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SPACE CURVES RELATED BY A TRANSFORMATION OF COMBESCURE

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Abstract. In this paper, the curves associated with the Combescure transform are discussed. With the help of the fact that these curves have a common Frenet frame, an equivalence relation is defined. The equivalence classes obtained by this equivalence relation have been examined for some special curves and it has been obtained that all curves in the equivalence class of a helix curve are also helix curves. This is also true for k-slant helix curves. The important part of this paper consists of the useful construction method to obtain a curve from the given curve α with the help of Combescure transformation. With this method, some special curves such as Bertrand, Mannheim, Salkowski, anti-Salkowski or spherical curve can be obtained from any curve related by a Combescure transform. For example, we obtain an example of Mannheim curves explicitly obtained from an anti-Salkowski curve. It is not easy to find an example of Mannheim curves except circular helix in the literature. In general, the conditions for being Bertrand, Mannheim, Salkowski, anti-Salkowski or spherical curve of the curve β obtained from the given curve α with the help of Combescure transformation were obtained.

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1. Introduction

In the curve theory, the geometry of a curve in 3-dimensional Euclidean space is essentially characterized by their invariants, curvature and torsion. When they are given, the problem of finding a corresponding curve is known as *solving natural equations*. Unfortunately, this is not an easy task. Explicit solutions are known only for a handful of curve classes, including notably the plane curves and circular helices. On the other hand, Frenet vectors of a curve, like curvature functions, play an important role in the study of differential geometric properties of the curve. Thanks to Frenet vectors and curvature functions, we can characterize curves. Another important subject in the geometry of curves is the study of curve pairs with the help of relations between Frenet vectors of two space curves. In searching for pairs of curves such that the tangents, principal normals or binormals of one may be the tangents, principal normals or binormals of the other, there are six cases to be considered [10].

In one of these cases, *Bertrand curves* are obtained. In 1845, Saint-Venant proposed [13], the question whether upon the ruled surface generated by the principal normals of a curve in the 3-dimensional Euclidean space $\mathbb{E}^3$, a second curve can exist having the same principal normals. This question was answered by Bertrand in the paper published in 1850 [2], who showed that a necessary and sufficient condition is that a linear relationship with constant coefficients shall exist between the curvature and torsion of the given original curve. In other words, we have $\lambda\kappa_1 + \mu\kappa_2 = 1$, $\lambda, \mu \in \mathbb{R}_0$ where $\kappa_1$and κ_2 is denoted by the first and second curvatures of a given curve respectively. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves [7].

Another case reveals *Mannheim curves* which have a property that their principal normal lines coincide with the binormal lines of the Mannheim mate (partner) curve at the corresponding points of the curves. It is known that the curvature functions of Mannheim curve in $\mathbb{E}^3$ satisfy the equality $\kappa_1 = a(\kappa_1^2 + \kappa_2^2)$ for some positive constant number a and its parametric equation is obtained in ([8], see also [10],[16]). Some characterizations of Mannheim curves in $\mathbb{E}^3$ can be found in ([7],[8]). Another notable case is evolutes and involutes curves. This situation has been discussed in many differential geometry books [7, 4].

Two curves, corresponding point for point so that the tangents in corresponding points are parallel, are said to be related by a transformation of *Combescure* [5, 15]. In this case, they share the same Frenet frames. Some properties of the curve β obtained from the given curve α by Combescure transformation in $\mathbb{E}^3$ are studied and the conditions for being a Mannheim curve are examined for β in [3].

The main purpose of examining the relations between Frenet vectors of space curve pairs is to construct new curves with the help of the given curve. For example, in [6], Izumiya and Takeuchi gave a new method for constructing Bertrand curves and proved that these curves can be constructed with the help of spherical curves. Another important study for the construction of these curves was by Monterde in [12]. In this work, Monterde has well explained two different methods (the first method was given by Serret and the second method was given by Struik) for constructing Bertrand curves and give Bertrand curve examples with the help of Salkowski curve in $\mathbb{E}^3$. He proved that if a Bertrand curve is also a slant helix, then it is the Bertrand curve associated to a Salkowski curve.

