A FUNDAMENTAL GROUP FOR DYNAMICAL SYSTEMS II: Z_p

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ABSTRACT. This paper continues exploration of a recent homotopy construction for group-sets. It proves that, for p prime, and with respect to $\mathbb{Z}_p$ regarded as a $\mathbb{Z}$-set, the homotopy group has a canonical identification with the kernel of the canonical projection from the profinite completion of $\mathbb{Z}$ to $\mathbb{Z}_p$.

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1. Introduction

This series of papers calculates examples of an invariant for dynamical systems. It is well-known (for example, see [2]) that many categories admit a construction similar to the fundamental group of algebraic geometry. In [1], the present author applied this universal method to symbolic dynamics. That paper established that a "fundamental group" exists. The current series computes actual examples.

In this paper, we prove the following:

Theorem A. *Let* $\mathbf{HT}[\mathbb{Z}]$ *denote the category of Hausdorff topological* $\mathbb{Z}$*-sets. Let* π_G *be the "fundamental group functor" for this category, as described in* [1].

Let p *be a prime number. Let* $\mathbb{Z}_p$ *denote the topological space of* p*-adic integers. Regard* $\mathbb{Z}_p$ *as a* $\mathbb{Z}$*-set by letting* $\mathbb{Z}$ *act by addition. Let* $\widehat{\mathbb{Z}}$ *be the profinite completion of* $\mathbb{Z}$*, and let*

$$\epsilon_p : \widehat{\mathbb{Z}} \longrightarrow \mathbb{Z}_p$$

be the canonical projection. Let E *denote a topological* $\mathbb{Z}$*-set which consists of a single point. Let* $e_p : \mathbb{Z}_p \longrightarrow E$ *be the unique function between these sets, regarded as a* $\mathbb{Z}$*-equivariant map. Then there is a topological group isomorphism* $\pi_G(E) \cong \widehat{\mathbb{Z}}$ *with respect to which* $\pi_G(e_p)$ *identifies with a closed imbedding of the kernel of* ϵ_p *into* $\widehat{\mathbb{Z}}$.

The perspective in this series changes from the perspective of its predecessors. The earlier works interpreted a categorical formulation in a specific category; the discussion relied on diagrams of objects and morphisms. Now, we study particular objects.

In any category, the specific value of an invariant for an object X depends on the details of what X is. Instead of universal theory, our calculations hinge on specific features. Analysis of each object reflects the quirks of that object. On the technical level, this paper and its sequels consist of disconnected observations. Behind the individual gimmicks is a universal topic: *what does this "fundamental group" measure?*

We begin with a review of the construction of a fundamental group in the category of Hausdorff topological $\mathbb{Z}$-sets. This portion of the paper is longer than the new analysis. However, this section lays out the language used in the subsequent papers.

The rest of the paper computes the fundamental group, which we denote by π_G, for a specific family of $\mathbb{Z}$-sets. For p a prime number, let $\mathbb{Z}_p$ be set of p-adic integers. Let $\mathbb{Z}$ act on $\mathbb{Z}_p$ by addition. In the standard topology, $\mathbb{Z}_p$ is totally disconnected in the sense that the only subsets of it which are connected consist of single points. Consequently, it lacks a fundamental group in the sense of point-set topology. However, the universal formulation of fundamental group applies to $\mathbb{Z}_p$.

2. Notation

Let $\mathbb{N}$ denote the natural numbers. For $n \in \mathbb{N}$, let $\mathbb{N}(n)$ be the set of natural numbers from 1 to n, inclusive.

Let $\mathbb{Z}$ denote the integers. For p a prime natural number, let $\mathbb{Z}_p$ be the topological space of p-adic integers. For $n \in \mathbb{Z}$, let $n \cdot \mathbb{Z}$ or (n) stand for the principal ideal generated by n. We represent the quotient for $\mathbb{Z}$ with respect to an ideal (n) by $\mathbb{Z}/(n)$. Throughout this paper, we interpret $\mathbb{Z}/(n)$ as an additive topological group by assigning it the discrete topology.

Let $n, m \in \mathbb{Z}$. We write "$n \mid m$" if n divides m in $\mathbb{Z}$.

Let $b : B \longrightarrow A$ be a group homorphism. Let $\mathrm{Ker}(b)$ denote the kernel of b. If b is a morphism of topological groups, then $\mathrm{Ker}(b)$ is a topological group.

