A FUNDAMENTAL GROUP FOR DYNAMICAL SYSTEMS III: SIMPLE COMPACTIFICATIONS

PAUL FEIT
UNIVERSITY OF TEXAS AT PERMIAN BASIN
ODESSA, TEXAS, 79762

(Received 24 November 2003)

ABSTRACT. This paper continues exploration of a recent homotopy construction for group-sets. It characterizes the fundamental group of the standard one point and two point compactifications of $\mathbb{Z}$. Each is a free profinite group on two generators. However, the canonical morphism between them does not induce an isomorphism on homotopy groups.

AMS Classification: 18B30, 18G55, 37B10, 54D99
Keywords: Dynamical Systems, Symbolic Dynamics, Homotopy, Profinite Topology.

INTRODUCTION

Two standard compactifications of $\mathbb{R}$ arise in Calculus. We apply these compactifications to $\mathbb{Z}$, to get two examples of disconnected topological $\mathbb{Z}$-sets. The universal homotopy construction developed in [1] applies to both spaces. In this paper, we calculate the invariants for these spaces, as well the the group homorphism induced from a canonical map between them.

We give a formal definition in this introduction. The objects are topological $\mathbb{Z}$-sets. Throughout this paper, we use "$\star$" to indicate the group action, which is on the left.

(1.a) Let OP be the one-point compactification of $\mathbb{Z}$. The underlying set consists of $\mathbb{Z}$ with a single formal symbol, ∞. The topology (of open subsets) consists of all subsets U such that

$$\infty \in U \;\Rightarrow$$

$$\exists r, s \in \mathbb{Z},\ \forall n \in \mathbb{Z},\ (n \leq r \text{ or } n \geq s) \;\Rightarrow\; n \in U\ .$$

Let $\mathbb{Z}$ act on OP by

$$\begin{aligned} \forall n, m \in \mathbb{Z}, &\quad n \star m &= (n+m) \\ \forall n \in \mathbb{Z}, &\quad n \star \infty &= \infty\ . \end{aligned}$$

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 2 (2004)

(1.b) Let TP be the two-point compactification of $\mathbb{Z}$. The underlying set consists of $\mathbb{Z}$ with two formal symbols, ∞ and $-\infty$. The topology (of open subsets) consists of all subsets U such that

$$\infty \in U \Rightarrow$$
$$[\exists s \in \mathbb{Z},\ \forall n \in \mathbb{Z},\ n \geq s \Rightarrow n \in U\ ]\ ,$$

and

$$-\infty \in U \Rightarrow$$
$$[\exists r \in \mathbb{Z},\ \forall n \in \mathbb{Z},\ n \leq r \Rightarrow n \in U\ ]\ .$$

Let $\mathbb{Z}$ act on OP by

$$\begin{array}{lrl} \forall n, m \in \mathbb{Z}, & n \star m & = (n+m), \\ \forall n \in \mathbb{Z}, & n \star \infty & = \infty, \text{ and} \\ \forall n \in \mathbb{Z}, & n \star (-\infty) & = (-\infty)\ . \end{array}$$

(1.c) Let $\nu : TP \longrightarrow OP$ be the $\mathbf{HT}[\mathbb{Z}]$-morphism which is the identity map on $\mathbb{Z}$, and satisfies

$$\nu(\infty) = \infty = \nu(-\infty)\ .$$

(1.d) Let E be any $\mathbf{HT}[\mathbb{Z}]$-object whose underlying set consists of a single point. Let $e_{OP} : OP \longrightarrow E$ be the unique function between these spaces.

The author formulates a functorial construction π_G on "pointed" Hausdorff topological $\mathbb{Z}$-sets in [1]. The functor is an example of the well-known categorical version of fundamental group (as described, for example, in [3]).

Let $r_1, \ldots, r_k$ be an irredundant list. We let $\widehat{Fr}[r_1, \ldots r_k]$ denote the free profinite group on (letters) $r_1, \ldots, r_k$. (The next section reviews the formulation of profinite free groups). Let $\widehat{\mathbb{Z}}$ be the profinite completion of the integers. This paper proves

Theorem A. *Let $\alpha^*, \beta^*, \gamma^*$ be three different symbols. Then there are isomorphisms*

$$\pi_G(TP, 0) \cong \widehat{Fr}[\alpha^*, \gamma^*],\ \ \pi_G(OP, 0) \cong \widehat{Fr}[\alpha^*, \beta^*] \text{ and } \pi_G(E) \cong \widehat{\mathbb{Z}}$$

with respect to which $\pi_G(\nu)$ maps

$$\alpha^* \mapsto \alpha^* \text{ and } \gamma^* \mapsto \beta^* \cdot \alpha^* \cdot (\beta^*)^{-1}$$

and, with respect to which, $\pi_G(e_{OP})$ maps

$$\alpha^* \mapsto 1 \text{ and } \beta^* \mapsto 0\ .$$

Our analysis provides some informal interpretation for the generators of the group.

Let X be a pathwise connected topological space, and let $x_0 \in X$. Members of the classical fundamental group at x_0 are (up to equivalence) paths which start and end at x_0. In contrast, the underlying spaces of OP and TP are topological disconnected; there is no way to start at 0 (which is the base point used in Theorem A) and to go anywhere. We must abandon the notion of path as a "progression" indexed by the closed interval $[0, 1]$. Instead, the $\mathbb{Z}$-action offers an alternative idea of "movement".

In OP, the $\mathbb{Z}$-action offers two "paths" to ∞. We can "go" from 0 to ∞ by passing through $n \star 0$ as $n \in \mathbb{N}$ goes to infinity, or we can go by way of $(-n) \star 0$ as $n \in \mathbb{N}$ goes to infinity. These casual descriptions refer to intuitions of motion; it is difficult to translate these ideas into the rigorous format. Unwinding the formalism, the element α^* is based on the loop

- Move from 0 to ∞ along the positive numbers.
- Once at ∞, apply 1 (as a member of $\mathbb{Z}$). In OP, $1 \star \infty = \infty$. However, if we think of paralleling these instructions in a covering space over OP, applying 1 may change the point.
- Move back, from ∞ to 0, along the positive numbers.

The element β^* is based on a different motion:

- Move from 0 to ∞ along the *negative* numbers.
- Move back, from ∞ to 0, along the positive numbers.

In $\pi_G(TP, 0)$, the α^* refers to essentially the same motion as in $\pi_G(OP, 0)$. That is, α^* refers to: move to ∞ via the positive numbers, apply 1, and then return home the same way. The γ^* represents the analogous motion: move to $-\infty$, action by 1, followed by a return.

1. Formulation and Statement

We assume that the reader is familiar with the immediate predecessor to this paper, *A Fundamental Group for Dynamical Systems II: $\mathbb{Z}_p$*. We continue in that context.

