A NUMERICAL CHARACTERIZATION OF EQUISINGULARITY FOR MAP GERMS FROM N-SPACE, (N≥3), TO THE PLANE

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ABSTRACT. By a celebrated 1976 result regarding the geometry of families of hypersurfaces with isolated singularities, Lê and Ramanujam showed that the constancy of a single numerical invariant, namely the Milnor number, implies that the family is locally topologically trivial. This classical result has led to generalizations of the Milnor number to various numerical invariants, including "polar multiplicities", "Lê numbers" and "Segre numbers", all designed to reflect the geometry of singularities in very general scenarios. In the present study, we consider families of finitely-determined holomorphic map germs from $\mathbf{C}^n$ to $\mathbf{C}^2$ for n $\geq$ 3, and characterize local topological triviality in terms of the constancy of (n-1) Lê numbers.

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0. INTRODUCTION

The key developments in approach and technique that underlie recent work in the area of singularities of analytic maps and hypersurfaces can be traced to the late 1960's, and to the seminal contributions of Mather (see [**18**], [**19**], [**20**]), and of Lê and Ramanujam ([**12**], [**16**]). A comprehensive exposition of the evolution of these and related ideas can be found in [**9**].

The present study builds upon an approach articulated in 1993 by the first author (see [**2**]), in which the topological triviality of a 1-parameter family of a class of map-germs is shown to be a consequence of the constancy of finitely many analytic invariants (*polar multiplicities*; see [**6**], [**22**]) associated to members of that family [the number of invariants depending upon the particular dimensions of source and of target space being considered], with a restriction on the type of admissible families. The analysis was applied to obtain specific results pertaining to (i) map germs from

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the plane to the plane [section 9 of [2]], and (ii) map germs from the plane to 3-space [section 8 of [2]]. In particular, the number of invariants required for Whitney equisingularity (see [15]) in each of these situations was shown to be smaller than the *a priori* number given by the general result of [2], by virtue of relationships specific to the particular case studied. More recently this approach has been used by Perez [21], to study map germs of corank 1 from $\mathbf{C}^3$ to $\mathbf{C}^3$; the present study of map germs from n-space (n$\geq$3) to the plane fits naturally into this framework.

From a technical standpoint, the present case is different chiefly in that non-isolated singularities in the source-space are permissible. We have been able to handle this by exploiting recent results of Massey [17] relative to Lê numbers, of Gaffney-Gassler [7] relative to Segre numbers, and Gaffney [3], [8] relative to aspects of the A_f and W_f conditions. These results were unavailable when [2] was written. Especially important is the fact that although the Lê numbers seem to be invariants of the top-dimensional stratum, they often control lower dimensional strata as well [See Theorem 6.6 of [9] for an example of this.] This leads to a considerable parsimony, in terms of the number of invariants which need to be controlled.

That we have been able to deal with the entire class of cases $\mathcal{O}$(n,2), n $\geq$ 3, "at one stroke" is a consequence of the fact that in each case, the stable singularities are precisely the same type, namely folds, cusps, and double folds. It turns out, in all these cases, that the inverse image X(t) of the discriminant carries all information necessary to determine if the family of map germs is Whitney equisingular.

The paper is organized as follows. Section 1 contains basic definitions and foundational results. Section 2 is the core of the study; the main result is a characterization of Whitney equisingualrity, relative to holomorphic map germs $\mathbf{C}^n$ to $\mathbf{C}^2$, for n$\geq$3, by (n−1) numerical invariants, namely a set of 0-dimensional Lê numbers associated to generic plane sections of the spaces X(t). Thus for the case of n=3, the number of invariants required is just two, compared to the eleven required *a priori* as per [2].

1. $\mathcal{O}$(N,2), N $\geq$ 3: BASIC CONSTRUCTIONS AND RESULTS

(1.0) NOTATION & DEFINITIONS We denote an origin-preserving holomorphic map germ f: $\mathbf{C}^n$ to $\mathbf{C}^p$ by writing f $\in \mathcal{O}$(n, p). Our interest is primarily in "finitely determined" map-germs; a standard reference in this context is [24]. We denote by $\mathcal{R}$ the group of local diffeomorphisms of the source ($\mathbf{C}^n$,0), and we denote by $\mathcal{L}$ the group of local diffeomorphisms of the target $\mathbf{C}^p, 0)$. Although these actions are intersesting individually (see e.g. [4]), our focus will be on the action of the product $(\mathcal{R} \times \mathcal{L}) =: \mathcal{A}$ which leads to **$\mathcal{A}$-equivalence** of map-germs: f, g $\in \mathcal{O}$(n, p) are $\mathcal{A}$-equivalent if they are equivalent by smooth coordinate changes of source and target. Similarly the action of the semi-direct product $\mathcal{K} := (\mathcal{R}.\mathcal{L})$ gives rise to **$\mathcal{K}$-equivalence** [also called **contact equivalence**] of map germs. A given map-germ f $\in \mathcal{O}$(n,p) is said to be **k-$\mathcal{G}$ -determined** ($\mathcal{G} = \mathcal{A}$ or $\mathcal{K}$) if whenever g $\in \mathcal{O}$(n,p) and $j^kf(0) = j^kg(0)$, then g is $\mathcal{G}$-equivalent to f; f is **finitely $\mathcal{G}$-determined** if it is k-$\mathcal{G}$-determined for some k$< \infty$. (In order to lighten notation, and in accordance with standard practice, whenever we say "finitely determined" (without specifying whether $\mathcal{A}$ or $\mathcal{K}$), we shall mean "finitely $\mathcal{A}$-determined").

Map germs which are finitely $\mathcal{K}$-determined are called map germs of **finite singularity type.** Using the symmetry between the theory of $\mathcal{A}$-equivalence and the theory of $\mathcal{K}$-equivalence [[**25**], p.201], we know that, in particular, if f$\in \mathcal{O}$(n,p) is finitely $\mathcal{A}$-determined, then f has finite singularity type.