We recall some generalities on profinite groups. Let G be a group, and let $\mathcal{N}$ be the set of normal subgroups of G of finite index. Order $\mathcal{N}$ by reverse containment. That is, for $L, M \in \mathcal{N}$, $L \geq M$ if and only if $L \subseteq M$. To each $L \in \mathcal{N}$, assign the topological group which is G/L in the discrete topology. To each pair $(L, M) \in \mathcal{N}^2$ such that $L \geq M$, assign the unique $\rho_{L,M}$ which factors the projection $G \longrightarrow G/M$ through $G \longrightarrow G/L$. Suppose that $\mathcal{L}$ is a subset of $\mathcal{N}$ such that

$$\forall P, Q \in \mathcal{L},\ \exists R \in \mathcal{L},\ R \geq P \text{ and } R \geq Q\ .$$

Then the diagram consisting of all G/L for $L \in \mathcal{L}$, along with all $\rho_{*,*}$ between members of $\mathcal{L}$, form an inversely directed system of compact Hausdorff groups.

We base several definitions on this construction.

(1.a) Let G be any group, and let $\mathcal{L}$ be the set of all normal subgroups of finite index. The inverse limit of the corresponding system is called the *profinite completion of G.*

(1.b) The profinite completion of $\mathbb{Z}$ is denoted $\widehat{\mathbb{Z}}$. For each $n \in \mathbb{N}$, let $\pi_n^{\mathbb{Z}}$ denote the canonical projection $\widehat{\mathbb{Z}} \longrightarrow \mathbb{Z}/(n)$. When discussing the inverse limit that defines $\widehat{\mathbb{Z}}$, we write $\rho_{n,m}$ or $\rho_{n,m}^{\mathbb{Z}}$ for $\rho_{(n),(m)}$ when $m \mid n$.

(1.c) Let p be a prime number. Consider $G = \mathbb{Z}$, and put

$$\mathcal{L} = \{ \, p^r \cdot \mathbb{Z} \; : \; r \in \mathbb{N} \} \; .$$

Then $\mathbb{Z}_p$ is inverse limit of the associated inversely directed system. For $r \in \mathbb{N}$, let π_r^p denote the canonical morphism $\mathbb{Z}_p \longrightarrow \mathbb{Z}/(p^r)$.

3. Established Features of π_G

This section reviews technical details for the fundamental group functor π_G.

Let $\mathbf{HT}[\mathbb{Z}]$ denote the catgeory of topological $\mathbb{Z}$-sets whose underlying topology is Hausdorff. (Morphisms are continuous, $\mathbb{Z}$-equivariant, functions.) All actions are assumed to be on the left.

We reserve "$\cdot$" for multiplication in $\mathbb{Z}$, in $\mathbb{Z}_p$, or in groups which arise. We use "$\star$" to represent the action by $\mathbb{Z}$ on an $\mathbf{HT}[\mathbb{Z}]$-object.

For $A \in \mathbf{HT}[\mathbb{Z}]$, define the *slice category* $\mathbf{HT}[\mathbb{Z}]/A$ as follows:

(2.a) An object is an $\mathbf{HT}[\mathbb{Z}]$-morphism whose codomain is A. We denote a member using either morphism notation, such as $b : B \longrightarrow A$, or as the morphism name, b, or a pair of the morphism and its domain, (B, b).

(2.b) A morphism $(B, b) \longrightarrow (C, c)$ is an $\mathbf{HT}[\mathbb{Z}]$-morphism $f : B \longrightarrow C$ such that $c \circ f = b$.

(2.c) Composition and identity morphism are induced from $\mathbf{HT}[\mathbb{Z}]$.

If $b : B \longrightarrow A$ is an $\mathbf{HT}[\mathbb{Z}]$-morphism, then let $\mathrm{Aut}_A(B, b)$, or $\mathrm{Aut}_A(b)$, denote the group of automorphisms of b with respect to $\mathbf{HT}[\mathbb{Z}]/A$.