In this paper, the curves associated with the Combescure transform are discussed. With the help of the fact that these curves have a common Frenet frame, an equivalence relation is defined. The equivalence classes obtained by this equivalence relation have been examined for some special curves and it has been obtained that all curves in the equivalence class of a helix curve are also helix curves. This is also true for k-slant helix curves. The important part of this paper consists of the useful construction method to obtain a curve from the given curve α with the help of Combescure transformation. With this method, some special curves such as Bertrand, Mannheim, Salkowski, anti-Salkowski or spherical curve can be obtained from any curve related by a Combescure transform. For example, we obtain an example of Mannheim curves explicitly obtained from an anti-Salkowski curve. It is not easy to find an example of Mannheim curves except circular helix in the literature. In generally, the conditions for being Bertrand, Mannheim, Salkowski, anti-Salkowski or spherical curve of the curve β obtained from the given curve α with the help of Combescure transformation were obtained.

2. **Space curves related by a transformation of Combescure**

In this section, we study space curves whose corresponding point for point so that the tangents in corresponding points are parallel. It is well-know that such curves is called as curves related by a transformation of Combescure by Bianchi [15].

The following theorem can be seen in [5].

Theorem 2.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be regular curves in Euclidean 3-space with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Then the tangent vector T of α is equal to the tangent vector T^* of β if and only if there exists $C : I \subset \mathbb{R} \to \mathbb{R}$ such that*

$$\beta(\varphi(s)) = \alpha(s) + \frac{d^2}{dr^2}\left(C(s) + \frac{d^2C(s)}{dt^2}\right)T - \frac{d^2C(s)}{dt^2}N + \frac{dC(s)}{dt}B$$

where $t = \int \tau(s)\,ds$, $r = \int \kappa(s)\,ds$ and $\varphi'(s) = ds^*/ds$ where s and s^* are arclength parameters of α and β, respectively.

Corollary 2.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be regular curves in Euclidean 3-space with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Then the tangent vector T of α is equal to the tangent vector T^* of β if and only if $N = N^*$ and $B = B^*$.*

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be regular curves in Euclidean 3-space with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Assume that the tangent vector T of α is equal to the tangent vector T^*. Then from above theorem, there exists $C : I \subset \mathbb{R} \to \mathbb{R}$ such that

$$\beta(\varphi(s)) = \alpha(s) + \frac{d^2}{dr^2}\left(C(s) + \frac{d^2C(s)}{dt^2}\right)T - \frac{d^2C(s)}{dt^2}N + \frac{dC(s)}{dt}B. \tag{1}$$

Differentiating (1) with respect to s, we have $N = N^*$ and $B = B^*$. The converse proof is clear. □

Theorem 2.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be regular curves in Euclidean 3-space with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$,* *respectively. Then the tangent vector T of α is equal to the tangent vector T^* of β if and only if*

$$\beta(\varphi(s)) = \varphi'(s)\,\alpha(s) - \int \varphi''(s)\,\alpha(s)\,ds \tag{2}$$

for $\varphi'(s) = ds^*/ds$ where s and s^* are arclength parameters of α and β, respectively.

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be regular curves in Euclidean 3-space with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Assume that the tangent vector T of α is equal to the tangent vector T^*. Then from above theorem, there exists $C : I \subset \mathbb{R} \to \mathbb{R}$ such that

$$\beta(\varphi(s)) = \alpha(s) + \frac{d^2}{dr^2}\left(C(s) + \frac{d^2C(s)}{dt^2}\right)T - \frac{d^2C(s)}{dt^2}N + \frac{dC(s)}{dt}B \tag{3}$$

where $t = \int \tau(s)\,ds$ and $r = \int \kappa(s)\,ds$. Here we have $ds^*/ds = \varphi'(s) = 1 + \kappa P(C)$ where s and s^* are arclength parameters of α and β, respectively. Thus we can easily see that

$$T^* ds^* = \varphi' T ds$$

which implies that

$$\beta(\varphi(s)) = \int \varphi'(s)\,T(s)\,ds$$

or

$$\beta(\varphi(s)) = \varphi'(s)\,\alpha(s) - \int \varphi''(s)\,\alpha(s)\,ds.$$

The converse proof is easily obtained by differentiating (2) with respect to s. □

Definition 2.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : J \subset \mathbb{R} \to \mathbb{E}^3$ be two regular curves with Frenet frames $\{T_1, N_1, B_1\}$ and $\{T_2, N_2, B_2\}$, respectively. If there is a diffeomorphism $h : I \to J$ such that the Frenet vectors T_1, N_1, B_1 are parallel to the Frenet vectors $T_2 \circ h, N_2 \circ h, B_2 \circ h$ respectively and $T_1 \circ h^{-1}, N_1 \circ h^{-1}, B_1 \circ h^{-1}$ are parallel to the Frenet vectors T_2, N_2, B_2 respectively, then α and β are said to be "equivalent".*

Remark 2.5. *It is clear that "Being equivalent curves" is an equivalence relation.*

Here we can choose $J = I$. So we have $\alpha, \beta : I \subset \mathbb{R} \to \mathbb{E}^3$.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$, and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be regular curves in Euclidean 3-space with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively.