We reserve "$\cdot$" for either multiplication in $\mathbb{Z}$ or the binary operation of a group. To avoid confusion, we indicate the action by $n \in \mathbb{Z}$ on a member x of some $\mathbf{HT}[\mathbb{Z}]$-object by $n \star x$.

Our result uses the *profinite free group* construction.

Let $\mathbf{Gr}$ be the category of groups. Let $\mathbf{HGr}$ be the category of Hausdorff topological groups.

Let S be a set. Define a category $\mathbf{Gr}[S]$ as follows:

(2.a) An object is a pair (H, h) in which H is a group and $h : S \longrightarrow H$ is a function (of sets).

(2.b) A morphism $(H, h) \longrightarrow (J, j)$ is a group homomorphism $\tau : H \longrightarrow J$ for which $j \circ \tau = h$.

(2.c) Composition and identity are induced from the category of sets.

Obviously, $(H, h) \mapsto H$ is a "forgetful functor" from $\mathbf{Gr}[S] \longrightarrow \mathbf{Gr}$.

A *free group on (letters) S* is a pair $(F, f) \in \mathbf{Gr}[S]$ such that, for each $(H, h) \in \mathbf{Gr}[S]$, there is a unique $\mathbf{Gr}[S]$-morphism $(F, f) \longrightarrow (H, h)$. General theory states that free groups exist. We refer to such a pair as $Fr[S]$; we use the same notation to refer to the group in the pair, and refer to the companion function as *canonical.*

Define a subcategory $\mathbf{Fin}[S]$ of $\mathbf{Gr}[S]$ by

(3.a) An object is a $\mathbf{Gr}[S]$-object (H, h) for which
- H is finite , and
- as a group, H is generated by the image of h.

(3.b) Every $\mathbf{Gr}[S]$-morphism between two $\mathbf{Fin}[S]$-objects is included as a $\mathbf{Fin}[S]$-morphism.

It is well-known that there is a set X of $\mathbf{Fin}[S]$-objects such that every $(H, h) \in \mathbf{Fin}[S]$ is isomorphic to exactly one member of X. In addition, the restriction of the forgetful functor to X, and to the morphisms between members of X, is an inversely directed system of finite groups. The inverse limit of this system, in the catgeory of profinite groups, is called the *profinite free group on (letters) S*. We denote it by $\widehat{Fr}[S]$.

This inverse limit is also an inverse limit in the sense of sets. Consequently, there is a unique $\iota : S \longrightarrow \widehat{Fr}[S]$ induced from function components of members of X. We refer to this ι as the *canonical map*. It has the following universal property: If H is a profinite group and $h : S \longrightarrow H$ is a function, then there is a unique continuous group homomorphism

$$h_1 : \widehat{Fr}[S] \longrightarrow H$$

such that $h = h_1 \circ \iota$.

It is well-known that the canonical functions from S to $Fr[S]$ and to $\widehat{Fr}[S]$, respectively, are injective. We freely identify each $s \in S$ with its images in these spaces. Also, if $r_1, \ldots, r_k$ is a list of the members of S, we freely use the list in place of "S" in the preceding notations.

2. Fixed Points

First, we refine the notion of leveling data near a point fixed by a member of $\mathbb{Z}$.

Lemma 1. *Let $b : B \longrightarrow A$ be separable* $\mathbf{HT}[\mathbb{Z}]$*-morphism. Let $z_0 \in A$ and $r_1, \cdots r_k \in \mathbb{Z}$ such that*

$$\forall j \in \mathbb{N}(k),\ r_j \star z_0 = z_0 \ .$$

Let F be the fiber $b^{-1}\{z_0\}$. Observe that action by each r_j permutes F.

Let $(V, \{U_z\}_{z\in F})$ be leveling data for z_0 in A. We say that the data is compatible with $r_1, \cdots, r_k$ *if*

$$\forall j \in \mathbb{N}(k),\ \forall z \in F,\ \forall x \in U_z,\ r_j \star b(x) \in V \ \Rightarrow\ (r_j \star x) \in U_{r_j \star z} \ .$$

Then there exists leveling data for z_0 in A which is compatible with $r_1, \cdots, r_k$.

Proof. Start with leveling data $(V_1, \{W_z\}_{z\in F})$. For each $z \in F$, put

$$Y_z = W_z \cap \bigcap_{j=1}^{k} ((-r_j) \star W_{r_j \star z}) \ .$$

Briefly, Y_z consist of all $x \in W_z$ such that, for any index j, if $r_j \star b(x) \in V$, then $r_j \star x \in W_{r_j \star z}$. For each $z \in F$, Y_z is an open neighborhood of z. Put

$$V = \cap_{z\in F}\, b(Y_z) \ ,$$

and, for each $z \in F$, put

$$U_z = W_z \cap (b^{-1}V) \ .$$

It is an easy exercise to check that $(V, \{U_z\}_{z\in F})$ is leveling data with the desired property. □

We proceed with a construction.

Proposition 2. *Let $b : B \longrightarrow A$ be a Galois* $\mathbf{HT}[\mathbb{Z}]$*-morphism. Let $y_0 \in A$, let $\epsilon \in \{-1, 1\}$, and let N be the index of b. Suppose that the sequence $y_n = (n \cdot \epsilon) \star y_0$ converges to some $z_0 \in A$. Let $F[y_0] = b^{-1}\{y_0\}$ and $F[z_0] = b^{-1}\{z_0\}$. Then there is a bijection $\theta : F[y_0] \longrightarrow F[z_0]$ with the property that, for $y \in F[y_0]$, $\theta(y)$ is the limit of the sequence $w_n = (N! \cdot n \cdot \epsilon) \star y$.*

In the remainder of this paper, we refer to this function θ as $\theta[b, y_0, \epsilon]$.

Proof. For convenience, let $F = F[z_0]$ denote the fiber over z_0. Let $(V, \{U_z\}_{z \in F})$ be leveling data at z_0 for b, which is compatible with the pair $1, -1$. Since the sequence $\{(n \cdot \epsilon) \star y_0\}$ converges to z_0, there exists $M \in I\!N$ such that

$$\forall n \in I\!N,\ n \geq M \ \Rightarrow\ (n \cdot \epsilon) \star y_0 \in V\ .$$

Let α denote $z \mapsto \epsilon \star \tilde{z}$ on F. Let $n, k \in I\!N$ such that $n \geq M$. The compatibilty of the leveling data implies that

$$\forall y \in F[y_0],\ \forall z \in F,\ (n \cdot \epsilon) \star y \in U_z \ \Rightarrow\ ((n + k)\epsilon) \star y \in U_{\alpha^k(z)}\ ,$$

by induction.

