

ROTATIONAL ENTROPY OF AN ANNULAR ITERATED FUNCTIONS SYSTEM

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ABSTRACT. In this article, we consider an iterated functions system (IFS) whose functions are homeomorphisms on an annulus. We define rotational spanning and separating sets for the IFS and then provide two definitions for the rotational entropy of the IFS. We show that in the IFSs, the rotational entropy is a topological invariant. We prove that the rotational entropy of an annular IFS is equal to the rotational entropy on its non-wandering set for sequences of functions with a specific condition.

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1. INTRODUCTION

The study of a non-trivial limiting characteristic of a dynamical system such as entropy is interesting for mathematicians. Entropy measures the rate of increase in dynamical complexity when time changes in a system. The appearance of entropy is the outcome of researches conducted by a French physicist and mathematician N. L. Sadi Carnot in 1824. But he died before finding answers to its questions. The

term entropy was introduced by a German physicist R. Clausius in 1850, [4]. Therefore, the word "entropy" was used in the texts of physics for the first time. Then the term entropy was invented in the probabilistic setup of statistical mechanics, quantum mechanics, probability and information theory, and dynamical systems, [5]. In dynamical systems, the metric or measure-theoretic entropy was used in a translation by A. N. Kolmogorov and his student, Y. G. Sinai, in 1959.(see [8, 10]). Their ideas come back to Shannon's information theory. In fact, the metric entropy measures the number of typical orbits. The first time, in 1965, the concept of topological entropy was created by R. L. Adler, A. G. Knoheim and, M. H. McAndrew for a continuous function which is defined on compact metric space, [1]. Topological entropy is simpler than metric entropy. Actually, topological entropy counts the numbers of distinguishable orbits. According to the definition, points with a distance less than ϵ aren't counted as distinct points. Based on the definition, if the numbers of distinguishable orbits of length n grow, approximately, in the form of e^{nh} then h is topological entropy. Sometimes, a homeomorphism is defined on an annulus, $S^1 \times [0, 1]$. For these homeomorphisms, various orbits may rotate at different speeds. Hence, orbits with rotational behavior exist, which are so chaotic. For these orbits, Botelho used Bowen's definition of topological entropy and measured the chaotic rotation of a given homeomorphism and designated "rotational entropy", see [2]. In this paper, we define the concept of the rotational entropy for an IFS whose functions are the homeomorphisms on the annulus. Our definition is on the basis of Botelho's interpretation, [2]. Also, We prove that the rotational entropy is topological invariant. This means that the rotational entropy for IFSs is a characteristic that doesn't change with the topological conjugacy. Non-wandering points have an important role in studying discrete dynamical system. One of the important theorems in the subject of entropy is the measurement of entropy on a non-wandering set. In 1970, Bowen considered a continuous function on a compact metric space and proved that the topological entropy of the function is equal to the topological entropy of the restriction of the function on its non-wandering set,[3]. In 1996, S. Kolyada and L. Snoha proved this theorem for a nonautonomous discrete dynamical system,[9]. We use their definition for non-wandering point and rewrite it with our notations. Then we prove this theorem for some of the sequences of IFS's functions.

2. ROTATIONAL ENTROPY OF AN ANNULAR IFS

In this paper we suppose that $\mathcal{F} = \{f_\lambda : \lambda \in \Lambda; A\}$ is an IFS where $A = S^1 \times [0, 1]$, Λ is a non-empty and finite set and for every $\lambda \in \Lambda$, f_λ is a homeomorphism from A to A . We call this IFS, an annular IFS. We know that the universal cover of A is $\mathbb{R} \times [0, 1]$. Let the function p from $\mathbb{R} \times [0, 1]$ to A be the usual covering map given by the criterion $p(x, y) = (e^{2\pi i x}, y)$. Let $\overline{\mathcal{F}} = \{\overline{f}_\lambda : \lambda \in \Lambda; \mathbb{R} \times [0, 1]\}$ be an IFS where the function $\overline{f}_\lambda$ is a lift of f_λ (that is, a continuous map such that $p\overline{f}_\lambda = f_\lambda p$) for every $\lambda \in \Lambda$.

Assume $\sigma = \{\lambda_1, \lambda_2, \dots\}$ be an element of $\Lambda^\mathbb{N}$. The composition $f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_2} \circ f_{\lambda_1}$ is denoted by F_{σ_n} and similarly, $\overline{F}_{\sigma_n} := \overline{f}_{\lambda_n} \circ \overline{f}_{\lambda_{n-1}} \circ \dots \circ \overline{f}_{\lambda_2} \circ \overline{f}_{\lambda_1}$. The sequences $\{f_{\lambda_i}\}_{i \in \mathbb{N}}$ and $\{\overline{f}_{\lambda_i}\}_{i \in \mathbb{N}}$ where $\lambda_i \in \sigma$ are shown respectively by the notations $\mathcal{F}_\sigma$ and $\overline{\mathcal{F}}_\sigma$.