Remark 2.6. *It is clear that "Having same Frenet frame for curves" is an equivalence relation. The equivalence class of a regular curve* $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ *is given by*

$$[\alpha] = \left\{ \beta | \ \beta : I \subset \mathbb{R} \to \mathbb{E}^3, \ \beta(\varphi(s)) = \varphi'(s)\,\alpha(s) - \int \varphi''(s)\,\alpha(s)\,ds \right\},$$

where $\varphi : I \to \mathbb{R}$ *is a non-constant differentiable function.*

Corollary 2.7. *If* $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ *is a helis then all curves in* $[\alpha]$ *are helices. Also if* $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ *is a k-slant helix then all curves in* $[\alpha]$ *are k-slant helices.*

Corollary 2.8. *Let* $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ *and* $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ *be regular curves in Euclidean* 3*-space with Frenet apparatus* $\{T, N, B, \kappa, \tau\}$ *and* $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, *respectively. If the tangent vector* T *of* α *is equal to the tangent vector* T^* *of* β, *then we get*

$$\varphi'(s) = \frac{ds^*}{ds} = \frac{\kappa}{\kappa^*} = \frac{\tau}{\tau^*}.$$

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ and $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be regular curves in Euclidean 3-space with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$, respectively. Assume that the tangent vector T of α is equal to the tangent vector T^*. Then differentiating $T = T^*$ with respect to s, we have

$$\frac{ds^*}{ds}\kappa^* N^* = \kappa N$$

which implies that $ds^*/ds = \kappa/\kappa^*$. Also differentiating $B = B^*$ with respect to s, we get

$$\frac{ds^*}{ds}\tau^* N^* = \tau N$$

which implies that $ds^*/ds = \tau/\tau^*$. □

Corollary 2.9. *Let* $\kappa = \lambda\kappa^*$ *and* $\tau = \lambda\tau^*$ *for a nonzero constant* λ. *From* (2), *we get*

$$\beta(\varphi(s)) = \lambda\alpha(s) + V$$

where V *is a constant vector in* $\mathbb{E}^3$. *Thus we obtain all curves which are congruent to* $\alpha(s)$.

Corollary 2.10. *Let* $\varphi'(s) = \lambda\kappa(s)$ *for a nonzero constant* λ. *From* (2), *we get*

$$\beta(\varphi(s)) = \lambda\kappa(s)\,\alpha(s) - \lambda\int \kappa'(s)\,\alpha(s)\,ds.$$

Here $\beta(\varphi(s))$ is a Salkowski curve in $\mathbb{E}^3$. Thus we obtain a family of Salkowski curves in $[\alpha]$.

In the following example, we obtain a Salkowski curve from an anti-Salkowski curve.

Example 2.11. *Let us consider the anti-Salkowski curve in $\mathbb{E}^3$ given by*

$$\alpha(s)=\begin{pmatrix} \frac{1}{37800}\left(24-s^2\right)^{3/2}\left(-144-9s^2+5s^4\right), \\ \\ \frac{s}{37800}\left(7560+1260s^2-189s^4+5s^6\right), \\ \\ -\frac{6}{5}\left[-2\arcsin\left(\frac{\sqrt{24-s^2}}{2\sqrt{6}}\right)+\sin\left(2\arcsin\left(\frac{\sqrt{24-s^2}}{2\sqrt{6}}\right)\right)\right] \end{pmatrix}$$

with the curvatures $\kappa=\frac{s}{\sqrt{24-s^2}}$, $\tau=1$ and the Frenet frame as

$$\begin{aligned} T &= \left(\frac{-s^3\sqrt{24-s^2}\left(s^2-15\right)}{1080},\frac{216+108s^2-27s^4+s^6}{1080},\frac{-\sqrt{24-s^2}}{5}\right), \\ N &= \left(s-\frac{s^3}{6}+\frac{s^5}{180},\frac{\sqrt{24-s^2}\left(36-18s^2+s^4\right)}{180},\frac{1}{5}\right), \\ B &= \left(1-\frac{s^2\left(540-45s^2+s^4\right)}{1080},\frac{s\left(24-s^2\right)^{3/2}\left(s^2-9\right)}{1080},-\frac{s}{5}\right). \end{aligned}$$