The permutation α acts on a set with N members. Therefore, $\alpha^{N!}$ is the identity permutation. We interpret this as follows. Let $y \in F[y_0]$ and let $n_1 \in I\!N$ be a multiple of $N!$ which is $\geq M$. Then there is $z \in F[z_0]$ such that $(n_1 \cdot \epsilon) \star y \in U_z$. Let $f : V \longrightarrow U_z$ be the inverse to the restriction of b to a function $U_z \longrightarrow V$. Then, for each $m \geq n_1$ which is a multiple of $N!$,

$$(m \cdot \epsilon) \star y = f((m \cdot \epsilon) \star y_0)\ .$$

Since $\{(n \cdot \epsilon) \star y_0\}$ converges to z_0, we conclude

$$(N! \cdot n \cdot \epsilon) \star y \to z\ .$$

Thus, there is a function θ that assigns each $y \in F[y_0]$ to the limit of $(N! \cdot n \cdot \epsilon) \star y$.

Our argument provides an alternative description for θ: each $y \in F[y_0]$ goes to the $z \in F[z_0]$ such that $(n_1 \cdot \epsilon) \star y \in U_z$. This characterization implies that θ is bijective. □

3. Separable Extensions of OP

We must classify the subcategory of Galois morphisms into OP. We introduce two categories, which help to index data.

We begin with notation.

(4) For F a finite set, let Σ_F denote the group of permutations on F (acting on the left).

We represent the binary operation either as composition or, when we wish to emphasize the group structure, by $\cdot$.

Define a category $\mathcal{C}_2$ as follows.

(5.a) An object is a triple (F, α, β) in which F is a finite set, and $\alpha, \beta \in \Sigma_F$.

(5.b) A morphism between objects $(F_1, \alpha_1, \beta_1) \longrightarrow (F_2, \alpha_2, \beta_2)$ is a function $f : F_1 \longrightarrow F_2$ such that

$$f \circ \alpha_1 = \alpha_2 \circ f \text{ and } f \circ \beta_1 = \beta_2 \circ f\ .$$

(5.c) Composition and identity are induced from the category of sets.

Let Sep$/OP$ be the subcategory of $\mathbf{HT}[\mathbb{Z}]/OP$ in which an object is any member which is separable as a morphism, and in which a morphism is any $\mathbf{HT}[\mathbb{Z}]/OP$-morphisms between such objects. There is a covariant functor Φ :Sep$/OP \longrightarrow \mathcal{C}_2$ characterized as follows:

(6.a) Let $b : B \longrightarrow OP$ belong to Sep$/OP$. The image is

$$\Phi(B, b) = (F, \alpha, \beta),$$

constructed as follows. Let $F = b^{-1}\{0\}$ be the fiber over 0. In the notation of Proposition 2, put

$$\theta_+ = \theta[b,0,1] \text{ and } \theta_- = \theta[b,0,-1] .$$

Define $\beta = \theta_+^{-1} \circ \theta_-$. Define α by

$$\forall x \in F,\ \alpha(x) = \theta_+^{-1}(1 \star \theta_+(x)) .$$

(6.b) Let $b : B \longrightarrow OP$ and $c : C \longrightarrow OP$ be separable morphisms, and let $f : (B,b) \longrightarrow (C,c)$ be a $\mathbf{HT}[\mathbb{Z}]/OP$-morphism. Express

$$\Phi(B,b) = (F_b, \alpha_b, \beta_b) \text{ and } \Phi(C,c) = (F_c, \alpha_c, \beta_c) .$$

Since F_b is a subset of B, f restricts to it. Then $f(F_b) \subseteq f(F_c)$. It is a trivial consequence of continuity that, as a function $F_b \longrightarrow F_c$, f is $\mathcal{C}_2$-morphism. We define $\Phi(f)$ to be the restriction.

It is trivial to check that Φ is a functor.

The following propositions classify all separable extensions of OP. We prove all assertions with a multistage argument.

Lemma 3. *Let $(F,\alpha,\beta) \in \mathcal{C}_2$. Then there is $b \in Sep/Op$ such that $\Phi(b)$ is isomorphic to (F,α,β) if and only if the subgroup of Σ_F generated by α and β acts transitively on F.*

Lemma 4. *Let $b, c \in Sep/OP$. Then the function, induced by Φ, from Sep/OP morphisms $b \longrightarrow c$, to the set of $\mathcal{C}_2$-morphisms $\Phi(b) \longrightarrow \Phi(c)$, is a bijection.*

Proof. We begin with a functorial construction on $\mathcal{C}_2$.

Let $\mu = (F,\alpha,\beta)$ belong to $\mathcal{C}_2$. Define $\mathrm{Ind}[\mu, OP]$ to be the Cartesian product $F \times OP$. We denote the canonical projection $F \times OP \longrightarrow OP$ by $\pi[\mu, OP]$. Define a binary operation $\mathbb{Z} \times \mathrm{Ind}[\mu, OP] \longrightarrow \mathrm{Ind}[\mu, OP]$ by

$$\begin{aligned} \forall x \in F,\ \forall n, m \in \mathbb{Z},\quad & n \star (x,m) &= (x, n+m) \\ \forall x \in F,\ \forall n \in \mathbb{Z},\quad & n \star (x,\infty) &= (\alpha^n(x), \infty) . \end{aligned}$$

We leave it to the reader to verify that $\star$ defines a left action by $\mathbb{Z}$ on $\mathrm{Ind}[\mu, OP]$, and that $\pi[\mu, OP]$ is equivariant with respect to this action.

For each $z \in F$, define a subset of $\mathrm{Ind}[\mu, OP]$ as follows:

$$\begin{aligned} W_z &= \{\ (x,k)\ :\ x \in F,\ k \in \mathbb{N} \text{ and } \alpha^k(x) = z\} \\ &\cup \{\ (x,-k)\ :\ x \in F,\ k \in \mathbb{N} \text{ and } (\alpha^{-k} \cdot \beta)(x) = z\} \\ &\cup \{(z,\infty)\} . \end{aligned}$$

For each $k \in \mathbb{N}$, let V_k be the subset of OP

(7) $$V_k = \{\ m\ :\ m \in \mathbb{N},\ m \geq k\} \cup \{\ -m\ :\ m \in \mathbb{N},\ m \geq k\} \cup \{\infty\} .$$

The family $\{V_k\}$ is a basis of open neighborhoods for ∞ in OP. The set of subsets of $\mathrm{Ind}[\mu, OP]$

$$\begin{aligned} &\{\ \{(x,k)\}\ :\ x \in F,\ k \in \mathbb{Z}\} \\ \cup &\{\ \pi[\mu, OP]^{-1} V_k\ \cap W_x\ :\ x \in F,\ k \in \mathbb{N}\} \end{aligned}$$

is a basis of a topology (of open sets) for $\mathrm{Ind}[\mu, OP]$. From here on, we assign this topology to $\mathrm{Ind}[\mu, OP]$.