Fibered products play a major role in the universal theory. Let $A, B, C \in \mathbf{HT}[\mathbb{Z}]$ and let $b : B \longrightarrow A$ and $c : C \longrightarrow A$ be $\mathbf{HT}[\mathbb{Z}]$-morphisms. Consider the subset of $B \times C$ given by

$$X = \{ \, (x, y) \in B \times C \; : \; b(x) = c(y) \} \; .$$

Assign to $B \times C$ the product topology and an action

$$\forall n \in \mathbb{Z}, \; \forall x \in B, \; \forall y \in C, \; n \star (x, y) = (n \star x, n \star y) \; .$$

This actions preserves the subset X. We let $B \times_A C$ refer to X with the subset topology and the subset action. Paired with the restrictions of the canonical projections from $B \times C$ to B and to C, this object has the mapping properties of a fibered product $b \times_A c$ in the categorical language.

In the next list, we present some definitions based on common topological conditions. The formulations have been modified to apply to the category of topological group sets.

(3.a) An object in $\mathbf{HT}[\mathbb{Z}]$ is *empty,* in the universal sense, if its underlying set is empty.

(3.b) Suppose that $A, B \in \mathbf{HT}[\mathbb{Z}]$. In this context, define a *disjoint union* of A with B as a disjoint union of the underlying sets, assigned the disjoint union topology, and assigned the usual extension of the $\mathbb{Z}$-action of the actions on A and B. Such a construction belongs to $\mathbf{HT}[\mathbb{Z}]$.

In an obvious manner, this binary version of disjoint union extends to a definition of disjoint union for any *finite* family of $\mathbf{HT}[\mathbb{Z}]$-objects.

(3.c) An object $X \in \mathbf{HT}[\mathbb{Z}]$ is said to be *$\mathbb{Z}$-disconnected* if it is isomorphic to a disjoint union of two non-empty objects. An object $Y \in \mathbf{HT}[\mathbb{Z}]$ is said to be *$\mathbb{Z}$-connected* if it is non-empty and it is not $\mathbb{Z}$-disconnected.

(3.d) Let $\{U_i\}_{i=1}^n$ be a finite list of non-empty $\mathbb{Z}$-connected $\mathbf{HT}[\mathbb{Z}]$-objects. Let X be a disjoint union. Then any other realization of X as a disjoint union of $\mathbb{Z}$-connected objects is isomorphic to some reordering of $\{U_i\}_{i=1}^n$. We say that X *decomposes into $\mathbb{Z}$-components.* In addition, we refer to the subsets of X which correspond to the various U_i's as the *$\mathbb{Z}$-components of X*. (Note: There are $\mathbf{HT}[\mathbb{Z}]$-objects which do not decompose into $\mathbb{Z}$-components.)

Suppose that $b : B \longrightarrow A$ is an $\mathbf{HT}[\mathbb{Z}]$-morphism. We extend some of the language to b.

(4.a) We say b is *non-empty* if B is non-empty.

(4.b) We say that b is *$\mathbb{Z}$-connected* if B is $\mathbb{Z}$-connected.

(4.c) Suppose that B decomposes into $\mathbb{Z}$-components. Each restriction of b to a $\mathbb{Z}$-component is referred to as a *component of b*.

We proceed to more convoluted properties.

Let $b : B \longrightarrow A$ be an $\mathbf{HT}[\mathbb{Z}]$-morphism, and let $y \in A$. Let $F = b^{-1}\{y\}$ be the fiber over y. By *leveling data for b at y*, we mean $(V, \{U_x\}_{x \in F})$ in which

(5.a) V is an open neighborhood of y in A,

(5.b) For each $x \in F$, U_x is an open neighborhood of x in B,

(5.c) $\cup_{x \in F} U_x = b^{-1}V$,

(5.d) $\forall x, z \in F,\ (\ U_x \cap U_z \neq \emptyset \ \Rightarrow\ x = z\ )$,

(5.e) for each $x \in F$, b restricts to a homeomorphism $U_x \longrightarrow V$.

For $N \in \mathbb{N}$, we say that b *is separable of index N* if and only if

(6.a) both A and B are $\mathbb{Z}$-connected objects,

(6.b) every fiber over a point $y \in A$ has N members, and

(6.c) there exists leveling data for each $y \in A$.

In this case, we refer to N as the *index* of b.

Suppose that $b : B \longrightarrow A$ is separable. Let N be the index of b. If $\mathrm{Aut}_A(b)$ has N members,

(7.a) we say that b is *Galois,*

(7.b) we refer to $\mathrm{Aut}_A(b)$ as the *Galois group of b,* and denote it by *Gal*(b).

The following assertions are established in [1].