For a given σ , we consider the sequence $\mathcal{F}_\sigma$. For $a \in A$ the orbit of a relative to σ is $\{F_{\sigma_1}(a), F_{\sigma_2}(a), \dots\}$. Now, we consider the orbits of length $n, (n \in \mathbb{N})$. For resolution, we choose a value $\epsilon > 0$. This means that points with mutual distances less than ϵ aren't considered distinct. Now, we define an (n, ϵ) -rotational spanning set relative to σ and an (n, ϵ) -rotational separating set relative to σ for an annular IFS.

Definition 2.1. *The subset E of A is called (n, ϵ) -rotational spanning set relative to σ iff for every $a \in A$ there exists $b \in E$ such that*

$$|\bar{a} - \bar{b}| < \epsilon \text{ and } \left| \frac{\pi_1(\overline{F}_{\sigma_j}(\bar{a})) - \pi_1(\overline{F}_{\sigma_j}(\bar{b}))}{j} \right| < \epsilon, \quad \forall j = 1, 2, \dots, n-1,$$

where $\bar{a}$ and $\bar{b}$ are the points of the set $[0, 1] \times [0, 1]$ with $p(\bar{a}) = a$ and $p(\bar{b}) = b$ and π_1 is the projection map on the first component.

Definition 2.2. *The subset D of A is said (n, ϵ) -rotational separating set relative to σ iff two distinct points a and b of D verify in the following the relations:*

$$|\bar{a} - \bar{b}| \geq \epsilon \text{ or } \exists j = 1, 2, \dots, n-1, \text{ s.t. } \left| \frac{\pi_1(\overline{F}_{\sigma_j}(\bar{a})) - \pi_1(\overline{F}_{\sigma_j}(\bar{b}))}{j} \right| \geq \epsilon,$$

where the notations $\bar{a}$, $\bar{b}$ and the function π_1 are the same as the previous definition.

In the next proposition, we prove that these sets exist.

Proposition 2.3. *For given σ , there exists a finite (n, ϵ) -rotational spanning set relative to σ with the least numbers of elements.*

Proof. We choose an arbitrary a in A . For every $\lambda \in \Lambda$, the lift of f_λ , $\bar{f}_\lambda$, is continuous. Also, we know that the projection map π_1 is continuous so the composition $\pi_1 \circ \bar{f}_{\sigma_j}$ ($j \in \mathbb{N}$) is continuous and it's uniformly continuous on compact space $[0, 1] \times [0, 1]$. For every j , $j = 1, 2, \dots, n-1$, therefore there exists $\delta_j > 0$ such that for every b with $|\bar{a} - \bar{b}| < \delta_j$ then $|\pi_1(\bar{f}_{\sigma_j}(\bar{a})) - \pi_1(\bar{f}_{\sigma_j}(\bar{b}))| < \epsilon$. Clearly, $\left| \frac{\pi_1(\bar{f}_{\sigma_j}(\bar{a})) - \pi_1(\bar{f}_{\sigma_j}(\bar{b}))}{j} \right| < \frac{\epsilon}{j} \leq \epsilon$. we can assume that $0 < \delta_j < \epsilon$. Now, we put $\delta_1 = \min \{\delta_j \mid j = 1, 2, \dots, n-1\}$. Clearly $\delta_1 < \epsilon$.

We know A is compact so we consider balls with centers $a_1, a_2, \dots, a_m$ and radiuses less than the value δ_1 such that inverse images of the balls under p cover the space $[0, 1] \times [0, 1]$. That is $A = \cup_{i=1}^m B_{\delta_i}(a_i)$ and $[0, 1] \times [0, 1] \subseteq \cup_{i=1}^m p^{-1}(B_{\delta_i}(a_i))$. Since p is continuous so for every i , $i = 1, 2, \dots, m$, $A_i := p^{-1}(B_{\delta_i}(a_i))$ is a open set in $[0, 1] \times [0, 1]$. Therefore for every $\bar{b} \in A_i$ there exists $\mu_{\bar{b}}$ such that $B_{\mu_{\bar{b}}}(\bar{b}) \subseteq A_i$. We can consider $\mu_{\bar{b}} < \delta_1$. As $A_i \subset [0, 1] \times [0, 1]$ and $[0, 1] \times [0, 1]$ is bounded so for some $\bar{b} \in A_i$, the union of the balls, $B_{\mu_{\bar{b}}}(\bar{b})$, cover A_i and in result there is a open cover for the compact space $[0, 1] \times [0, 1]$. Then there exist $\bar{b}_1, \bar{b}_2, \dots, \bar{b}_k$ such that $[0, 1] \times [0, 1] = \cup_{i=1}^k B_{\mu_i}(\bar{b}_i)$ where $\mu_i < \delta_1$ for every i , $i = 1, 2, \dots, k$. We claim that the set $\{b_1, b_2, \dots, b_k : p(\bar{b}_i) = b_i, i = 1, 2, \dots, k\}$ is an $(n - \epsilon)$ -rotational spanning set relative to σ .