Suppose $x, z \in F$. Let $k, n \in \mathbb{N}$. By inspection of the sets,

$$\begin{aligned} (x,k) &\in W_z &\Rightarrow\quad & n \star (x,k) &\in W_{\alpha^n(z)} , \\ (x,-k) &\in W_z &\Rightarrow\quad & (-n) \star (x,-k) &\in W_{\alpha^{-n}(z)} . \end{aligned}$$

Next, suppose, in addition, that $k > n$. Then

$$\begin{array}{rlrl} (x,-k) & \in W_z & \Rightarrow \quad n \star (x,-k) & \in W_{\alpha^n(z)} \ , \\ (x,k) & \in W_z & \Rightarrow \quad (-n) \star (x,k) & \in W_{\alpha^{-n}(z)} \ . \end{array}$$

The following assertions follow easily.

(8.a) With this topology and this action, $\mathrm{Ind}[\mu, OP]$ is an $\mathbf{HT}[\mathbb{Z}]$-object and $\pi[\mu, OP]$ is an $\mathbf{HT}[\mathbb{Z}]$-morphism. We regard $\mathrm{Ind}[\mu, OP]$ and $\pi[\mu, OP]$ as object and morphism in $\mathbf{HT}[\mathbb{Z}]$.

(8.b) For each $y \in OP$, there is leveling data for $\pi[\mu, OP]$ at y.

(8.c) Let N be the size of F. Every fiber of $\pi[\mu, OP]$ has N members.

(8.d) The function $\pi[\mu, OP]$ is an open mapping. In addition, it is a closed mapping.

Let N be the size of F. For $k \in \mathbb{N}$,

$$\begin{array}{rl} (N! \cdot k) \star (x,0) & = (x, N! \cdot k) \in W_x, \\ (-N! \cdot k) \star (x,0) & = (x, -N! \cdot k) \in W_{\beta(x)} \ . \end{array} \tag{9}$$

Consequently,

(10.a) the sequence $(N! \cdot k) \star (x,0)$ converges to (x, ∞), and

(10.b) the sequence $(-N! \cdot k) \star (x,0)$ converges to $(\beta(x), \infty)$.

We may conclude that $\pi[\mu, OP]$ is separable *if and only if* $\mathrm{Ind}[\mu, OP]$ is $\mathbb{Z}$-connected.

Next, we address the issue of being $\mathbb{Z}$-connected. Suppose that $T \subseteq F$ is a non-empty subset such that $\alpha(T) \subseteq T$ and $\beta(T) \subseteq T$. We leave it to the reader to verify that $T \times OP$ is an open and closed subset of $\mathrm{Ind}[\mu, OP]$, and that $T \times OP$ is closed under the $\mathbb{Z}$-action.

Conversely, suppose that Y is a non-empty, open and closed, $\mathbb{Z}$-invariant, subset of $\mathrm{Ind}[\mu, OP]$. The image of Y under $\pi[\mu, OP]$ is a non-empty, open and closed, $\mathbb{Z}$-invariant subset of OP. Consequently, $\pi[\mu, OP](Y) = OP$. Let

$$J = \{ \ x \in F \ : \ (x, \infty) \} \ .$$

we claim that

(11.a) $\alpha(J) \subseteq J$,

(11.b) $\beta(J) \subseteq J$, and

(11.c) $J \times OP \subseteq Y$.

Let $x \in J$. Then $1 \star (x, \infty) = (\alpha(x), \infty)$ is in Y, which implies (11.a). Take $k \in \mathbb{N}$ such that $\pi[\mu, OP]^{-1} V_k \cap W_x \subseteq Y$. Let N be the size of F, and let n be a multiple of $N!$ such that $n \geq k$. Then

$$(x, n) \in \pi[\mu, OP]^{-1} V_k \cap W_x \subseteq Y \ .$$

The $\mathbb{Z}$-orbit of (x, n) is $\{x\} \times \mathbb{Z}$. Therefore, $\{x\} \times OP \subseteq Y$. By (9), $(\beta(x), \infty) \in Y$ as well. This completes the proof of (11).

We have established $\mathrm{Ind}[\mu, OP]$ as a class-theoretic function from objects of $\mathcal{C}_2$ to objects of $\mathbf{HT}[\mathbb{Z}]/OP$. We are ready to extend it to a functor.

For the next phase of the argument, let $\mu_1 = (F_1, \alpha_1, \beta_1)$ and $\mu_2 = (F_2, \alpha_2, \beta_2)$ be two $\mathcal{C}_2$-objects. First, suppose that $f : \mu_1 \longrightarrow \mu_2$ is a $\mathcal{C}_2$-morphism. Consider the function $f^* = \mathrm{Ind}[f, OP]$ defined by

$$f^*(x, n) = (f(x), n)$$

from $\mathrm{Ind}[\mu_1, OP] \longrightarrow \mathrm{Ind}[\mu_2, OP]$. We leave it to the reader to verify that f^* is an $\mathbf{HT}[\mathbb{Z}]/OP$-morphism. Furthermore, we leave it as an exercise that these two $\mathrm{Ind}[*, OP]$ constructions determine a functor, which we denote in this manner.

Conversely, suppose that ϕ is an $\mathbf{HT}[\mathbb{Z}]/OP$-morphism

$$\mathrm{Ind}[\mu_1, OP] \longrightarrow \mathrm{Ind}[\mu_2, OP] .$$

Since ϕ commutes with the canonical projections to OP, it must send the fiber over 0 of $\mathrm{Ind}[\mu_1, OP]$ into the fiber of $\mathrm{Ind}[\mu_2, OP]$. Consequently, there is a function $f : F_1 \longrightarrow F_2$ such that

$$\forall x \in F_1,\ \phi(x, 0) = (f(x), 0) .$$

For $n \in \mathbb{Z}$ and $z \in \bar{\pi}[\mu_1, OP]^{-1}\{0\}$, it must be true that $\phi(n \star z) = n \star \phi(z)$. This can be paraphrased as

$$\forall x \in F_1,\ \forall n \in \mathbb{Z},\ \phi(x, n) = (f(x), n) .$$

This identity, in tandem with the continuity of ϕ and (10) imply that

$$\forall x \in F_1,\ \phi(x, \infty) = (f(x), \infty) . \tag{12}$$

Once we accept (12), property (10) also implies that $f \circ \beta_1 = \beta_2 \circ f$. The identity

$$\forall x \in F_1,\ \phi(1 \star (x, \infty)) = 1 \star \phi(x, \infty)$$

translates as $f \circ \alpha_1 = \alpha_2 \circ f$. We conclude that

(13.a) f is a $\mathcal{C}_2$-morphism $\mu_1 \longrightarrow \mu_2$, and
(13.b) ϕ is the f^* construction of the previous paragraph.

We conclude that, for any two $\mu_1, \mu_2 \in OP$, the construction $\mathrm{Ind}[*, OP]$ determines a bijection from $\mathcal{C}_2$-morphisms, from μ_1 to μ_2, to $\mathbf{HT}[\mathbb{Z}]/OP$-morphisms, from $\mathrm{Ind}[\mu_1, OP]$ to $\mathrm{Ind}[\mu_2, OP]$.