Lemma 1. *Let $b : B \longrightarrow A$ be a Galois* $\mathbf{HT}[\mathbb{Z}]$*-morphism. Let $y \in A$. Then Gal(b) acts regularly on $b^{-1}\{y\}$. (That is, the action is transitive and the stabilizer of each member is trivial.)*

Lemma 2. *Let $A, B, C, D \in$* $\mathbf{HT}[\mathbb{Z}]$. *Let $b : B \longrightarrow A$ and $d : D \longrightarrow C$ be two Galois* $\mathbf{HT}[\mathbb{Z}]$*-morphisms, and let $f : C \longrightarrow A$ be any* $\mathbf{HT}[\mathbb{Z}]$*-morphism. Let S be the set of* $\mathbf{HT}[\mathbb{Z}]$*-morphisms $g : D \longrightarrow B$ such that $b \circ g = f \circ d$. Assume that S is a non-empty set.*

(A) *Suppose that $x \in D$ and $z \in B$ such that $(f \circ d)(x) = b(z)$. Then there is a unique $g \in S$ such that $g(x) = z$.*

(B) *Suppose $g \in S$. Then there is a unique group homomorphism $g_* : Gal(d) \longrightarrow Gal(b)$ with the property that*

$$\forall \delta \in Gal(d),\ [g_*(\delta)] \circ g = g \circ \delta\ .$$

We can now sketch the fundamental group construction. The easiest version is to introduce the notion of "base point".

Let $\mathcal{P}$ be the category determined as follows:

(8.a) An object is a pair (A, y) in which A is a non-empty $\mathbb{Z}$-connected $\mathbf{HT}[\mathbb{Z}]$-object and $y \in A$.

(8.b) A $\mathcal{P}$-morphism $(B, x) \longrightarrow (A, y)$ is any $\mathbf{HT}[\mathbb{Z}]$-morphism $b : B \longrightarrow A$ such that $y = b(x)$.

(8.c) Composition and identity are induced from $\mathbf{HT}[\mathbb{Z}]$.

Let $(A, y) \in \mathcal{P}$. Recall the slice category construction for $\mathcal{P}/(A, y)$: objects are $\mathcal{P}$-morphisms into (A, y), and a morphism from object $b : (B, x) \longrightarrow (A, y)$ to $c : (C, z) \longrightarrow (A, y)$ is a $\mathcal{P}$-morphism $f : (B, x) \longrightarrow (C, z)$ such that $c = b \circ f$.

For each $(A, y) \in \mathcal{P}$, choose a set of $\mathcal{P}/(A, y)$ objects $S[A, y]$ such that

(9.a) each $b \in S[A, y]$ is, as an $\mathbf{HT}[\mathbb{Z}]$-morphism, Galois, and

(9.b) if $c \in \mathcal{P}/(A, y)$ is, as an $\mathbf{HT}[\mathbb{Z}]$-morphism, Galois, then there is a unique $b \in S[A, y]$ which is $\mathcal{P}/(A, y)$-isomorphic to c.

Define an inversely directed system of finite groups, denoted $\Pi[A, y]$, indexed by $S[A, y]$, as follows.

(10.a) Assign to each $b \in S[A, y]$ the group $\mathrm{Gal}(b)$, and

(10.b) to each $\mathcal{P}/(A, y)$-morphism g between two members of $S[A, y]$, assign g_* as defined in Lemma 2 (with respect to the identity map on A for f).

By definition, $\pi_G(A, y)$ is the profinite limit of $\Pi[A, y]$.

The functorial extension to morphisms is also based on Lemma 2. Suppose that $f : (C, w) \longrightarrow (A, y)$ is a $\mathcal{P}$-morphism. Then $\pi_G(f)$ is the inverse limit of all g_*, as described in Lemma 2, where g is a $\mathcal{P}$-morphism from a member of $S[C, w]$ to a member of $S[A, y]$.

Let f be as in the previous paragraph. One details in this theory is that for each $b \in S[A, y]$, there must some $d \in S[C, w]$ for which there is a morphism g such that $b \circ g = f \circ d$. This assertion is a consequence of an established proposition which we need in this paper.