We say f is a **stable germ** if every "nearby" germ is $\mathcal{A}$-equivalent to f. A **stable type** is an $\mathcal{A}$-equivalence class of stable germs. A finitely determined germ f has **discrete stable type** if there exists a versal unfolding of f in which only a finite number of stable types occur. In particular, if the numbers (n, p) are in Mather's "nice dimensions" [which is true for (n, 2), n$\geq$3, our focus here] or on the boundary thereof, then every finitely determined germ f $\in \mathcal{O}$(n, p) has discrete stable type.

(1.1) SETUP Let f $\in \mathcal{O}$(n, 2), n$\geq$3 be a finitely determined map germ, and let F: $\mathbf{C}\times\mathbf{C}^n \to \mathbf{C}\times\mathbf{C}^2$ be a one-parameter unfolding of f $= f_0$, such that F is stratified by stable type, and by the parameter axis T. For each $f_t : (\mathbf{C}^n,0) \to (\mathbf{C}^2,0)$ in the family $\{f_t\}$, let the discriminant (the image of the singular locus), be denoted Δf_t, and let the equation that defines the discriminant,with reduced structure, be denoted by $g_t = 0$, where $g_t : (\mathbf{C}^2,0) \to (\mathbf{C},0)$. Let $\mathfrak{X} = \{$X(t)$\}$ denote the family of reduced hypersurfaces X(t):= $f_t^{-1}(\Delta f_t)$ in $\mathbf{C}^n$.

As a first step, we shall collect several fundamental facts relative to this setup in a preliminary lemma, namely Lemma (1.4) below. We begin with

(1.2) LEMMA If f $\in \mathcal{O}$(n,p), n$\geq$p, then the singular locus $\mathcal{S}$(f) of f satisfies the relation dim $\mathcal{S}$(f) $\geq$ p$-$1, provided $\mathcal{S}$(f) $\neq \emptyset$.

Proof We use notation as in [**10**]. Consider the 1-jet map
$j^1 f : \mathbf{C}^n \to J^1((\mathbf{C}^n,0),(\mathbf{C}^p,0))$. We use the fact that $\mathcal{S}$(f) is precisely the preimage, under this map $j^1 f$, of the closure $\bar{\mathcal{S}}_1$ of the subset $\mathcal{S}_1 \subseteq J^1((\mathbf{C}^n,0),(\mathbf{C}^p,0))$. It therefore suffices to show that if $\mathcal{S}$(f) is nonempty, then dim $(j^1 f(\mathbf{C}^n) \cap \bar{\mathcal{S}}_1) \geq$ (p - 1). Recall that the codimension of a nonempty intersection of two varieties is no greater than the sum of the codimensions of the two varieties; [see, e.g. [**11**], p.47 for this]. Now dim $J^1((\mathbf{C}^n,0),(\mathbf{C}^p,0))$ = n+p+np, so that the codimension of $\bar{\mathcal{S}}_1$ in $J^1((\mathbf{C}^n,0),(\mathbf{C}^p,0))$ is n$-$p+1, and that of $j^1 f(\mathbf{C}^n)$is p+np. It follows that dim $(j^1 f(\mathbf{C}^n) \cap \bar{\mathcal{S}}_1)$ is at least p$-$1. $\square$

(1.3) Remark Recall that if f $\in \mathcal{O}$(n,p) is finitely $\mathcal{A}$-determined, then f has finite singularity type. If f has finite singularity type, then the restriction $f|_{\mathcal{S}(f)} : \mathcal{S}$(f) $\to \Delta$f is a **finite map** [any point in the target has at most a finite number of preimages], and hence dim $\mathcal{S}$(f) = dim Δf.

(1.4) LEMMA In the setup (1.1):
(i) $\mathcal{S}(f_t)$, the singular locus of f_t, is either empty or 1-dimensional;
(ii) $\mathcal{S}(f_t) \cap f_t^{-1}(0) = \{0\}$, if $\mathcal{S}(f_t) \neq \emptyset$;

(iii) $f_t^{-1}(0)$ is $(n-2)$-dimensional;
(iv) g_t is a submersion at each point of $\Delta f \backslash 0$;
(v) $\mathcal{S}(f_t) \subseteq \mathcal{S}(g_t \circ f_t) \subseteq \mathcal{S}(f_t) \cup f_t^{-1}(0)$;
(vi) If $h_t := (g_t \circ f_t) : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ then X(t) $= \mathcal{V}(h_t)$, and $\mathcal{I}(\mathrm{X(t)}) = (h_t)$.

Proof See Appendix below. □

(1.5) Remark The proof of part (v) above [and using (vi)] shows that the inclusion $\mathcal{S}(\mathrm{X(t)}) \subseteq \mathcal{S}(f_t) \cup f_t^{-1}(0)$ is in fact an equality if the defining equation g_t of $\Delta(f_t)$ satisfies $\mathrm{D}g_t(0) = 0$; in this case dim $\mathcal{S}(h_t) = (\mathrm{n}-2)$. We know that for the case $\mathcal{O}$(n,2), n$\geq$3 at hand, there are precisely four different types of stable map-germs, namely (i) submersions, (ii) folds, [stable type of dimension 1 in the target] (iii) cusps, [stable type of dimension 0 in target] and (iv) double-folds, i.e. normal crossings of the discriminant in the target [stable type of dimension 0 in target]. The (topological) invariants corresponding to the cusps and the double-folds are denoted **c**(f) and **d**(f) respectively. If we denote the polar multiplicity of $\Delta(f_t)$ at 0 by $m_0\,(\Delta f_t)$, and denote the first polar multiplicity associated to the degree of f_t by $m_1\,(\Delta f_t)$ [see [**9**]], then we have the following

(1.6) PROPOSITION. In the setup above, let p: $(\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a generic projection for Δf_t. Then $\mu\,(\Delta f_t) + m_0\,(\Delta f_t) - 1 = 3\mathbf{c}(f_t) + 2\mathbf{d}(f_t) + m_1\,(\Delta f_t)$.