Next, fix $b : B \longrightarrow OP$ a separable $\mathbf{HT}[\mathbb{Z}]$-morphism into OP. We claim that b is isomorphic to $\mathrm{Ind}[\Phi(b), OP]$.

Expand $\Phi(b) = (F, \alpha, \beta)$. In addition, let

$$\theta_+ = \theta[b, 0, 1] \text{ and } \theta_- = \theta[b, 0, -1] .$$

Define a set-theoretic function

$$\tau : F \times OP \longrightarrow B \text{ by}$$

$$\tau(x, n) = \begin{cases} n \star x & \text{if } n \in \mathbb{Z}, \\ \theta_+(x) & \text{if } n = \infty . \end{cases}$$

We make a series of simple observations.

(14.a) τ induces a bijection from $F \times \mathbb{Z} \longrightarrow b^{-1}\mathbb{Z}$.
(14.b) By Proposition 2, τ is bijective.
(14.c) From the definition of τ,

$$\forall x \in F,\ \forall n, m \in \mathbb{Z},\ \tau(m \star (x, n)) = m \star \tau(x, n) .$$

(14.d) The definition of Φ (specifically, of the α component,) implies that

$$\forall x \in F,\ \tau(1 \star (x, \infty)) = 1 \star \tau(x, \infty) .$$

Put $F_1 = b^{-1}\{\infty\}$. By Lemma 1, there is leveling data $(V, \{U_z\}_{z\in F_1})$ for b at ∞, compatible with $1, -1$. Obviously, we may replace V with any open subset (by replacing each U_z with its intersection with b^{-1} of this subset). We may assume that, in the notation of (7), $V = V_K$ for some fixed $K \in \mathbb{N}$.

Let $x \in F$ and $n \in \mathbb{N}$ such that $n \geq K$. There is $z \in F_1$ such that $\tau(x, n) \in U_z$. Induction on $m \in \mathbb{N}$ and compatibility with 1 implies that,

$$\forall m \in \mathbb{N},\ \tau(x, n+m) = m \star \tau(x,n) \in U_{m\star z}\ . \tag{15}$$

Let N be the size of F.

First apply (15) when x is fixed, $n \geq K$ is a fixed multiple on $N!$, and m is any positive multiple on $N!$. Then $z \mapsto m \star z$ is the identity permutation on $z \in F_1$. Consequently,

$$(n+m) \star x \in U_z$$

for all such m. Now $\theta_+(x)$ is the limit of the sequence from $(n+m)\star x$, indexed by the parameter m (which is restricted to positive multiples of $N!$). Consequently, this limit must be z. We deduce that $z = \theta_+(x)$.

Next, consider $x \in F$ and $n \in \mathbb{N}$ such that $n \geq K$. Again let $z \in F_1$ such that $\tau(x, n) \in U_z$. Let $m \in \mathbb{N}$ such that $n+m$ is a multiple of $N!$. Then

(16.a) $\tau(x, n+m) \in U_{m\star z}$ and
(16.b) $\tau(x, n+m) \in U_{\theta_+(x)}$.

We conclude that

$$\theta_+(x) = m \star z \ \Rightarrow\ n \star \theta_+(x) = (n+m) \star z$$
$$\Rightarrow\ \theta_+(\alpha^n(x)) = (n+m) \star z\ .$$

The formula $z \mapsto 1 \star z$ induces a permutation on F_1. Since b is separable, F_1 has N members. Consequently, $(n+m) \star z = z$. We conclude that

$$\forall x \in F,\ \forall n \in \mathbb{N},\ n \geq K\ \Rightarrow\ \tau(x,n) \in U_{\theta_+(\alpha^n(x))}\ .$$

We have characterized half of each subset U_z.

Let $x \in F$. Suppose that $n \in \mathbb{N}$ such that $n \geq K$. This time, let $z \in F_1$ such that

$$\tau(x, -n) \in U_z\ . \tag{17}$$

For $m \in \mathbb{N}$, compatibility yields

$$\tau(x, -n-m) \in U_{(-m)\star z}\ .$$

Reasoning as before, we conclude that

(18) If n is a multiple of $N!$, then $z = \theta_-(x)$.

For reasons that will become clear shortly, we prefer to characterize z in terms of θ_+. The definition of β implies

(19) If n is a multiple of $N!$, then $z = \theta_+(\beta(x))$.

Next, consider x, n and z related by (17). Take $m \in \mathbb{N}$ such that $n+m$ is a multiple of $N!$. It follows that

$$(-m) \star z = \theta_+(\beta(x)) \ \Rightarrow\ z = \theta_+((\alpha^{-n} \cdot \beta)(x))\ .$$

The sets U_z are characterized.

Compare these above comments on U_z with the definition of sets W_y of $\mathrm{Ind}[\mu, OP]$, where $z = \theta_+(y)$. It follows that τ is continuous.

Either by inspection, or by the general theory of separable morphisms, it follows that τ is an $\mathbf{HT}[\mathbb{Z}]$-isomorphism. $\square$

4. Galois Extensions

We recall some standard terms concerning group sets. In these definitions, "$\cdot$" indicates the binary operation of a group, or, when the context makes this clear, for the left action by a group on a set.

Let G be any group. We review elementary concerning left G-sets.

Let G_L denote the G-set whose underlying set is G, and on which the action is multiplication (ie., by the operation of G) on the left.

Let X be a G-set. We say that the action on X is *regular* if it is transitive and the stabilizer of any point is trivial. This is equivalent to the condition that, as a G-set, X is isomorphic to G_L.

Define *opposite* group of G, denoted by G°, to be the set of G with the binary operation $*$ characterized by

$$\forall g, h \in G,\ g * h = h \cdot g\ .$$

As the name suggests, G° is a group. It is trivial to check that the rule

$$g \wedge h = h \cdot g$$

defines a *left* action by G° on G_L. This action induces a group isomorphism between G-equivariant automorphisms of G_L and G°. A consequence of this structure theory is that, for $g, h \in G_L$, there is a unique G-equivariant automorphism of G_L which takes g to h. This can be paraphrased as follows: If X is a regular G-set and $x \in X$, then there is a unique isomorphism (of G-sets) $f : X \longrightarrow G_L$ which maps x to the identity element of G.

Lemma 5. *Let X be a finite set, and let G be a subgroup of the set of permutations of X (which we treat as acting on the left). Let A be the group of permutations of X which commute with each member of G. Suppose that the action of A on X is regular, and that G is not the trivial group.*

(A) *As a G-set, X is regular.*

(B) *Fix $x_0 \in G$. Then there is a unique group isomorphism $\phi : G^\circ \longrightarrow A$ characterized by*

$$\forall g \in G,\ \phi[g](x_0) = g \cdot x_0\ .$$

Proof. The second assertion is an easy consequence on the first. We prove Part (A).