By taking $\varphi'(s)=\kappa=s/\sqrt{24-s^2}$, from (2), we find

$$\beta(\varphi(s))=\left(\frac{s^5\left(21-s^2\right)}{7560},\frac{s\sqrt{24-s^2}\left(216+36s^2+9s^4-s^6\right)}{7560s},-\frac{s^2}{10}\right)$$

with the curvatures $\kappa^*=1$, $\tau^*=\sqrt{24-s^2}/s$ and the Frenet frame as

$$\begin{aligned} T^* &= \left(\frac{-s^3\sqrt{24-s^2}\left(s^2-15\right)}{1080},\frac{216+108s^2-27s^4+s^6}{1080},\frac{-\sqrt{24-s^2}}{5}\right), \\ N^* &= \left(s-\frac{s^3}{6}+\frac{s^5}{180},\frac{\sqrt{24-s^2}\left(36-18s^2+s^4\right)}{180},\frac{1}{5}\right), \\ B^* &= \left(1-\frac{s^2\left(540-45s^2+s^4\right)}{1080},\frac{s\left(24-s^2\right)^{3/2}\left(s^2-9\right)}{1080},-\frac{s}{5}\right). \end{aligned}$$

Here the curve $\beta(\varphi(s))$ is a Salkowski curve in $\mathbb{E}^3$. Also the curve β is in the equivalence class $[\alpha]$.

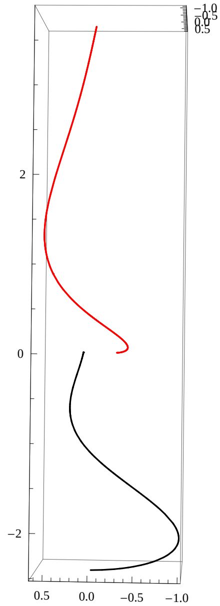

FIGURE 1. The red graphic is the anti-Slakowski curve α and the black graphic is the Salkowski curve β in Example 2.1

Corollary 2.12. *Let $\varphi'(s) = \lambda\tau(s)$ for a nonzero constant λ. From (2), we get*

$$\beta(\varphi(s)) = \lambda\tau(s)\,\alpha(s) - \lambda\int\tau'(s)\,\alpha(s)\,ds.$$

Here $\beta(\varphi(s))$ is an anti-Salkowski curve in $\mathbb{E}^3$. Thus we obtain a family of anti-Salkowski curves in $[\alpha]$.

In the following example, we obtain an anti-Salkowski curve from a Salkowski curve.

Example 2.13. *Let us consider the Salkowski curve in $\mathbb{E}^3$ given by*

$$\alpha(s) = \begin{pmatrix} \frac{78s\sqrt{25-s^2}\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)+\sqrt{26}(28s^2-625)\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{2860}, \\ \\ \frac{\sqrt{26}\left(625-28s^2\right)\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)+78s\sqrt{25-s^2}\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{2860}, \\ \\ \frac{25-2s^2}{4\sqrt{26}} \end{pmatrix}$$

with the curvatures $\kappa = 1$, $\tau = s/\sqrt{25-s^2}$ *and the Frenet frame as*

$$\begin{aligned}
T &= \begin{pmatrix} \frac{-\sqrt{25-s^2}\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5} - \frac{s\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}}, \\ \frac{s\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}} - \frac{\sqrt{25-s^2}\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5}, \\ -\frac{2}{\sqrt{26}} \end{pmatrix}, \\
N &= \left(\frac{5\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{\sqrt{26}}, -\frac{5\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{\sqrt{26}}, -\frac{1}{\sqrt{26}}\right), \\
B &= \begin{pmatrix} \frac{s\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5} - \frac{\sqrt{25-s^2}\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}}, \\ \frac{\sqrt{25-s^2}\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}} + \frac{s\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5}, \\ -\frac{\sqrt{25-s^2}}{\sqrt{26}} \end{pmatrix}.
\end{aligned}$$