Proposition 3. *Let $A, B, C \in \mathbf{HT}[\mathbb{Z}]$, $b : B \longrightarrow A$ be a Galois morphism and let $c : C \longrightarrow A$ be a $\mathbb{Z}$-connected morphism. Let the triple D, $d_1 : D \longrightarrow C$ and $g_1 : D \longrightarrow A$ be a choice of fibered product $c \times_A b$. Then D decomposes into $\mathbb{Z}$-components. Moreover,*

(11.a) *restriction of d_1 to any component is a Galois morphism into C,*

(11.b) *restrictions of d_1 to any two components are $\mathbf{HT}[\mathbb{Z}]/C$-isomorphic, and*

(11.c) *if d is restriction of d_1 to a $\mathbb{Z}$-component, and if g is the restriction of g_1 to that same component, then $b \circ g = f \circ d$.*

Any restriction of d_1 to a $\mathbb{Z}$-component of D is called a Galois pullback*of c. A consequence of this theorem is that Galois pullbacks are unique up to $\mathbf{HT}[\mathbb{Z}]/C$-isomorphism.*

4. Paraphrase of the Main Theorem

For the remainder of this paper, fix p a prime number.

Every $\pi_G(*)$ is characterized as an inverse limit. To establish our theorem, we must describe the $\mathrm{Ker}(\epsilon_p)$ as an inverse limit.

For this discussion, let Q be the set of pairs (n,r) such that

$$n, r \in I\!N,\ p^r \mid n\ .$$

Order Q by

$$(m,s) \geq (n,r)\ \Leftrightarrow\ (\ n \mid m \text{ and } s \geq r\ )\ .$$

Also, for $(n,r) \in Q$, let

$$\begin{aligned} Q[n,r] &= \{\ (m,s) \in Q\ :\ (m,s) \geq (n,r)\}\ , \\ R &= \{\ (n,r,m,s)\ :\ (n,r) \in Q,\ (m,s) \geq (n,r)\}\ . \end{aligned}$$

Let $(n,r,m,s) \in R$. In a standard manner, $\rho_{m,n}$ and ρ_{p^s,p^r} induce a group homorphism

$$\kappa[n,r,m,s] : \mathrm{Ker}(\rho_{m,p^s}) \longrightarrow \mathrm{Ker}(\rho_{n,p^r})\ .$$

$$\begin{tikzcd} \mathrm{Ker}(\rho_{m,p^s}) \arrow[r] \arrow[d, "{\kappa[n,r,m,s]}"] & \mathbb{Z}/(m) \arrow[r] \arrow[d] & \mathbb{Z}/(p^s) \arrow[d] \\ \mathrm{Ker}(\rho_{n,p^r}) \arrow[r] & \mathbb{Z}/(n) \arrow[r] & \mathbb{Z}/(p^r) \end{tikzcd}$$

It is tempting to use the spaces $\mathrm{Ker}(\rho_{n,p^r})$ to build an inversely directed system whose inverse limit is $\mathrm{Ker}(\epsilon_p)$. However, there are $\alpha = (n,r,m,s) \in R$ for which $\kappa[\alpha]$ is not surjective. Much of the structure theory of profinite groups hinges on surjectivity of every component morphism.

Let $(n,r) \in Q$. In terms of $\widehat{\mathbb{Z}}$, the obstruction is this: If $p^{r+1} \mid n$, then there exists $x \in\mathrm{Ker}(\rho_{n,p^r})$ such that no member of $(\pi_n^Z)^{-1}\{x\}$ lies in the kernel of ϵ_p. In terms of the finite groups that define $\widehat{\mathbb{Z}}$ and $\mathbb{Z}_p$, the failure translates as existence of a companion $(m,s) \in Q[n,r]$ such that x is not in the image of $\kappa[n,r,m,s]$. Conversely, and just as important, if r is the greatest power of p that divides n, then $\kappa[n,r,m,s]$ is surjective for every $(m,s) \in Q[n,r]$.

Now let Q_1 be the subset of $(n,r) \in Q$ such that r is the largest number such that $p^r \mid n$. Let $\mathcal{K}$ be the inversely directed system indexed by Q_1, under the assignments

- for objects, $(n,r) \mapsto \mathrm{Ker}(\rho_{n,p^r})$,
- for morphisms, each pair $(n,r),(m,s) \in Q_1$ for which $(m,s) \geq (n,r)$ is assigned $\kappa[n,r,m,s]$

It requires an easy diagram chase to verify that an inverse limit of this system, paired with the canonical morphism from it to $\widehat{\mathbb{Z}}$, has the mapping property of a kernel. Once that is accepted, it is easy to prove that the morphism is a closed embedding.