Let $x \in X$. Let H be the stabilizer of x under the action by G. Then

(20.a) For $g \in G$, the stabilizer of $g \star x$ under G is $g \cdot H \cdot g^{-1}$.

(20.b) For $\alpha \in A$, the stabilizer of $\alpha(x)$ is H.

Since the action by A is transitive, we conclude that

(21.a) H is the stabilizer of each $x \in X$.

(21.b) H is a normal subgroup of G.

We conclude that each $h \in H$ acts as the identity on X. By assumption, $H \subseteq \Sigma_X$ and the action is function application. Hence, H consists exactly of the identity map on X. The G-stabilizer of each and every point is trivial.

It follows that, as a G-set, X is a disjoint union of copies of the left regular action. That is, any two non-empty orbits correspond to isomorphic G-sets. We claim that there is just one orbit. Proof is by contradiction.

Suppose that X has two or more orbits. Let O be one orbit in X. Recall that G has at least two members. Clearly, there is $\alpha \in A$ such that $\alpha(x) = x$ for $x \notin O$, and $\alpha(x) \neq x$ for every $x \in O$. Existence of such an α violates the assumption that A acts regularly. □

We apply this lemma to Galois morphisms into OP. Let $b : B \longrightarrow OP$ be such a morphism, and expand $\Phi(b) = (F, \alpha, \beta)$. Put $A =\mathrm{Gal}(b)$. Restriction of the action to F determines an injection $A \longrightarrow \Sigma_F$; for this discussion, we freely identify A with its image under this map. By the general theory of [1], the action by A is regular. By Lemma 4, A is also the group of $\mathcal{C}_2$-automorphisms. Consequently, A can be described as the group of permutations that commute with both α and β.

Let H be the subgroup of Σ_F generated by $\{\alpha, \beta\}$. Lemma 5 applies, with $X = F$. Our data happens to satisfy one more condition: by Lemma 3, H acts transitively. Consequently, if G is trivial, then F has just one member. It follows that, as a G-set, F is the left regular action, and that H is isomorphic to action by G on the right. The exact correspondence depends on which member of F is identified with the identity of G. Fix a correspondence. If $\alpha_1, \beta_1 \in G$ correspond to α, β, we conclude that G is generated by α_1 and β_1.

We are ready to identify $\pi_G(OP, 0)$ with a profinite free group.

Let S be a set with two members, which we represent by formal symbols α^* and β^*. Assign to each $(H, h) \in \mathbf{Fin}[S]$ the $\mathcal{C}_2$-object

$$\Gamma(H, h) = (H, \alpha, \beta)$$

in which

(22.a) α is the permutation on H given by $y \mapsto y \cdot h(\alpha^*)$,
(22.b) β is the permutation on H given by $y \mapsto y \cdot h(\beta^*)$,

In addition, for $(H, h) \in \mathbf{Gr}[S]$, let e_H be the identity element of the group H.

The following observations are justified by the structure theory established so far.

(23) Let $b : B \longrightarrow OP$ be a Galois morphism into OP, and let $x \in b^{-1}\{0\}$. Expand $\Phi(b) = (F, \alpha, \beta)$. Then there is $(H, h) \in \mathbf{Fin}[S]$ and a function $f : F \longrightarrow H$ such that
- f is a $\mathcal{C}_2$-isomorphism $\Phi(b) \longrightarrow \Gamma(H, h)$,
- $f(x) = e_H$.

For this discussion, we say that the tuple (b, x, H, h, f) is a *classification choice.*

This construction is subject to the following uniqueness conditions:

(24) In the context of (23),
- The function f is uniquely determined by the choice of (H, h),
- Any two choices of (H_1, h_1) and (H_2, h_2) for a given pair b, x are $\mathbf{Fin}[S]$-isomorphic.

Every member of $\mathbf{Fin}[S]$ corresponds to a Galois morphism.

(25) For (H,h) $\in$**Fin**$[S]$, for b =Ind$[\Gamma(H,h),OP]$ and for x any member of $b^{-1}\{0\}$, there is a function f such that (b,x,H,h,f) is a classification choice.

Classification choices extend to a functorial identification of Galois group with **Fin**$[S]$ component.

(26) Let $\delta = (b,x,H,h,f)$ be a classification choice. Then there is a unique group isomorphism $\omega_\delta : H \longrightarrow$Gal$(b)$ with the property that

$$\forall z \in F,\ \forall d \in H,\ f((\omega_\delta(d))(z)) = d \cdot f(z)\ .$$

Finally, we note that the canonical morphisms between Galois groups are **Fin**$[S]$-morphisms.

(27) Suppose that

$$\delta_1 = (b_1,x_1,H_1,h_1,f_1) \text{ and } \delta_2 = (b_2,x_2,H_2,h_2,f_2)$$

are two classification choices. Let ω_1 and ω_2 be the respective group homomorphisms, as described in (26). Suppose that $g : b_1 \longrightarrow b_2$ be a **HT**$[\mathbb{Z}]/OP$-morphism such that $g(x_1) = g(x_2)$. Then the group homomorphism induced by g from Gal$(b_1) \longrightarrow$Gal(b_2) is $\omega_2 \circ \zeta \circ \omega_1^{-1}$, where ζ is a **Fin**$[S]$-morphism $(H_1,h_1) \longrightarrow (H_2,h_2)$. More precisely, ζ is the unique group homomorphism $H_1 \longrightarrow H_2$ such that $h_1 = h_2 \circ \zeta$.

In this manner, we identify the inversely directed system that defines $\pi_G(OP,0)$ with the system for $\widehat{Fr}[\alpha^*,\beta^*]$.

We present the identification in gory detail because we refer to it later. We need it immediately, in order to characterize $\pi_G(e)$.

Recall (from [2], for example) that every Galois extension of E is isomorphic to $\mathbb{Z}/(n) \longrightarrow E$ for some $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For $k \in \mathbb{Z}$, denote the equivalence class of k mod(n) by $\bar{k}$. The pullback of $\mathbb{Z}/(n)$ along the morphism e_{OP} is $OP\times(\mathbb{Z}/(n))$, where the $\times$ is taken with respect to the category **HT**$[\mathbb{Z}]$. Let $b : OP\times(\mathbb{Z}/(n)) \longrightarrow OP$ be the canonical projection. General theory tells us that b is a disjoint union of Galois morphisms.