Proof. The proof is exactly as in [**13**]. (See also p.218 of [**2**] for the case $\mathcal{O}$(2,2)). □

(1.7) Remark Using the upper-semicontinuity of invariants, we see from (1.6) that if the Milnor number $\mu\,(\Delta f_t)$ is independent of t [which is equivalent to Whitney equisingularity of the family of plane curves $\{\Delta f_t\}$ over the parameter axis T], then, in particular, the invariants $\mathbf{c}(f_t)$ and $\mathbf{d}(f_t)$ are also independent of t, and that the unfolding F has no 1-dimensional stratum other than the parameter axis T. Note also that since the non-stable jets are in the closure of the cusps & double-folds, it suffices to control $\mathbf{c}(f_t)$ and $\mathbf{d}(f_t)$.

We will now use **Lê cycles** [see [**17**] and [**3**]] to decompose the locus $\mathcal{S}(h_t)$. In order to lighten notation, subscripts will be omitted whenever possible. We begin by recalling some general facts pertaining to our study.

(1.8) BLOWUP ALONG JACOBIAN IDEAL Let h:$(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ define an (n$-$1)-dimensional, reduced analytic space germ X $\subseteq \mathbf{C}^n$. We consider the **blowup** B of $\mathbf{C}^n$ along the Jacobian ideal J(h), namely the map

b: B=$Bl_{J(h)}\mathbf{C}^n \to \mathbf{C}^n$

with **exceptional divisor** E:= $\mathrm{b}^{-1}(\mathcal{V}(\mathrm{J(h)})) \equiv \mathrm{b}^{-1}(\mathcal{S}(\mathrm{h}))$. Note that (i) the map b, called the **structure map** of the blowup, is a proper morphism; & (ii) B equals the closure in $\mathbf{C}^n \times \mathbf{P}^{n-1}$ of the graph of the map $\mathbf{C}^n - \mathcal{V}\mathrm{J((h))} \to \mathbf{P}^{n-1}$, given by

$$z \mapsto ((\partial h/\partial z_1)(z) : ... : (\partial h/\partial z_n)(z)).$$

Recall also that if Y is a given analytic space germ (with its reduced structure), then an **analytic cycle in** Y is a finite formal sum $\sum_i c_i W_i$, where each W_i is an irreducible analytic subset of Y, and the coefficient c_i associated to W_i, is an integer called the **algebraic multiplicity** of W_i. It will often be useful to regard the exceptional divisor E as a cycle, properly denoted [E] and defined as $\sum_i c_i [W_i]$, where the integer c_i is the order of vanishing of J(h) on the irreducible component W_i of E. The morphism b induces [by linearity] a homomorphism b_* [called the **pushforward**] of the group of cycles on B to the group of cycles on the target [see (1.10), part (ii) below].

(1.9) DEFINITION/REMARK The **relative polar variety** of h, of dimension k, [k=1,...,n], denoted $\Gamma^k(h)$, is defined by:

$$\Gamma^n(h) := \mathbf{C}^n, \ \& \ \Gamma^k(h) := b(H_1 \cap \ \ldots \ \cap H_{n-k} \cap Bl_{J(h)}\mathbf{C}^n) \text{ for k =1,...,n}-1,$$

where the H_j are hyperplanes in $\mathbf{P}^{n-1}$. The kth **relative polar multiplicity** of h, $[k = 1, ..., n-1]$, denoted $m_k\,(h)$, is defined by: $m_k\,(h) := \mathrm{mult}_0\left(\Gamma^k\,(h)\right)$. The Lê cycle of h of dimension i, [i.e of codimension (n−i)] is denoted $\Lambda^i\,(h)$, and is defined as the pushforward by the structure map of the cycle formed by intersecting the exceptional divisor E in the blowup B with $(n-i-1)$ generic hyperplanes; [see (1.10), (iii) below for formula]. Note that each component of $\Lambda^i\,(h)$ has dimension at least i [i.e. codimension ≤(n−i)]. The associated **Lê number** of dimension i [i.e. of codimension (n−i)] is $\lambda^i\,(h) \equiv \lambda_{n-i}\,(h) := \mathrm{mult}_0\Lambda^i\,(h)$. When the singular locus of h has dimension s, say, then the Lê numbers of larger dimension, namely $\lambda^{s+1}, ..., \lambda^n$, are zero. For the sake of uniformity, we shall work exclusively with *dimension* [for Lê cycles & numbers] rather than codimension, in the sequel.

We collect, below, the relevant computational rules, derivations of which are based upon [**1**], and developed in [**7**].

(1.10) RULES
(i) The notation $[E]\cdot H^n$ refers to the cycle $[E \cap H_1 \cap \ldots \cap H_n]$ obtained when E is intersected by n hyperplanes in $\mathbf{P}^{n-1}$ that are **general with respect to B**; [see section 2 of [**7**] for detail]. By convention, $[E]\cdot H^0$ is just [E].

(ii) For a given cycle $[S] := \sum_i c_i W_i \subseteq B$, we define $b_*([S]) := \sum_i c_i \cdot b(W_i)$ if $\dim W_i = \dim(b(W_i))$; and 0 otherwise.

(iii) $\Lambda^i(h) := b_*([E]\cdot H^i)$, for i = 0,..., n−1.

(iv) The multiplicity [at the origin] of a given cycle $[S] := \sum_i c_i W_i$ is defined as $\mathrm{mult}_0 S := \sum_i c_i(\mathrm{mult}_0(W_i))$.

(v) $\lambda^i := \mathrm{mult}_0\left(\Lambda^i\right)$, for i = 0,..., n−1.