By taking $\varphi'(s) = \tau = s/\sqrt{25-s^2}$, *from* (2), *we find*

$$\beta(\varphi(s)) = \begin{pmatrix} \frac{\left(39s^2-350\right)\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right) - 14\sqrt{26}s\sqrt{25-s^2}\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{1430}, \\ \frac{14\sqrt{26}s\sqrt{25-s^2}\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right) + \left(39s^2-350\right)\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{1430}, \\ \frac{s\sqrt{25-s^2} - 25\arcsin\left(\frac{s}{5}\right)}{2\sqrt{26}} \end{pmatrix}$$

with the curvatures $\kappa^* = \sqrt{25-s^2}/s$, $\tau^* = 1$ *and the Frenet frame as*

$$T^* = \begin{pmatrix} \frac{-\sqrt{25-s^2}\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5} - \frac{s\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}}, \\ \\ \frac{s\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}} - \frac{\sqrt{25-s^2}\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5}, \\ \\ -\frac{2}{\sqrt{26}} \end{pmatrix},$$

$$N^* = \left(\frac{5\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{\sqrt{26}}, -\frac{5\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{\sqrt{26}}, -\frac{1}{\sqrt{26}}\right),$$

$$B^* = \begin{pmatrix} \frac{s\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5} - \frac{\sqrt{25-s^2}\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}}, \\ \\ \frac{\sqrt{25-s^2}\cos\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5\sqrt{26}} + \frac{s\sin\left(\sqrt{26}\arcsin\left(\frac{s}{5}\right)\right)}{5}, \\ \\ -\frac{\sqrt{25-s^2}}{\sqrt{26}} \end{pmatrix}.$$

Here the curve $\beta(\varphi(s))$ *is an anti-Salkowski curve in* $\mathbb{E}^3$. *Also the curve* β *is in the equivalence class* $[\alpha]$.

Corollary 2.14. *Let* $\varphi'(s) = \lambda_1\kappa(s) + \lambda_2\tau(s)$ *for constants* λ_1 *and* λ_2. *From* (2), *we get*

$$\beta(\varphi(s)) = (\lambda_1\kappa(s) + \lambda_2\tau(s))\,\alpha(s) - \int (\lambda_1\kappa'(s) + \lambda_2\tau'(s))\,\alpha(s)\,ds$$

whose curvatures κ^* *and* τ^* *is obtained as*

$$\kappa^* = \frac{\kappa(s)}{\lambda_1\kappa(s) + \lambda_2\tau(s)} \quad \textit{and} \quad \tau^* = \frac{\tau(s)}{\lambda_1\kappa(s) + \lambda_2\tau(s)}.$$

So we see that $\lambda_1\kappa^*(s) + \lambda_2\tau^*(s) = 1$ *which implies that* β *is a Bertrand curve. Thus we obtain a family of Bertrand curves in* $[\alpha]$.

Example 2.15. *Let us consider the curve in* $\mathbb{E}^3$ *given by*

$$\alpha(s) = \begin{pmatrix} \frac{-3}{\sqrt{2}}\cos\left(\sqrt{2}s\right)\sin s + 2\cos s\sin\left(\sqrt{2}s\right), \\ \\ \frac{-3}{\sqrt{2}}\sin\left(\sqrt{2}s\right)\sin s - 2\cos s\cos\left(\sqrt{2}s\right), \\ \\ \frac{1}{\sqrt{2}}\sin s \end{pmatrix}$$

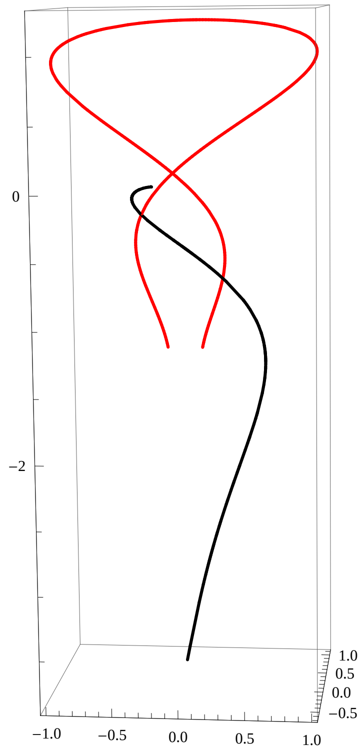

FIGURE 2. The red graphic is the Slakowski curve α and the black graphic is the anti-Salkowski curve β in Example 2.2