Let E be any $\mathbf{HT}[\mathbb{Z}]$-object whose underlying set consists of a single point, e_0. For each $N \in I\!N$, let $C_N = \mathbb{Z}/(N)$. Inside C_N, for $m \in \mathbb{Z}$ we write $\bar{m}$ to denote the equivalence class of m. We interpet C_N in several ways. First, it is a cyclic group. Second, we treat it as an $\mathbf{HT}[\mathbb{Z}]$-object by assigning it the discrete topology and the canonical $\mathbb{Z}$-module structure. With this convention, let c_N denote the unique morphism $C_N \longrightarrow E$. Also, we freely identify C_N with $\mathcal{P}$-object $(C_N, \bar{0})$.

Clearly, c_N is Galois and its Galois group can be identified with C_N, the group, acting by C_N's addition. Moreover, every Galois morphism into E is $\mathbf{HT}[\mathbb{Z}]/E$-isomorphic to c_N for some N. Choose $S[E, e_0] = \{c_N \ : \ N \in \mathbb{N}\}$. In this way, $\pi_G(E, e_0)$ becomes identified with $\widehat{\mathbb{Z}}$.

The pullback of c_N along $e : \mathbb{Z}_p \longrightarrow E$ is $\mathbb{Z}_p \times C_N$. This leads to an elementary

Lemma 4. *Let $N \in \mathbb{N}$, and let π be the canonical projection $\mathbb{Z}_p \times C_N \longrightarrow \mathbb{Z}_p$. If N is relatively prime to p, then π is Galois. If $N = p^r$ for some $r \in \mathbb{N}$, then*

(12.a) *$\mathbb{Z}_p \times C_N$ decomposes into $\mathbb{Z}$-components, and*

(12.b) *each restriction of π to a $\mathbb{Z}$-component is an isomorphism.*

Proof. The general theory tells us that $\mathbb{Z}_p \times C_N$ decomposes into $\mathbb{Z}$-components, and that restriction of π to each is Galois. It remains to classify components.

Suppose that N is relatively prime to p. The Chinese Remainder Theorem easily implies that the $\mathbb{Z}$-orbit of $(0, \bar{0})$ is dense in $\mathbb{Z}_p \times C_N$. It follows that the entire space is the only $\mathbb{Z}$-component.

Suppose that $r \in \mathbb{N}$ and $N = p^r$. Let μ be the canonical projection

$$\mathbb{Z}_p \longrightarrow C_N = \mathbb{Z}/(p^r) .$$

Consider

$$X = \{(x, \alpha) \in \mathbb{Z}_p \times C_N \ : \ \mu(x) = \alpha\} .$$

Obviously, X is a closed subset. We leave it to the reader to check that this set is also open and that it is closed under the action by $\mathbb{Z}$. Also, restriction of π to X is a homeomorphism. □

Let $M \in \mathbb{N}$. Then C_M is isomorphic to a product $C_N \times C_{p^r}$ in which N is relatively prime to p and $r \in \mathbb{N} \cup \{0\}$. Consequently, Lemma 4 characterizes every pullback of c_M, for any $M \in \mathbb{N}$, along e.

Consider our characterization of $\mathrm{Ker}(\epsilon_p)$ at the start of this section. The lemma, and an elementary diagram chase, leads to the conclusion that the image of $\pi_G(e)$ is exactly the kernel of ϵ_p.

The rest of the assertions of Theorem A reduce to

Proposition 5. *Let $b : B \longrightarrow \mathbb{Z}_p$ be a Galois $\mathbf{HT}[\mathbb{Z}]$-morphism. Then there is an $N \in \mathbb{N}$, which is relatively prime to p, such that b is $\mathbf{HT}[\mathbb{Z}]/\mathbb{Z}_p$-isomorphic to the canonical projection $\mathbb{Z}_p \times C_N \longrightarrow \mathbb{Z}_p$.*

The remainder of this paper consists of the proof of Proposition 5.