We now specialize to the case of interest, namely the case of finitely determined map germs f∈ $\mathcal{O}$(n,2), n≥3.

(1.11) Remark From [part (v) of Lemma (1.4) above], and Remark (1.5), we can see that precisely one of the two inclusions in the nesting $\mathcal{S}(f) \subseteq \mathcal{S}(h) \subseteq \mathcal{S}(f) \cup f^{-1}(0)$ is an equality, for a given f. The inclusion on the left side is an equality iff f is a fold, and in this case the analysis is particularly simple, as the singular locus of X, namely $\mathcal{S}(h)$, is necessarily 1-dimensional, irrespective of how large the integer n [of the source space $\mathbf{C}^n$] is. In all other situations, the inclusion on the right is an equality, and in the sequel, we shall assume that it is this equality $\mathcal{S}(h) = \mathcal{S}(f) \cup f^{-1}(0)$ that holds.

(1.12) DEFINITION The **conormal** of a given analytic space $Y \subseteq \mathbf{C}^n$, denoted $\mathbf{C}(Y)$, is the closure in $\mathbf{C}^n \times \mathbf{P}^{n-1}$ of the pairs (y, H), where y is a smooth point of Y, and H is a tangent hyperplane to Y at such a point.

(1.13) Remark Since the conormal $\mathbf{C}(Y)$ encodes information about limiting tangent hyperplanes to Y, it plays a role [as also does its relativized version] in stating equisingularity conditions, such as the Whitney conditions, or Thom's A_f condition.

(1.14) LEMMA Suppose f: $(\mathbf{C}^n, 0) \to (\mathbf{C}^2, 0)$ is a finitely determined germ. Then for any given representative of f [also denoted f], we can find an open neighborhood U of 0 in $\mathbf{C}^n$, such that in the above setup, we have:

$$\overline{b^{-1}(f^{-1}(0) \cap U) \backslash b^{-1}(0)} \quad = \quad \mathbf{C}(f^{-1}(0) \cap U).$$

Proof Note that since Thom's A_f condition holds generically, it holds for h for $(U\backslash 0, f^{-1}(0)\backslash 0)$, for suitable choice of neighborhood U of 0, because f is a submersion for each $z \in \left(f^{-1}(0)\backslash 0\right) \cap U$. We then have the inclusion of linear spaces $T_z(f^{-1}(0)) \subseteq P$, where P is any element of the fiber $b^{-1}(z)$; this implies in turn that we have the inclusion $\overline{(b^{-1}(f^{-1}(0) \cap U)\backslash b^{-1}(0))} \subseteq \mathbf{C}(f^{-1}(0) \cap U)$. Equality then follows from dimensional considerations [see section 6.2 of [**7**]]. □

We denote by $\mathcal{V}_{f^{-1}(0)}$ the [germ of the] variety which is given set-theoretically by $\overline{b^{-1}(f^{-1}(0)\backslash 0)}$, when it is endowed with reduced structure. By the preceding Lemma (1.14), we see that the conormal $\mathbf{C}(f^{-1}(0))$ is precisely equal to [the component] $\mathcal{V}_{f^{-1}(0)}$. Note also that $b(\mathbf{C}(f^{-1}(0)) \cap H^i)$ is the **i-th codimension absolute polar variety** of $f^{-1}(0)$ in the sense of [**2**], p.195.

(1.15) LEMMA Keep the setup of (1.1) above. Let $\mathcal{V}_{\mathcal{S}(f)}$ and $\mathcal{V}_0$ denote the reduced varieties defined set theoretically by $\mathcal{V}_{\mathcal{S}(f)} := b^{-1}(\mathcal{S}(f)\backslash 0) \subseteq E$, and $\mathcal{V}_0 := b^{-1}(0) \subseteq E$ respectively. Then the cycle [E] associated to the exceptional divisor E of the blowup $B = Bl_{J(h)}\mathbf{C}^n$ is given by : $[E] = \mu(g)[\mathcal{V}_{f^{-1}(0)}] + c_{\mathcal{S}(f)}[\mathcal{V}_{\mathcal{S}(f)}] + c_0[\mathcal{V}_0]$.

Proof In view of the decomposition $E = \mathcal{V}_{f^{-1}(0)} \cup \mathcal{V}_{\mathcal{S}(f)} \cup \mathcal{V}_0$ that follows from the definition of E as the preimage in B of the singular locus $\mathcal{S}(h)$, together with (1.8) above, it suffices to show that the integer coefficient $c_{\mathcal{V}_{f^{-1}(0)}}$ associated to [or the multiplicity of] the component $\mathcal{V}_{f^{-1}(0)}$, namely the order of vanishing of J(h) along $b^{-1}(f^{-1}(0))$, is the Milnor number $\mu(g)$ of g. We note first that since at any point z on $f^{-1}(0)\setminus\{0\}$, f is a submersion, it follows that the ideal generated by Dh is the same as that generated by Dg. Thus the order of vanishing of J(h) along $b^{-1}(f^{-1}(0))$ is the same as the order of vanishing of J(g) along $f^{-1}(0)$, and we can [see [14]] calculate this as the multiplicity of the ideal on a normal slice to $f^{-1}(0)$ at z; since f is a diffeomorphism $\mathbf{C}^2 \to \mathbf{C}^2$ on such a slice, we obtain the desired result that the multiplicity of J(g) is precisely $\mu(g)$. □

(1.16) Remark One checks that in the formula for [E] in the preceding Lemma, the coefficient $c_{\mathcal{S}(f)}$ is just 1, and that the coefficient c_0 is λ^0.

2. $\mathcal{O}$(N,2), N $\geq$ 3 : NUMERICAL CHARACTERIZATION OF EQUISINGULARITY

We turn first to the Lê numbers $\lambda^0(h), ..., \lambda^{n-1}(h)$ associated to the $(n-1)$-dimensional variety $X \subseteq \mathbf{C}^n$ defined by h.