with the curvatures $\kappa = \sin s$, $\tau = \cos s$ *and the Frenet frame as*

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos\left(\sqrt{2}s\right) \cos s + \sin\left(\sqrt{2}s\right) \sin s, \\ \\ \frac{1}{\sqrt{2}} \sin\left(\sqrt{2}s\right) \cos s - \cos\left(\sqrt{2}s\right) \sin s, \\ \\ \frac{1}{\sqrt{2}} \cos(s) \end{pmatrix},$$

$$N = \left(\frac{\cos\left(\sqrt{2}s\right)}{\sqrt{2}}, \frac{\sin\left(\sqrt{2}s\right)}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right),$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos\left(\sqrt{2}s\right) \sin(s) - \cos(s) \sin\left(\sqrt{2}s\right), \\ \\ \frac{1}{\sqrt{2}} \sin\left(\sqrt{2}s\right) \sin(s) + \cos(s) \cos\left(\sqrt{2}s\right), \\ \\ \frac{1}{\sqrt{2}} \sin(s) \end{pmatrix}.$$

It can be easily obtained that

$$\sigma = \left(\frac{\kappa_2}{\kappa_1}\right)' \frac{\kappa_1^2}{\left(\kappa_1^2+\kappa_2^2\right)^{3/2}} = 1.$$

So α *is a slant helix. By taking* $\varphi'(s) = \kappa + \tau = \sin s + \cos s$, *from* (2), *we find*

$$\beta(\varphi(s)) = \begin{pmatrix} \frac{(1-3\cos(2s)-3\sin(2s))\sin\left(\sqrt{2}s\right)-\sqrt{2}(1+2\cos(2s)-2\sin(2s))\cos\left(\sqrt{2}s\right)}{4}, \\ \\ \frac{(-1+3\cos(2s)+3\sin(2s))\cos\left(\sqrt{2}s\right)-\sqrt{2}(1+2\cos(2s)-2\sin(2s))\sin\left(\sqrt{2}s\right)}{4}, \\ \\ \frac{2+2s-\cos(2s)+\sin(2s)}{4\sqrt{2}} \end{pmatrix}$$

with the curvatures

$$\kappa^* = \frac{1}{1+\cot s}, \quad \tau^* = \frac{1}{1+\tan s}$$

and the Frenet frame as

$$\begin{aligned} T^* &= \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}s\right)\cos s + \sin\left(\sqrt{2}s\right)\sin s, \\ \\ \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}s\right)\cos s - \cos\left(\sqrt{2}s\right)\sin s, \\ \\ \frac{1}{\sqrt{2}}\cos(s) \end{pmatrix}, \\ N^* &= \left(\frac{\cos\left(\sqrt{2}s\right)}{\sqrt{2}}, \frac{\sin\left(\sqrt{2}s\right)}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \\ B^* &= \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}s\right)\sin(s) - \cos(s)\sin\left(\sqrt{2}s\right), \\ \\ \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}s\right)\sin(s) + \cos(s)\cos\left(\sqrt{2}s\right), \\ \\ \frac{1}{\sqrt{2}}\sin(s) \end{pmatrix}. \end{aligned}$$

It is clear that $\lambda_1\kappa^*(s) + \lambda_2\tau^*(s) = 1$ *for* $\lambda_1 = \lambda_2 = 1$. *So the curve* $\beta(\varphi(s))$ *is a Bertrand curve as well as slant helix in* $\mathbb{E}^3$. *Also the curve* β *is in the equivalence class* $[\alpha]$.

Corollary 2.16. *Let*

$$\varphi'(s) = \frac{\lambda\left(\kappa^2(s)+\tau^2(s)\right)}{\kappa(s)}$$

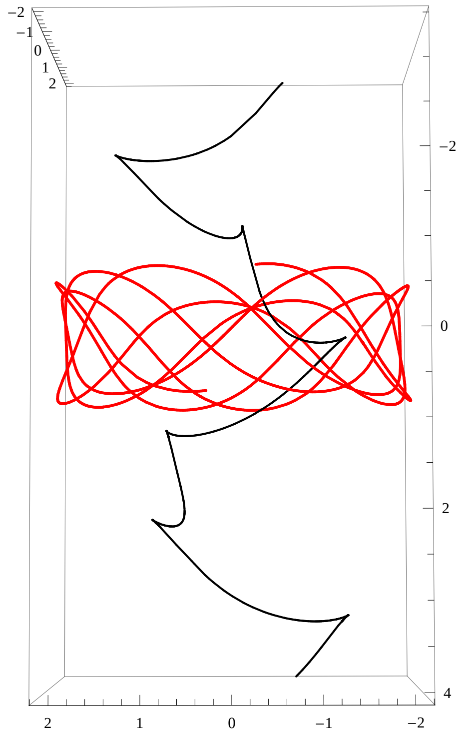

FIGURE 3. The red graphic is the slant helix α and the black graphic is the Bertrand curve β in Example 2.3

