities

$$(7) \quad \begin{aligned} a_{11}a_{22} &> 2a_{12}a_{21}, \\ a_{ii}a_{i+1,i+1} &> 4a_{i,i+1}a_{i+1,i}, \quad 2 \leq i \leq n-2 \\ a_{n-1,n-1}a_{nn} &> 2a_{n-1,n}a_{n,n-1}, \end{aligned}$$

are fulfilled, then, for any given initial function $u^0(x)$, the Cauchy problem (2)-(3) for cooperation chain systems and for competition chain systems, has a unique global positive solution $u(t, x)$ such that

$$\lim_{t \rightarrow \infty} u(t, x) = u^*$$

uniformly for $x \in E$ (where E is any bounded subset of R^m).

Proof: Because of Theorem 8, the Corollary is easy to obtain by the definition of degree of species and the definition of tree system.

Lotka-Volterra Models Between One and Multi-Species

For competition systems (if $a_{ii} > 0$, $a_{1j} > 0$, $a_{j1} > 0$, $i, j \in \mathcal{N}$), for cooperation systems (if $a_{ii} > 0$, $a_{1j} < 0$, $a_{j1} < 0$, $i, j \in \mathcal{N}$), and for prey-predator systems (if $a_{ii} > 0$, $a_{1j}a_{j1} < 0$, $i, j \in \mathcal{N}$) between one and multi-species, the community matrix is

$$(8) \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Corollary 10. *Assume that $a_{ii} > 0$, $i \in \mathcal{N}$. If the inequalities*

$$(9) \quad a_{11}a_{jj} > (n-1)a_{1j}a_{j1}, \quad j \in \mathcal{N} - \{1\}$$

are fulfilled, then, for any given initial function $u^o(x)$, the Cauchy problem (2)-(3) for cooperation systems and for competition systems between one and multi-species, has a unique global positive solution $u(t, x)$ such that

$$\lim_{t \rightarrow \infty} u(t, x) = u^*$$

uniformly for $x \in E$ (where E is any bounded subset of R^m).

Proof: Because of Theorem 8, the Corollary is obvious by the definition of degree of species and the definition of tree system.

Lotka-Volterra Prey-Predator Tree Systems

As we know any tree system is composed with a number of chain systems and/or a number systems with models between one and multi-species. The prey-predator tree system is composed with a number of subsystems whose interaction matrices are given by (6) and/or (8).

Corollary 11. *Assume that $a_{ii} > 0$, $i \in \mathcal{N}$. Then, for any given initial function $u^o(x)$, the Cauchy problem (2)-(3), for all Lotka-Volterra prey-predator tree systems of reaction-diffusion equations, always has a unique global positive solution $u(t, x)$ such that*

$$\lim_{t \rightarrow \infty} u(t, x) = u^*$$

uniformly for $x \in E$ (where E is any bounded subset of R^m).

Proof: Because $a_{ii} > 0$, $i \in \mathcal{N}$ and by definition of prey-predator model we have $a_{ij}a_{ji} < 0$ which implies (5) then applying the definition of tree system we get the Corollary.

3. PROOF OF THE MAIN RESULT

Before proving, let us give an interpretation on the conditions of our results. First, for prey-predator systems, since we have $a_{ij}a_{ji} < 0$ (by definition of prey-predator model) and $a_{ii}a_{jj} > 0$, therefore, the inequalities (5), (7), and (9) are always true. So, prey-predator tree system is always globally stable. As to cooperation tree system and to competition tree system, these inequalities imply that, for global stability, the system has strong self regulating (or resource limited) negative feedback, in other words, the intraspecific competition is such stronger than the interspecific interaction that it can be measured in a way given by inequalities (5) for tree systems, by inequalities (7) for chain systems, and by inequalities (9) for systems between one and multi-species.

To prove Theorem 8, we need more lemmas.

Lemma 1. [8] *If system (2) admits a global Liapunov function $G(u)$ satisfying Definition 3, then for any given initial function $u^0(x)$, the Cauchy problem (2)-(3) has a unique global positive solution $u(t, x)$ such that*

$$\lim_{t \rightarrow \infty} u(t, x) = u^*$$

uniformly for $x \in E$ (where E is any bounded subset of R^m).

Lemma 2. [8] *For the interaction matrix $A = (a_{ij})_{n \times n}$, if there exists a positive diagonal matrix $W = \text{diag}(w_1, \dots, w_n)$ such than $WA + A^T W$ is positive definite, then system (2) admits a global Liapunov function $G(u)$ satisfying Definition 3 which is given by*

$$G(u) = \sum_{i=1}^n w_i \left(u_i - u_i^* - u_i^* \ln \frac{u_i}{u_i^*} \right).$$

Definition We define matrix $A = (a_{ij})_{n \times n} \in S_w$ if there exists a positive diagonal matrix $W = \text{diag}(w_1, \dots, w_n)$ such than $WA + A^T W$ is positive definite.

Therefore, in order to prove Theorem 8, we only have to prove that if the inequalities (5) are fulfilled, then matrix $A = (a_{ij})_{n \times n} \in S_w$.