(2.1) Remark Since the singular locus $\mathcal{S}$(h) has dimension (n − 2) [see 1.5 &1.11] , we know [see (1.9)] that $\lambda^{n-1}(h) = 0$. Also, since dim $\mathcal{S}(f) = 1$ & dim $\{0\} = 0$, we know that only $\mathcal{V}_{f^{-1}(0)}$ can contribute to the cyles $\Lambda^2(h), ..., \Lambda^{n-2}(h)$, for n > 3.

(2.2) FORMULAS: HYPERPLANE SECTIONS AND LÊ NUMBERS
Let $\lambda^j(h_t|L^k)$ denote the j-dimension Lê number of the restriction of X(t) to a generic k-dimensional plane section L^k. By [(3.4) &(6.1) of [7]], the following relations hold between the Lê numbers $\lambda^j(h_t)$, the relative polar multiplicities $m_t(h_t)$ [see 1.9 above], and the Lê numbers $\lambda^j(h_t|L^k)$:

(i). $\lambda^0(h_t|L^j) \;=\; \lambda^{n-j}(h_t) \;+\; m_{n-j}(h_t)$ for j=2,...,n−1; &

(ii). $\lambda^{k-i}\left(h_t|L^k\right) = \lambda^{n-i}(h_t)$ for $i = 1,...,k-1$;

We will now study the upper-semicontinuity property for the relative polar multiplicity $m_k(h_t)$ of the maps h_t, [i.e. the upper-semicontinuity of the map $t \mapsto m_k(h_t)$], and we fix notation as follows. Let F: $(\mathbf{C} \times \mathbf{C}^n, 0) \to (\mathbf{C} \times \mathbf{C}^2, 0)$ be given by F(t,z):= $(t, f_t(z))$, $f_t \in \mathcal{O}(n,2), n \geq 3$. Let G be the defining equation of $\Delta(F) \subseteq \mathbf{C} \times \mathbf{C}^2$, and put H := G∘$F$: $(\mathbf{C} \times \mathbf{C}^n, 0) \to (\mathbf{C}, 0)$.

We use a variation of the blowup described in (1.8) above: here we work with the blowup b: $\mathfrak{B} := Bl_{J_z(H)}(\mathbf{C} \times \mathbf{C}^n) \to \mathbf{C} \times \mathbf{C}^n$, with associated exceptional divisor $\mathcal{E}$. Let $\pi : \mathfrak{B} \to T \cong \mathbf{C}$ be the map obtained by composing the structure map b with the canonical projection of $\mathbf{C} \times \mathbf{C}^n$ to the first factor; the **fiber** over t∈ T shall refer to the preimage of t under this map π. Consider the set $\Gamma_z^k(H) = b(\mathfrak{B} \cap H_1 \cap ... \cap H_{n+1-k})$, and note that in general [see (2.2) of [7]] the following inclusion holds:

$$\Gamma^k(h_t) \subseteq \Gamma_z^{k+1}(H) \cap \{t\} \times \mathbf{C}^n. \qquad (*)$$

(2.3) Remark We wish to show that the preceding inclusion is in fact an equality for t = 0, [i.e. that the fiber $\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbf{C}^n$ of $\Gamma_z^{k+1}(H)$ over 0 **specializes** to $\Gamma^k(h_0)$] because if so, we can write $m_k(h_t) = \mathrm{e}(\mathbf{m}_z, \mathcal{O}_{\Gamma^{k+1}(H)\cap\{t\}\times\mathbf{C}^n})$, [here $\mathbf{m}_z$ denotes the maximal ideal $(z_1,...,z_n)$], and the desired result follows from the upper-semicontinuity of the classical ideal multiplicity e [see **[14]**,(3.5)]. The problem is that there may well exist a component $C \subseteq \mathcal{S}(h_0)$ such that $\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbf{C}^n = \Gamma^k(h_0) \cup C$, and the inclusion (*) is strict. More precisely, if there is no component of $[\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbf{C}^n]$ in $\mathcal{S}(h_0)$, then we have the desired specialization, i.e. $\Gamma^k(h_0) = \Gamma_z^{k+1}(H) \cap (\{0\} \times \mathbf{C}^n)$, and hence the upper-semicontinuity of the relative polar multiplicities $m_k(h_t)$, for k = 0,...,n−2.

(2.4) LEMMA In the setup above, if the Milnor number $\mu(g_t)$ is independent of t, then the map $t \mapsto m_k(h_t)$, k = 0,...,n−2, is upper-semicontinuous.