for a nonzero constant λ. *From* (2), *we get*

$$\beta(\varphi(s)) = \frac{\lambda\left(\kappa^2(s)+\tau^2(s)\right)}{\kappa(s)}\alpha(s) - \lambda\int\left(\frac{\kappa^2(s)+\tau^2(s)}{\kappa(s)}\right)'\alpha(s)\,ds$$

whose curvatures κ^* *and* τ^* *is obtained as*

$$\kappa^* = \frac{\kappa^2(s)}{\lambda\left(\kappa^2(s)+\tau^2(s)\right)} \quad \text{and} \quad \tau^* = \frac{\kappa(s)\tau(s)}{\lambda\left(\kappa^2(s)+\tau^2(s)\right)}.$$

So we see that $\kappa^*(s) = \lambda\left(\kappa^*(s)^2 + \tau^*(s)^2\right)$ *which implies that* $\beta(\varphi(s))$ *is a Mannheim curve in* $\mathbb{E}^3$. *Thus we obtain a family of Mannheim curves in* $[\alpha]$.

Example 2.17. *Let us consider the anti-Salkowski curve in Example 2.11. By taking*

$$\varphi'(s) = \frac{\kappa^2+\tau^2}{\kappa} = 2\sqrt{6}\ln\left(\frac{12+\sqrt{6}\sqrt{24-s^2}}{s}\right),$$

from (2) , *we find*

$$\beta\left(\varphi\left(s\right)\right)=\left(\frac{s^3\left(25-s^2\right)}{225},\frac{\sqrt{24-s^2}\left(84+13s^2-s^4\right)-90\sqrt{6}\ln\left(\frac{12+\sqrt{6}\sqrt{24-s^2}}{s}\right)}{225},\frac{-12\left(\ln\left(\frac{s^2}{4}\right)+1\right)}{5}\right.$$

with the curvatures

$$\kappa^*=\frac{s^2}{24},\qquad \tau^*=\frac{s\sqrt{24-s^2}}{24}$$

and the Frenet frame as

$$\begin{aligned}
T^* &= \left(\frac{-s^3\sqrt{24-s^2}\left(s^2-15\right)}{1080},\frac{216+108s^2-27s^4+s^6}{1080},\frac{-\sqrt{24-s^2}}{5}\right),\\
N^* &= \left(s-\frac{s^3}{6}+\frac{s^5}{180},\frac{\sqrt{24-s^2}\left(36-18s^2+s^4\right)}{180},\frac{1}{5}\right),\\
B^* &= \left(1-\frac{s^2\left(540-45s^2+s^4\right)}{1080},\frac{s\left(24-s^2\right)^{3/2}\left(s^2-9\right)}{1080},-\frac{s}{5}\right).
\end{aligned}$$

Here the curve $\beta\left(\varphi\left(s\right)\right)$ *is a Mannheim curve in* $\mathbb{E}^3$. *Also the curve* β *is in the equivalence class* $[\alpha]$.

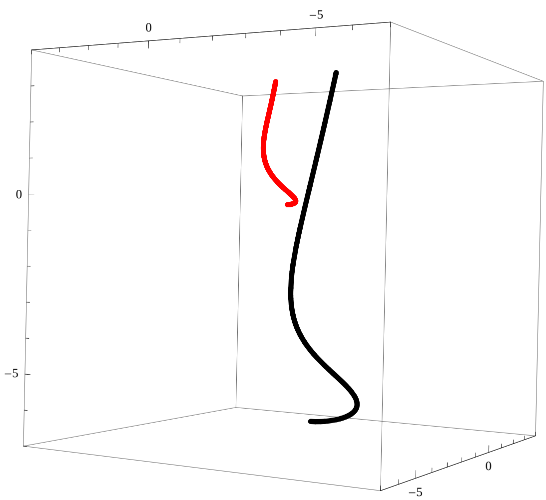

FIGURE 4. The red graphic is the anti-Slakowski curve α and the black graphic is the Mannheim curve β in Example 2.4