By inspection, for $k \in \mathbb{Z}/(n)$, the sequence

$$\{(n!\cdot m)\star(0,\bar{k})\}_{m=1}^{\infty}$$

converges to $(\infty,\bar{k})$. In addition, the sequence

$$\{(-n!\cdot m)\star(0,\bar{k})\}_{m=1}^{\infty}$$

converges to $(\infty,\bar{k})$. The action obeys

$$\forall m \in \mathbb{Z},\ \forall y \in OP,\ \forall k \in \mathbb{Z},\ m\star(x,\bar{k}) = (m\star x, \overline{m+k})\ .$$

It follows that

(28.a) $OP \times (\mathbb{Z}/(n))$ is $\mathbb{Z}$-connected,

(28.b) b is Galois,

(28.c) both functions $\theta[b,0,1]$ and $\theta[b,0,-1]$ are given by

$$\forall k \in \mathbb{N},\ (0,\bar{k}) \mapsto (\infty,\bar{k})\ .$$

Expand $\Phi(b) = (F,\alpha,\beta)$. Then the permutations are

$$\forall k \in \mathbb{Z},\ \alpha(0,\bar{k}) = (0,\overline{k+1}) \text{ and } \beta(0,\bar{k}) = (0,\bar{k})\ .$$

It follows that there is a choice of classification data $(b, (0, \bar{0}), \mathbb{Z}/(n), h, f)$ in which

$$h(\alpha^*) = \bar{1} \text{ and } h(\beta^*) = \bar{0} .$$

Next, one must examine this classification as the parameter n varies over all natural numbers. Happily, our choices are preserved by all projection maps used in construction of $\pi_G(OP, 0)$ and $\pi_G(E)$. We conclude

(29) Under the choices made here, $\pi_G(e_O P)$ is the unique profinite group homomorphim $\widehat{Fr}[\alpha^*, \beta^*] \longrightarrow \widehat{\mathbb{Z}}$ which satisfies $\alpha^* \mapsto 1$ and $\beta^* \mapsto 0$.

5. Analysis of TP

Our classification of Galois extensions of TP strongly resembles the classification of OP. We begin with a functor Ψ :Sep$/TP \longrightarrow \mathcal{C}_2$.

(30.a) Let $b : B \longrightarrow TP$ belong to Sep$/TP$. The image is

$$\Phi(B, b) = (F, \alpha, \gamma),$$

constructed as follows. Let $F = b^{-1}\{0\}$ be the fiber over 0. In the notation of Proposition 2, put

$$\theta_+ = \theta[b, 0, 1] \text{ and } \theta_- = \theta[b, 0, -1] .$$

Define α by

$$\forall x \in F, \ \alpha(x) = \theta_+^{-1}(1 \star \theta_+(x)) .$$

Define γ by

$$\forall x \in F, \ \gamma(x) = \theta_-^{-1}(1 \star \theta_-(x)) .$$

(30.b) Let $b : B \longrightarrow TP$ and $c : C \longrightarrow TP$ be separable morphisms, and let $f : (B, b) \longrightarrow (C, c)$ be a $\mathbf{HT}[\mathbb{Z}]/TP$-morphism. Express

$$\Phi(B, b) = (F_b, \alpha_b, \gamma_b) \text{ and } \Phi(C, c) = (F_c, \alpha_c, \gamma_c) .$$

Since F_b is a subset of B, f restricts to it. Then $f(F_b) \subseteq f(F_c)$. It is a trivial consequence of definitions that, as a function $F_b \longrightarrow F_c$, f is $\mathcal{C}_2$-morphism. We define $\Psi(f)$ to be the restriction.

The following propositions classify all separable extensions of TP.

Lemma 6. *Let $(F, \alpha, \gamma) \in \mathcal{C}_2$. Then there is $b \in$Sep$/TP$ such that $\Psi(b)$ is isomorphic to (F, α, γ) if and only the subgroup of Σ_F generated by α and γ acts transitively on F.*

Lemma 7. *Let $b, c \in$Sep$/TP$. Then the function, induced by Ψ, from Sep$/TP$ morphisms $b \longrightarrow c$, to the set of $\mathcal{C}_2$-morphisms $\Psi(b) \longrightarrow \Psi(c)$, is a bijection.*

Proof. As in Section 3, the basis of the proof is an explicit construction of morphisms from $\mathcal{C}_2$-objects. We give this explicitly. The rest of the argument is an easy variation on the analysis of OP. That portion is left to the reader.

Let $\mu = (F, \alpha, \gamma) \in \mathcal{C}_2$. To begin, let Ind$[\mu, TP]$ be the set $F \times TP$. We denote the canonical projection $F \times TP \longrightarrow TP$ by $\pi[\mu, TP]$. Define a binary operation $\mathbb{Z} \times$Ind$[\mu, TP] \longrightarrow$Ind$[\mu, TP]$ by

$$\begin{aligned} \forall x \in F, \ \forall n, m \in \mathbb{Z}, \quad & n \star (x, m) &= (x, n+m) \\ \forall x \in F, \ \forall n \in \mathbb{Z}, \quad & n \star (x, \infty) &= (\alpha^n(x), \infty) \\ \forall x \in F, \ \forall n \in \mathbb{Z}, \quad & n \star (x, -\infty) &= (\gamma^n(x), -\infty) . \end{aligned}$$

We leave it to the reader to verify that $\star$ defines a left action by $\mathbb{Z}$ on $\mathrm{Ind}[\mu, TP]$, and that $\pi[\mu, TP]$ is equivariant with respect to this action.

For each $z \in F$, define two subset of $\mathrm{Ind}[\mu, TP]$ as follows:

$$\begin{aligned} W_z^+ &= \{\ (x,k)\ :\ x \in F,\ k \in \mathbb{N} \text{ and } \alpha^k(x) = z\} \cup \{(z,\infty)\}\ , \\ W_z^- &= \{\ (x,-k)\ : x \in F,\ k \in \mathbb{N} \text{ and } \gamma^{-k}(x) = z\} \cup \{(z,-\infty)\}\ . \end{aligned}$$

For each $k \in \mathbb{N}$, define two subsets of TP as follows

$$V_k^+ = \{\ m\ :\ m \in \mathbb{N} \text{ such that } m \geq k\} \cup \{\infty\}\ ,$$
$$V_k^- = \{\ -m\ :\ m \in \mathbb{N} \text{ such that } m \geq k\} \cup \{-\infty\}\ .$$

The family $\{V_k^+\}$ is a basis of open neighborhoods for ∞ in TP, and the family $\{V_k^-\}$ is a basis of open neighborhoods for $-\infty$ in TP. The set of subsets of $\mathrm{Ind}[\mu, TP]$

$$\begin{aligned} &\{\ \{(x,k)\}\ :\ x \in F,\ k \in \mathbb{Z}\} \\ \cup\ &\{\ \pi[\mu,TP]^{-1}V_k^+\ \cap W_x^+\ :\ x \in F,\ k \in \mathbb{N}\} \\ \cup\ &\{\ \pi[\mu,TP]^{-1}V_k^-\ \cap W_x^-\ :\ x \in F,\ k \in \mathbb{N}\} \end{aligned}$$

is a basis of a topology (of open sets) for $\mathrm{Ind}[\mu, TP]$. From here on, we assign this topology to $\mathrm{Ind}[\mu, TP]$.