Proof Since $\mathcal{S}(h_0) = f_0^{-1}(0) \cup \mathcal{S}(f_0)$, [by (1.11)], it suffices, by (2.3) above, to show that for k = 0,...,n−2, (i) no component of $[\Gamma^{k+1}(H) \cap \{0\} \times \mathbf{C}^n]$ belongs to $f_0^{-1}(0)$; and (ii) no component of $[\Gamma^{k+1}(H) \cap \{0\} \times \mathbf{C}^n]$ belongs to $\mathcal{S}(f_0)$.
(i) Fix k. Note first that at any point $z \in f_0^{-1}(0) \setminus \{0\}$ $(=f_0^{-1}(0)\setminus\{\mathcal{S}(f_0)\})$, the map f_0 [and hence F at (0,z)] is a submersion. Also, the hypothesis of constancy of $\mu(g_t)$ implies [by the Lê-Saito theorem; see **[17]**, p.70] that Thom's condition A_G is satisfied by G relative to the pair $(\mathbf{C}\times\mathbf{C}^2, T)$, so it follows that the map H:= (G∘F) satisfies the A_H condition for the pair $(\mathbf{C}\times\mathbf{C}^n, F^{-1}(T))$. This means that every limiting tangent plane to fibers of H contains the tangent space to $F^{-1}(T)$; in terms of the blowup $b{:}B{=}Bl_{J(H)}\mathbf{C} \times \mathbf{C}^n \to \mathbf{C} \times \mathbf{C}^n$, it follows [since $\Gamma^{k+1}(H) := b(B \cap H_{n-k})$; see (1.9) above] that: there is a k-dimensional component of $[\Gamma^{k+1}(H) \cap \{0\} \times \mathbf{C}^n]$ in $f_0^{-1}(0)$ iff $H_{n-k} \cap \mathcal{E} \cap b^{-1}(f_0^{-1}(0))$ has dimension k. But since at a generic point, the dimension of $(b^{-1}(f_0^{-1}(0)))$ is n−1, we get that $[H_{n-k} \cap \mathcal{E} \cap b^{-1}(f_0^{-1}(0))]$ has dimension (k - 1), so we are done.
(ii) We use a more direct argument here. Note that at a generic point z of $\mathcal{S}(f_0)$ [i.e. for any point $z \in \mathcal{S}(f_0)\setminus\{0\}$], f_0 is a fold, and that at $f_0(z)$, the map g_0 is linear submersion. Using the hypothesis of constancy of $\mu(g_t)$, we see that [for t small], H is then [by suitable choice of coordinates] the map $(t, z_1, ..., z_n) \to (z_2^2 + ... + z_n^2)$, so that $J(H) = (z_2, ..., z_n)$, from which it is clear that the relative polar variety $\Gamma^{k+1}(H)$ is independent of t [for each k], and hence specializes as desired, so that $\Gamma^k(h_0) = \Gamma^{k+1}(H) \cap (\{0\} \times \mathbf{C}^n)$. □

(2.5) LEMMA In the setup of (1.1), if the 0-dimensional Lê number $\lambda^0(h_t|L^2)$ is independent of t, for small $t \in T$, then the Milnor number $\mu(\Delta f_t) \equiv \mu(g_t)$ is likewise independent of t.

Proof Suppose that the hypothesis of constancy of the invariant $\lambda^0(h_t|L^2)$ holds, but that the value of the Milnor number changes at 0; we will derive a contradiction. We may suppose that the singular sets of the maps g_t are of the form $\bigcup_t(\mathcal{S}(g_t)) = T \cup C$, where C is a curve in $\mathbf{C} \times \mathbf{C}^2$, such that C is distinct from T, and that $T \cap C = (0,0)$. Consider $F^{-1}(C) \cap (L^2 \times T)$; at each point $(t,c) \in F^{-1}(C)$, we know that $Dg_t(F(t,c))) = 0$. This implies that the maps $h_t|L^2$ are singular along $F^{-1}(C)$, which in turn means that $\lambda^0(h_t|L^2)$ cannot be independent of t, a contradiction of the hypothesis. □

(2.6) Remark Lemmas (2.4) & (2.5) together imply that if the Lê number $\lambda^0(h_t|L^2)$ is independent of t, then the relative polar multiplicities $m_k(h_t)$, for k = 1,...,n−2, are upper-semicontinuous.

(2.7) PROPOSITION Keep the setup of (1.1) above, & fix n≥3. Then the (n−1) zero-dimensional Lê numbers $\lambda^0(h_t|L^2), ..., \lambda^0(h_t|L^n)$ are independent of t, iff all the Lê numbers of all generic plane sections of X(t), as well as the relative polar multiplicities $m_k(h_t)$, for k = 1,...,n−2, are independent of t.

Proof (⇐:) (trivial); (⇒:) Recall our notation that $\lambda^i(h_t) \equiv \lambda^i(h_t|L^n)$, and the fact that only the Lê numbers of dimension less than or equal to the dimension of the singular locus, can be non-zero. The hypothesis regarding (in particular) the constancy of the Lê number $\lambda^0(h_t|L^2)$ implies, [see Remark (2.6)] that the relative polar multiplicities $m_k(h_t)$, for k = 1,...,n−2, are upper-semicontinuous. Using this upper-semicontinuity property, and recalling that the Lê numbers are lexicographically upper-semicontinuous [see [7], Cor.(4.5)], it follows from formula (i) of (2.2) above, that if the invariants $\lambda^0(h_t|L^j)$ are independent of t, then so also are all the relevant Lê numbers of h_t, namely $\lambda^0(h_t), ..., \lambda^{n-2}(h_t)$, as well as the relative polar multiplicities $m_k(h_t)$, for k = 1,...,n−2. The remaining [possibly non-zero] Lê numbers are $\lambda^d(h_t|L^s)$, [where d = 1, 2,..., n−2; s = 2, 3,..., n−1; and d<(s−1)]; by formula (ii) of (2.2), Lê number $\lambda^d(h_t|L^s)$ can be seen to be equal to the Lê number $\lambda^{n-s+d}(h_t|L^n) \equiv \lambda^{n-s+d}(h_t)$, and the latter is already known to be independent of t. □

This leads to our main result.

(2.8) THEOREM Keep the setup of (1.1) above, and fix n ≥ 3. Then the unfolding F is Whitney equisingular iff the (n−1) zero-dimensional Lê numbers $\lambda^0(h_t|L^2), ..., \lambda^0(h_t|L^n)$, are independent of t, for small t ∈ T.

Proof Suppose first that the numerical invariants are all independent of t. We rely upon the following result, due to the first author:

FOUNDATIONAL THEOREM Suppose F: $\mathbf{C} \times \mathbf{C}^n \to \mathbf{C} \times \mathbf{C}^p$ is an excellent unfolding of $f \in \mathcal{O}(n,p)$. Then F is topologically trivial, provided the polar invariants of all stable types which occur in the discriminant $\Delta(f_0)$, in $\sum(f_0)$, and in $f_0^{-1}(\Delta(f_0)) \setminus \sum(f_0)$ are constant at the origin for t.