Lemma 3 (in [5, 6]). *Let $i, j \in \mathcal{N}$ be fixed and $i \neq j$. Let $a_{ii}, a_{jj} \in R_+$ and $a_{ij}, a_{ji} \in R$. If there is a positive real number μ_{ij} satisfying the inequality*

$$(10) \quad a_{ii}a_{jj} > \mu_{ij}a_{ij}a_{ji}, \quad i \neq j, \quad i, j \in \mathcal{N},$$

then there exist positive constants $w_i, w_j \in R_+$ such that

$$w_i w_j a_{ii} a_{jj} > \frac{1}{4} \mu_{ij} (w_i a_{ij} + w_j a_{ji})^2.$$

Proof: Given $w_i > 0$. Let

$$w_j = \frac{2a_{ii}a_{jj} - \mu_{ij}a_{ij}a_{ji}}{\mu_{ij}a_{ji}^2} w_i$$

Then, we have

$$\begin{aligned}
& w_i w_j a_{ii} a_{jj} - \frac{\mu_{ij}}{4} (w_i a_{ij} + w_j a_{ji})^2 \\
&= w_i^2 \frac{2a_{ii}^2 a_{jj}^2 - \mu_{ij} a_{ij} a_{ji} a_{ii} a_{jj}}{\mu_{ij} a_{ji}^2} - \frac{\mu_{ij}}{4} \left(\frac{w_i \mu_{ij} a_{ji}^2 a_{ij} + 2a_{ji} a_{ii} a_{jj} w_i - \mu_{ij} a_{ij} a_{ji}^2 w_i}{\mu_{ij} a_{ji}^2} \right)^2 \\
&= w_i^2 \frac{2a_{ii}^2 a_{jj}^2 - \mu_{ij} a_{ij} a_{ji} a_{ii} a_{jj}}{\mu_{ij} a_{ji}^2} - \frac{a_{ii}^2 a_{jj}^2 w_i^2}{\mu_{ij} a_{ji}^2} \\
&= \frac{w_i^2 a_{ii} a_{jj} (a_{ii} a_{jj} - \mu_{ij} a_{ij} a_{ji})}{\mu_{ij} a_{ji}^2} > 0 \\
&\text{by the assumption (10).}
\end{aligned}$$

Lemma 4. For an n -species tree system, if the inequalities (10)

$$a_{ii} a_{jj} > \mu_{ij} a_{ij} a_{ji}, \quad i \neq j, \quad i, j \in \mathcal{N},$$

are satisfied and a positive number w_{i_0} is given, then there exist $n - 1$ positive numbers w_j ($j \neq i_0$) such that

$$w_i w_j a_{ii} a_{jj} > \frac{1}{4} \mu_{ij} (w_i a_{ij} + w_j a_{ji})^2.$$

Lemma 5 (in [7]). For an arbitrary n -species tree system, if the inequalities (5), i.e. the conditions in Theorem 8,

$$(11) \quad a_{ii} a_{jj} > m_i m_j a_{ij} a_{ji}, \quad i, j \in \mathcal{N}$$

are fulfilled, where m_i and m_j are the degrees of i^{th} -species and j^{th} -species in $G(A)$ respectively, and $A = (a_{ij})_{n \times n}$ is the community matrix of n -species tree system. Then there exists a positive diagonal matrix $W = \text{diag}(w_1, w_2, \dots, w_n)$ such that $WA + A^T W$ is positive definite.

Proof: Let $x = \text{col}(x_1, x_2, \dots, x_n) \in R^n$, $x \neq \text{col}(0, \dots, 0) \in R^n$. It is easy to show that for an arbitrary Lotka-Volterra tree system with community matrix $A = A(a_{ij})_{n \times n}$, the quadratic form corresponding to $\frac{1}{2}(WA + A^T W)$ is

$$\begin{aligned}
f(x_1, x_2, \dots, x_n) &= \frac{1}{2} x^T (WA + A^T W) x \\
&= \sum_{i=1}^n w_i a_{ii} x_i^2 + \sum_{i < j, a_{ij} \neq 0} (w_i a_{ij} + w_j a_{ji}) x_i x_j \\
(12) \quad &= \sum_{1 \leq i < j, a_{ij} \neq 0} \left[\frac{w_i}{m_i} a_{ii} (x_i)^2 + (w_i a_{ij} + w_j a_{ji}) x_i x_j + \frac{w_j}{m_j} a_{jj} (x_j)^2 \right]
\end{aligned}$$

by lemma 4 the discriminate of each term of the right hand side of (12) is negative, that is

$$\Delta = (w_i a_{ij} + w_j a_{ji})^2 - \frac{4w_i w_j}{m_i m_j} a_{ii} a_{jj} < 0.$$

By Lemma 4 and inequalities (11), we know that there exists a positive constant diagonal matrix $W = \text{col}(w_1, w_2, \dots, w_n)$ whose elements can be given by induction from the formula

$$w_j = \frac{2a_{ii} a_{jj} - m_i m_j a_{ij} a_{ji}}{m_i m_j a_{ji}^2} w_i$$

(with $w_1 = 1$, for instance) such that the quadratic form

$$f(x_1, x_2, \dots, x_n) > 0, \quad \forall x \neq (0, 0, \dots, 0) \in R^n,$$

That is, the matrix $WA + A^T W$ is positive definite. Therefore, the interacting matrix $A \in S_w$. The proof of this lemma is completed. This implies that we have proved Theorem 8.

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