One can modify the earlier argument to deduce:

(31.a) $\mathrm{Ind}[\mu, TP]$, with this action and this topology, is an $\mathbf{HT}[\mathbb{Z}]$-object.

(31.b) $\pi[\mu, TP]$ is an $\mathbf{HT}[\mathbb{Z}]$-morphism.

(31.c) $\mathrm{Ind}[\mu, TP]$ is $\mathbb{Z}$-connected if and only if there does not exists a non-empty subset $T \subset F$ such that

$$T \neq F,\ \alpha(T) \subseteq T \text{ and } \gamma(T) \subseteq T\ .$$

(31.d) If $\mathrm{Ind}[\mu, TP]$ is $\mathbb{Z}$-connected, then $\pi[\mu, TP]$ is separable.

(31.e) There is a functor $\mathcal{C}_2 \longrightarrow \mathbf{HT}[\mathbb{Z}]/TP$ which assigns an object μ to the morphism $\pi[\mu, TP]$ on $\mathrm{Ind}[\mu, TP]$, and, for $f : \mu_1 \longrightarrow \mu_2$ a $\mathcal{C}_2$. Hereafter, we refer to this functor as $\mathrm{Ind}[*, TP]$.

(31.f) For $b \in \mathrm{Sep}/TP$, there is an isomorphism from b to $\mathrm{Ind}[\Psi(b), TP]$. (In fact, there is a natural isomorphism between the two functors.)□

Let α^* and γ^* be formal symbols. We can replicate the steps of Section 4 to identify $\pi_G(TP, 0)$ with $\widehat{Fr}[\alpha^*, \gamma^*]$.

Our final topic is to characterize $\pi_G(\nu)$, where $\nu : TP \longrightarrow OP$ is the morphism of (1). Let $b : B \longrightarrow OP$ be a Galois morphism, and expand $\Phi(b)$ as (F, α, β). The key step is to characterize the Galois pullback of b along ν.

Put $\theta_+ = \theta[b, 0, 1]$ and $\theta_- = \theta[b, 0, -1]$. Let C be the standard choice of $\nu \times_{OP} b$, taken in the category $\mathbf{HT}[\mathbb{Z}]$. That is, the underlying set of C is a subset of $TP \times B$ with the subset topology of that product. Let $c : C \longrightarrow TP$ be the canonical projection. Note that

$$c^{-1}\{0\} = \{\ (0,x)\ :\ x \in F\}\ .$$

Let N be the size of F. Define two functions $F \longrightarrow C$ by

$$\forall x \in F,\ \theta_1(x) = (\infty, \theta_+(x)) \text{ and } \theta_2(x) = (-\infty, \theta_-(x))\ .$$

By inspection, for $x \in F$, and in the topology of C,

(32.a) the sequence $\{(N! \cdot k) \star (0,x)\}_{m=1}^{\infty}$ converges to $\theta_1(x)$, and

(32.b) the sequence $\{(-N! \cdot k) \star (0,x)\}_{m=1}^{\infty}$ converges to $\theta_2(x)$.

By choice of α,

$$\forall x \in F,\ \theta_+(\alpha(x)) = 1 \star \theta_+(x)\ .$$

By definition of β, $\theta_+ = \theta_- \circ \beta$. Since β is a permutation, we deduce that

$$\forall x \in F,\ \theta_-((\beta \cdot \alpha \cdot \beta^{-1})(x) = 1 \star \theta_-(x)\ .$$

We can paraphrase the last two identities by

$$\forall x \in F,\ \theta_1(\alpha(x)) = 1 \star \theta_1(x) \text{ and } \theta_2((\beta \cdot \alpha \cdot \beta^{-1})(x)) = 1 \star \theta_2(x)\ .$$

General nonsense (see [1]) tells us that C decomposes into $\mathbb{Z}$-components, and that restriction of c to any component is Galois.

For each $J \subseteq F$, define subsets J_1 and $C[J]$ of C by

$$\begin{aligned} J_1 &= \{\ (0,x)\ :\ x \in J\}\ , \\ C[J] &= \{\ (n, n \star x)\ :\ x \in J,\ n \in \mathbb{Z}\} \\ &\quad \cup \{\theta_1(x)\ :\ x \in J\} \cup \{\theta_2(x)\ :\ x \in J\}\ . \end{aligned}$$

An easy consequence of (32) is that if $Y \subseteq C$ is a (topologically) closed subset which is closed under the action by $\mathbb{Z}$, then

$$\forall J \subseteq F,\ J_1 \subseteq Y\ \Rightarrow\ C[J] \subseteq Y\ .$$

Let H be the subset of Σ_F generated by the permutations $\alpha, (\beta \cdot \alpha \cdot \beta^{-1})$. The following assertions are left to the reader.

(33.a) If $Y \subseteq C$ is (topologically) open and closed in C, and is closed under the action by $\mathbb{Z}$, then there is $J \subseteq F$ such that $Y = C[J]$.

(33.b) Suppose that $J \subset F$. Then $C[J]$ is (topologically) open and closed in C, and is closed under the action by $\mathbb{Z}$, if and only if J is closed under the action by H.

Let O be the H-orbit of any member of F. Restriction of c to $C[O]$ must be a Galois pullback of b.

Now, we make choices based on the classification theory. Let $G = \mathrm{Gal}(b)$. Earlier propositions provide a canonical identification of (F, α, β) with the the set G and right multiplication by canonical images of α^* and β^*, respectively, in G. Fix such an identification. Let e_F be the member of F which corresponds to the identity of the group G. Let O be the H-orbit of e_F. Denote restriction of c to $C[O]$ by c_1.

The Galois morphism c_1 is isomorphic to $\mathrm{Ind}[\mu, TP]$ where

$$\mu = (O, \alpha_{|O}, (\beta \cdot \alpha \cdot \beta^{-1})_{|O}),$$

in which the subscript "$|O$" indicates restriction to O. Navigating our numerous identifications, it follows that the group homorphism $\mathrm{Gal}(c_1) \longrightarrow G$, induced by projection $C[O] \longrightarrow B$, sends the image of α^* and of γ^*, in $\mathrm{Gal}(c_1)$, to the image of α^* and of $\beta^* \cdot \alpha^* \cdot (\beta^*)^{-1}$, in G, respectively.

This proves our assertion on $\pi_G(\nu)$.

References

[1] Feit P., A Fundamental Group for Dynamical Systems I: Definitions, *Proc. Sym. Pure Math.*, Vol. 66, Part I (1999) 51–86.

[2] Feit P., A Fundamental Group for Dynamical Systems II: $\mathbf{Z}_p$, *Journal of Dynamical Systems and Geometric Theories*, Vol. 2 (2004) 81-90.

[3] Lubkin S., Theory of Covering Spaces, *Trans. Amer. Math. Soc.*, **104** (1962) 205–238.