Corollary 2.18. *Let*

$$\varphi'\left(s\right)=R\kappa\left(s\right)\cos\left(\int\tau\left(s\right)ds\right)$$

for a positive constant R. From (2), *we get*

$$\beta(\varphi(s)) = R\kappa(s)\cos\left(\int \tau(s)\,ds\right)\alpha(s) - R\int\left(\kappa(s)\cos\left(\int \tau(s)\,ds\right)\right)'\alpha(s)\,ds$$

whose curvatures κ^* *and* τ^* *is obtained as*

$$\kappa^* = \frac{1}{R\cos\left(\int \tau(s)\,ds\right)} \quad \text{and} \quad \tau^* = \frac{\tau(s)}{R\kappa(s)\cos\left(\int \tau(s)\,ds\right)}.$$

So we see that

$$\left(\frac{1}{\kappa^*}\right)^2 + \left(\left(\frac{1}{\kappa^*}\right)'\frac{1}{\tau^*}\right)^2 = R^2$$

which implies that β *is a spherical curve in* $\mathbb{E}^3$. *Thus we obtain a family of spherical curves in* $[\alpha]$.

Example 2.19. *Let us consider the curve in* $\mathbb{E}^3$ *given by*

$$\alpha(s) = \left(2\cos\frac{s}{\sqrt{5}}, 2\sin\frac{s}{\sqrt{5}}, \frac{s}{\sqrt{5}}\right)$$

with the curvatures $\kappa = 2/5$, $\tau = 1/5$ *and the Frenet frame as*

$$\begin{aligned} T &= \left(-\frac{2}{\sqrt{5}}\sin\frac{s}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \\ N &= \left(-\cos\frac{s}{\sqrt{5}}, -\sin\frac{s}{\sqrt{5}}, 0\right), \\ B &= \left(\frac{1}{\sqrt{5}}\sin\frac{s}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right). \end{aligned}$$

By taking $\varphi'(s) = \kappa(s)\cos\left(\int \tau(s)\,ds\right) = \frac{2}{5}\cos\frac{s}{5}$, *from* (2), *we find*

$$\beta(\varphi(s)) = \left(\cos\frac{s}{5}\cos\frac{s}{\sqrt{5}} + \frac{1}{\sqrt{5}}\sin\frac{s}{5}\sin\frac{s}{\sqrt{5}}, \cos\frac{s}{5}\sin\frac{s}{\sqrt{5}} - \frac{1}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}\sin\frac{s}{5}, \frac{2}{\sqrt{5}}\sin\frac{s}{5}\right)$$

with the curvatures

$$\kappa^* = \sec\frac{s}{5}, \quad \tau^* = \frac{1}{2}\sec\frac{s}{5}$$

and the Frenet frame as

$$\begin{aligned} T^* &= \left(-\frac{2}{\sqrt{5}}\sin\frac{s}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \\ N^* &= \left(-\cos\frac{s}{\sqrt{5}}, -\sin\frac{s}{\sqrt{5}}, 0\right), \\ B^* &= \left(\frac{1}{\sqrt{5}}\sin\frac{s}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\cos\frac{s}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right). \end{aligned}$$

It is clear that the curve $\beta(\varphi(s))$ *is a spherical general helix lying the unit sphere in* $\mathbb{E}^3$*. Also the curve* β *is in the equivalence class* $[\alpha]$.

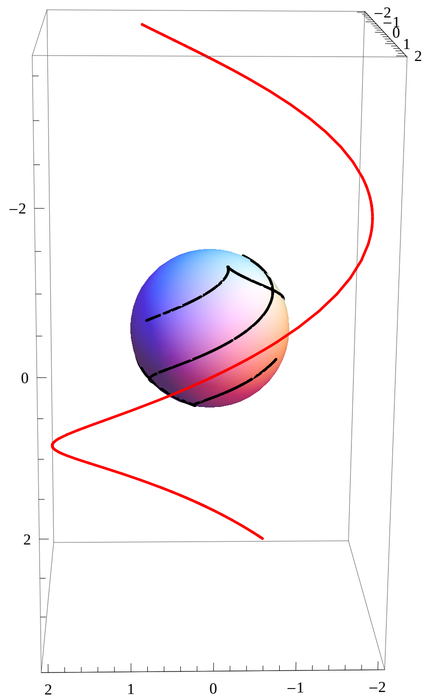

FIGURE 5. The red graphic is the circular helix α and the black graphic is the spherical curve β in Example 2.5

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