5. The Calculation

Existence of leveling data easily implies that

Lemma 6. *Let $b : B \longrightarrow A$ be a separable $\mathbf{HT}[\mathbb{Z}]$-morphism. Suppose that the underlying topological space of A is second Baire category. If $T \subseteq B$ is a subset such that $\overline{b(T)}$, the closure of $b(T)$, has non-empty interior, then $\bar{T}$, the closure of T, has non-empty interior.*

Proof. Since b is Galois, leveling data exists for each $y \in A$. This effectively reduces the implication to the analogous statement for projection $A \times F \longrightarrow A$. The argument is elementary. □

Corollary 7. *Let $b : B \longrightarrow \mathbb{Z}_p$ be a Galois* **HT**$[\mathbb{Z}]$*-morphism. For each $x \in B$, the $\mathbb{Z}$-orbit of x is dense in B.*

Proof. For $x \in B$, let $T(x)$ be the closure of the $\mathbb{Z}$-orbit of x in B. Then $b(T(x))$ is the closure of the orbit of $b(x)$. Regardless of x, the latter set is all of $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is a complete metric space, Lemma 6 applies. Hence, for all $x \in B$, $T(x)$ has non-empty interior.

Take an arbitrary $x \in B$. Let U be the interior of $T(x)$. The subset $T(x) \setminus U$

(13.a) is closed in the topology,
(13.b) has no interior points, and
(13.c) is closed under the action by $\mathbb{Z}$.

Suppose $z \in T(x) \setminus U$. Clearly, $T(z) \subseteq T(x) \setminus U$. The last paragraph states that $T(z)$ has non-empty interior. But this would contradict (13.b). Therefore, $U = T(x)$. Consequently, $T(x)$ is both open and closed. Since B is **HT**$[\mathbb{Z}]$-connected, $U = T(x) = B$. $\square$

We are ready to prove Proposition 5. Let $b : B \longrightarrow \mathbb{Z}_p$ be a Galois **HT**$[\mathbb{Z}]$-morphism.

Proof. Let $F = b^{-1}\{0\}$, and let $(V, \{U_x\})$ be leveling data at 0. We are free to pass to an open subneighborhood of V. Hence, we may assume $V = p^k \cdot \mathbb{Z}_p$ for some $k \in \mathbb{N}$. Fix a complete and irredundant list of representatives $0 = a_1, \ldots, a_{p^k}$ for $\mathbb{Z}/p^k\mathbb{Z}$, which begins with 0. For each $x \in F$, let

$$Z_x = \cup_{i=1}^{p^k} (a_i + U_x) .$$

The following consequences are elementary:

(14.a) $\forall x, z \in F,\ (\ Z_x \cap Z_z \neq \emptyset\ \Rightarrow\ x = z\ )$
(14.b) $B = \cup_{x \in F} Z_x$.
(14.c) For each $x \in F$, Z_x is an open and closed subset of B.
(14.d) For each $x \in F$, the restriction of b to $Z_x \longrightarrow \mathbb{Z}_p$ is a homeomorphism.

Without loss of generality, we assume

(15) There is a finite set F such that the underlying topological space of B is $\mathbb{Z}_p \times F$ and such that $b(x, s) = x$ for all $x \in \mathbb{Z}_p$ and $s \in F$.

Note that the $\mathbb{Z}$-action on B remains unspecified.

Let Σ_F be the permutation group on F. Define $\sigma : \mathbb{Z}_p \longrightarrow \Sigma_F$ by

$$\forall x \in \mathbb{Z}_p,\ \forall s \in S,\ 1 \star (x, s) = (1 + x, \sigma[x](s)).$$

It is an exercise in point set topology that continuity of the $\mathbb{Z}$-action forces σ to be locally constant.

The function $\epsilon \mapsto \sigma^{-1}\{\epsilon\}$ is a partition of $\mathbb{Z}_p$, indexed by by Σ_F, into open subsets. Consequently, each member of the partition is closed, and compact. Fix a p-adic metric for $\mathbb{Z}_p$. Let r be the smallest of distances between any two non-empty members of this partition. The fact that $r > 0$ implies existence of $n \in \mathbb{N}$ with the property that

$$\forall x, d \in \mathbb{Z}_p,\ \sigma[x + p^n d] = \sigma[x] . \tag{16}$$

For $k \in \mathbb{N}(p^n)$, let

$$\tau_k = \sigma[k-1] \circ \sigma[k-2] \circ \cdots \circ \sigma[0] .$$

We adopt the convention τ_0 is the identity. It follows that

(17) $$\forall x \in p^n \cdot \mathbb{Z}_p,\ \forall s \in F,\ \forall k \in \mathbb{N}(p^n),\ k \star (x,s) = (k + x, \tau_k(s)) \ .$$

We analyze $\tau = \tau_{p^n}$ next.