Proof [c.f.[2], p.207]

This foundational theorem remains valid if we replace the term "an excellent unfolding" in the hypothesis by "a 1-parameter unfolding which, when stratified by stable type & by the parameter axis T, has only the parameter axis T as 1-dimensional stratum at the origin". Also, the proof of this theorem shows that F is Whitney equisingular.

Now by Proposition (2.7) above, it follows from the hypothesis that the Lê numbers of h_t, and of the restriction of X(t) to generic plane sections at the origin, as well as the relative polar multiplicities $m_j\,(h_t)$, for j = 1,..., n−2, are all likewise independent of t. The result that $\mathfrak{X}$ is Whitney equisingular along T now follows from [Theorem (6.2) of [**7**]], provided we can establish the claim that $\mathfrak{X}\backslash\mathcal{S}(\mathfrak{X})$ is Whitney along $\mathcal{S}(\mathfrak{X})\backslash$T. This claim is valid in our setup because of the general fact that off the parameter axis T, each finitely determined map is stable. Finally, the constancy of the Milnor number $\mu\,(\Delta f_t)$ [implied by the hypothesis: see (2.5)] implies that the 0-stable invariants are constant, and T is the only 1-parameter stratum of the unfolding F in the target [see [**23**], section(1.7)], so that we can apply the foundational theorem stated above to conclude that F is Whitney equisingular.

Conversely, suppose that F is Whitney equisingular. Then $\mathfrak{X}$ is Whitney equisingular along T, which is equivalent [see e.g. [**15**]] to the Lê numbers of X(t) and all its plane sections being independent of t, so that in particular, the (n−1) zero-dimensional Lê numbers are independent of t. □

Appendix Proof of Lemma (1.4):

(i) Assume $\mathcal{S}(f_t) \neq \emptyset$. We will use the fact [see Appendix A(1)of [**23**] for this] that the singular locus, if nonempty, of any smooth stable map germ from $\mathbf{C}^n$ to $\mathbf{C}^2$, $n \geq 3$, is necessarily of dimension 1. We recall also the following result that relates finitely determined map germs and stable map germs:
Theorem (Mather): *If $f_t \in \mathcal{O}(n,p)$ is finitely determined, then there is an open neighborhood U of 0 in the source such that for each point z in $U\backslash\{0\}$, the germ of f_t at z is stable.* [This result is proved in [**5**]]

The desired result now follows, because at each point z different from 0 in the singular locus, the singular locus of f_t is locally 1-dimensional. □

(ii) We know [see (i)] that $\mathcal{S}(f_t)$, if nonempty, is 1-dimensional, and it is clear that both $\mathcal{S}(f_t)$ and $f_t^{-1}(0)$ contain the point $\{0\}$ in the source. The desired result now holds because $f_t|_{\mathcal{S}(f_t)}$ is a finite map [see (1.3)]. □

(iii) Suppose $f_t{:=}(\phi,\varphi)$; since $f_t^{-1}(0) = \mathcal{V}(\phi,\varphi) \subseteq \mathbf{C}^n$, it follows that $\dim f_t^{-1}(0) = \dim\mathcal{V}(\phi,\varphi) \geq (n-2)$. We claim that $\dim\mathcal{V}(\phi,\varphi)$ cannot be n or (n−1). To see this, note that if $\dim\mathcal{V}(\phi,\varphi)$ is either n or (n−1), then the kernel rank of f_t must be strictly greater than (n−2) at points of $f_t^{-1}(0)$, implying that $f_t^{-1}(0) \subseteq \mathcal{S}(f_t)$, and hence that $\dim\mathcal{S}(f_t) > 1$, a contradiction of (i) above. □

(iv) For all points $\bar{z} \in \Delta(f_t)\backslash 0$ sufficiently close to 0, we know Δf_t is smooth at $\bar{z}$; since g_t defines Δf_t with reduced structure, we conclude that $Dg_t(\bar{z}) \neq 0$. □

(v) We first show that $\mathcal{S}(f_t) \subseteq \mathcal{S}(g_t \circ f_t)$, where $h_t := (g_t \circ f_t)$. It is clear that this inclusion holds if $\mathcal{S}(f_t) = \emptyset$; consider the case of $\mathcal{S}(f_t) \neq \emptyset$. Note that at any nonzero point in $\mathcal{S}(f_t)$ close to 0, f_t is locally a fold, so it follows immediately that locally the inclusion $\mathcal{S}(f_t) \subseteq \mathcal{S}(g_t \circ f_t)$ holds. As regards the point $0 \in \mathcal{S}(f_t)$, the connectedness of the curve $\mathcal{S}(f_t)$, together with the inclusion just established for nonzero points in $\mathcal{S}(f_t)$ shows that $0 \in \mathcal{S}(g_t \circ f_t)$ As regards the second inclusion $\mathcal{S}(g_t \circ f_t) \subseteq f_t^{-1}(0) \cup \mathcal{S}(f_t)$ asserted in (v), consider a point $z \in \mathcal{S}(g_t \circ f_t)\backslash\mathcal{S}(f_t)$; it suffices to show that z belongs to $f_t^{-1}(0)$. At such a point z, we know f_t is a submersion, and $Dh_t(z) = 0$; it follows from the chain rule that we then must have $Dg_t(f_t(z)) = 0$. Since we also know that g_t is a submersion at each point of $\Delta f_t\backslash 0$, [see part (iv) of (2.4)], we conclude that $f_t(z) = 0$; the desired conclusion follows. □