First, for $x \in p^n \cdot \mathbb{Z}_p$, we may apply (17) with $x - p^n$ in place of x and $\tau^{-1}(s)$ in place of s. It follows that

(18) $$\forall x \in p^n \cdot \mathbb{Z}_p,\ \forall s \in F,\ (-p^n) \star (x,s) = (-p^n + x, \tau^{-1}(s)) \ .$$

For the rest of our discussion, fix $s_0 \in F$, and put $z_0 = (0, s_0) \in B$.

Let $m \in \mathbb{N} \cup \{0\}$. By Corollary 7, the orbit $\mathbb{Z} \star z_0$ is dense in B. The only members of the orbit in the subset $U = b^{-1}(p^{n+m} \cdot \mathbb{Z}_p)$ are those of the form $(kp^{n+m}) \star z_0$, where $k \in \mathbb{Z}$. Since U is both open and closed, this subset of the orbit must be dense within U. By (17) and (18), for $k \in \mathbb{Z}$, $(kp^{n+m}) \star z_0$ belongs to $\mathbb{Z}_p \times \{(\tau^{p^m})^k(s_0)\}$. Density requires that τ^{p^m} acts transitively on F.

We conclude that, for all $m \in \mathbb{N}$, τ^{p^m} act transitively on F. Consequently,

(19) The order of F is relatively prime to p, and τ is a cyclic permutation of F.

We refine our model for B. Let N be the size of F. We may identify F with $\mathbb{Z}/(N)$ in such a manner that

(20.a) s_0 identifes with $\bar{0}$, and

(20.b) the permutation τ identifies with $\bar{r} \mapsto \overline{r + p^n}$.

Here, and hereafter, we use overbars to indicate equivalence class mod(N).

Define $\theta : B \longrightarrow \mathbb{Z}_p \times F$ as follows. Suppose $(x,s) \in B$. There is a unique $r \in \mathbb{N}(p^n - 1) \cup \{0\}$ such that $x - r \in p^n \cdot \mathbb{Z}_p$. Let

$$\theta(x,s) = (x, \tau_r^{-1}(s) + \bar{r}) \ .$$

It is a trivial exercise that θ is a homeomorphism.

Suppose $r \in \mathbb{N}(p^n - 1) \cup \{0\}$ and $x \in r + (p^n \cdot \mathbb{Z}_p)$. Let $s \in F$, and let $t \in F$ such that $1 \star (x,s) = (x+1, t)$, The following three relations follow from established structure:

(21) $$\begin{aligned} (x,s) &= r + (x - r, \tau_r^{-1}(s)), \\ 1 \star (x,s) &= (r+1) + (x - r, \tau_r^{-1}(s)), \\ 1 \star (x,s) &= (r+1) + (x - r, \tau_{r+1}^{-1}(t)) \ . \end{aligned}$$

The last two imply that $\tau_r^{-1}(s) = \tau_{r+1}^{-1}(t)$.

Put $u = \tau_r^{-1}(s) + \bar{r}$. That is, $\theta(x,s) = (x,u)$. If $r < p^n - 1$, the relations (21) imply that $\theta(1 \star (x,s)) = (x + 1, u + \bar{1})$. Now suppose $r = p^n - 1$. It is still true that $\tau_r^{-1}(s) = \tau_{r+1}^{-1}(t) = \tau^{-1}(t)$. By our initial normalization,

$$t = \tau(\tau^{-1}(t)) = \tau^{-1}(t) + \overline{p^n} \ .$$

Now rewrite using u and $r = p^n - 1$:

$$t = \tau_r^{-1}(s) + (\bar{r} + \bar{1}) = u + \bar{1} \ .$$

The logic differs, but the conclusion remains that $\theta(1 \star (x,s)) = \theta(x + 1, u + \bar{1})$.

To summarize: It is now clear that

$$\theta : B \longrightarrow (\mathbb{Z}_p \times (\mathbb{Z}/(N))$$

is an $\mathbf{HT}[\mathbb{Z}]/\mathbb{Z}_p$-isomorphism. Existence of such an isomorphism is the conclusion of Proposition 5. □

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