(vi) As $X(t) \subset (\mathbf{C}^n, 0)$ is a hypersurface germ, the ideal $\mathcal{I}(X(t))$ is principal; say the principal ideal $(\tilde{h}_t)$ corresponds to X(t), $\tilde{h}_t$: $(\mathbf{C}^n,0) \to (\mathbf{C},0)$ non-constant; in particular, $(\tilde{h}_t) := \mathcal{I}(X(t))$. We know also that by definition, $\mathcal{V}(h_t) = X(t)$, so the ideal containment $(h_t) \subseteq (\tilde{h}_t)$ is clear. We claim that strict containment is untenable; this will give us the desired result. Now if ideal (h_t) is strictly contained in $(\tilde{h}_t)$, then there must exist a component X' of X such that the defining equation $\tilde{h}_t'$ of X' is related related to $h_t' := \left(g_t' \circ f_t\right)$ by $h_t' = (\tilde{h}_t')^m)$ for some integer $m \geq 2$. In order to lighten notation, we may assume, without loss of generality, that X is irreducible, and hence that we must have $(g_t \circ f_t) = \left(\tilde{h}_t^m\right)$. We now consider the singular set $\mathcal{S}(g_t \circ f_t)$ of $h_t \equiv (g_t \circ f_t)$, namely $\mathcal{S}(g_t \circ f_t) := \mathcal{V}(g_t \circ$

$f_t, \quad \partial(g_t \circ f_t)/\partial z_1, \ldots, \quad \partial(g_t \circ f_t)/\partial z_n)$ and derive a contradiction. Since we have $(g_t \circ f_t) = (\tilde{h}_t^m)$, it follows that $\tilde{h}_t$ divides each of the functions defining $\mathcal{S}(g_t \circ f_t)$, implying that these functions belong to the ideal $(\tilde{h}_t)$ and thus must vanish on X(t). But then $\mathcal{S}(g_t \circ f_t)$ is *all* of X(t), which contradicts the fact [see (i), (ii) & (v)] that $\mathcal{S}(g_t \circ f_t)$ is at most (n−2)-dimensional, whereas X(t) is (n−1)-dimensional, n≥3. □

REFERENCES

[1] Fulton William, Intersection Theory, *Springer Verlag*, 1984.

[2] Gaffney Terence, Polar Multiplicities and Equisingularity of Map Germs, *Topology*, vol.32, No.1, (1993) pp. 185-223.

[3] Gaffney Terence, Plane Sections, W_f and A_f, in Real and Complex Singularities, edited by Bruce, J.W., and F. Tari; *Research Notes in Mathematics* 412, Chapman & Hall (2000).

[4] Gaffney Terence, $\mathcal{L}^0$-equivalence of Maps, in *Math. Proc. Camb. Phil. Soc.* **128**, (2000) pp. 479-496.

[5] Gaffney Terence, Properties of Finitely Determined Germs, *Ph.D. thesis*, Brandeis University, 1975.

[6] Gaffney Terence, Multiplicities and Equisingularities of ICIS Germs, *Invent. Math.* **123**, (1996) pp. 200-220.

[7] Gaffney Terence and Gassler Robert, Segre Numbers and Hypersurface Singularities, *Algebraic Geometry*, 8, (1999) pp. 695-736.

[8] Gaffney Terence and Kleiman S.L., W_f and Integral Dependence, in Real and Complex Singularities, edited by Bruce, J.W., and F. Tari; *Research Notes in Mathematics* 412, Chapman & Hall (2000).

[9] Gaffney Terence and Massey David, Trends in Equisingularity Theory, in Singularity Theory, Edited by Bruce, J.W., and D. Mond, *Cambridge University Press*, 1999.

[10] Golubitsky Martin and Guillemin V., Stable Mappings and Their Singularities, *Springer Verlag*, 1973.

[11] Hartshorne Robin, Algebraic Geometry, *Springer Verlag*, 1977.

[12] Lê Dung Trang, Une Application d'un Theoreme d'A'Campo a L'equisingularite, in *jour. Indag. Math*, vol. 35, (1973) pp.403-409.

[13] Lê Dung Trang and Spitzen G. Greuel, Doppelpunkte unde verticale tangenten in der diskriminante verseller deformationen von vollständigen durchschmitten, *Math. Ann.* 222, (1976) pp. 71-99.

[14] Lipman Joseph, Equimultiplicity, Reduction, and Blowing Up, *Lecture Notes Pure Appl. Math. 68*, Marcel Dekker, New York, 1982.

[15] Lê Dung Trang and Teissier B., Cycles evanescents, sections planes et conditions de Whitney. II *Proc. Symposia pure math.*, vol 40, part 2, Amer. Math. Soc., (1983) pp. 65-104.

[16] Lê Dung Trang and Ramanujam C.P., The invariance of Milnor's number implies the invariance of the topological type, Amer. J. Math, **98**, (1976) 67-78.

[17] Massey David, Lê Cycles and Hypersurface Singularities, *Springer Lecture Notes in Mathematics* #1615 (1995).

[18] Mather John N., Stability of C^∞ Mappings, IV: Classification of Stable Germs by **R**-Algebras, *Publ. Math. I.H.E.S.* **37**, (1969) pp. 223-248.

[19] Mather John N., Stability of C^∞ Mappings, V: Transversality, *Advances in Mathematics* 4 (1970).

[20] Mather John N., Stability of C^∞ Mappings, III: Finitely Determined Map Germs, *Publ. Math. I.H.E.S.* **35**, (1968) pp. 127-156.

[21] Perez V.H.J., Polar Multiplicities and Equisingularity of map germs from $\mathbf{C}^3$ to $\mathbf{C}^3$, *Ph.D. thesis*, Universidade de Sao Paulo (2000).

[22] Teissier Bernard, Varieties Polaires Locales: quelques resultats, *Journels Complexes*, Institut Elie Cartan (March 1981).

[23] Vohra Ranjit, Equisingularity of Map Germs from n-Space, (n≥3), to the Plane, *Ph.D. thesis*, Northeastern University, 2003.

[24] Wall C.T.C., Finite Determinacy of Smooth Map-Germa, *Bull. London Math. Soc.*, 13, (1981) pp. 481-539.

[25] Wall C.T.C., Lectures on C^∞-Stability and Classifications, *Springer Lecture Notes in Mathematics*, #192, (1971) pp. 178-206.