Proof of the claim: We know that p is a onto map so for every $a \in A$, there is $\bar{a} \in [0, 1] \times [0, 1]$ such that $p^{-1}(a) = \bar{a}$, by the above relations, there exist $\bar{b}_i, i = 1, \dots, k$, such that $\bar{a} \in B_{\mu_i}(\bar{b}_i)$, that is $|\bar{a} - \bar{b}_i| < \mu_i < \delta_1 < \epsilon$. Therefore by considering the above, we see that $\left| \frac{\pi_1(\bar{f}_{\sigma_j}(\bar{a})) - \pi_1(\bar{f}_{\sigma_j}(\bar{a}_i))}{j} \right| < \epsilon$, for every j , $j = 1, 2, \dots, n-1$. By choice least numbers of balls with given properties, the set $\{b_1, b_2, \dots, b_k\}$ has least numbers and the assertion is proved. $\square$

Let $E_{n,\epsilon,\sigma}^r$ denote the set guaranteed by the previous proposition and the notation $\#E_{n,\epsilon,\sigma}^r$ introduce the number of elements of this set. That is, it's the minimal cardinality.

Definition 2.4. ϵ -rotational entropy of IFS $\mathcal{F}$ relative to σ , denoted by $h_{r,\epsilon,\sigma}(\mathcal{F})$ or $h_{r,\epsilon}(\mathcal{F}_\sigma)$, is defined to be $\limsup_{n \rightarrow \infty} \frac{\log(\#E_{n,\epsilon,\sigma}^r)}{n}$.

Notation 2.5. By considering the definition of the set $E_{n,\epsilon,\sigma}^r$, we see that

- if $m \leq n$ then $\#E_{m,\epsilon,\sigma}^r \leq \#E_{n,\epsilon,\sigma}^r$

- the value of $\limsup$ exists as $n \rightarrow \infty$ because denominator increases faster than numerator as $n \rightarrow \infty$

Definition 2.6. Rotational entropy of IFS $\mathcal{F}$ relative to σ is defined to be the limit of ϵ -rotational entropy of IFS $\mathcal{F}$ relative to σ as $\epsilon \rightarrow 0$. It will be denoted by $h_{r,\sigma}(\mathcal{F})$ or $h_r(\mathcal{F}_\sigma)$. That is $h_{r,\sigma}(\mathcal{F}) := \lim_{\epsilon \rightarrow 0} h_{r,\epsilon,\sigma}(\mathcal{F})$

Notation 2.7. By the definition of the set $E_{n,\epsilon,\sigma}^r$, we can see that if $\epsilon_2 \leq \epsilon_1$ then $\#E_{n,\epsilon_1,\sigma}^r \leq \#E_{n,\epsilon_2,\sigma}^r$. Therefore $h_{r,\epsilon_1,\sigma}(\mathcal{F}) \leq h_{r,\epsilon_2,\sigma}(\mathcal{F})$.

(n, ϵ) -rotational separating set relative to σ with the most numbers of elements will be denoted by $D_{n,\epsilon,\sigma}^r$ and we will use the symbol $\#D_{n,\epsilon,\sigma}^r$ to denote the number of elements of this set.

Now, we will give another definition from ϵ -rotational entropy and rotational entropy of IFS $\mathcal{F}$ relative to σ . That is, it's the maximal cardinality.

Definition 2.8. ϵ -rotational entropy of IFS $\mathcal{F}$ relative to σ , denoted by $h_{r,\epsilon,\sigma}(\mathcal{F})$ or $h_{r,\epsilon}(\mathcal{F}_\sigma)$, is defined to be $\limsup_{n \rightarrow \infty} \frac{\log(\#D_{n,\epsilon,\sigma}^r)}{n}$.

Notation 2.9. One has

- if $m \leq n$ then $\#D_{m,\epsilon,\sigma}^r \leq \#D_{n,\epsilon,\sigma}^r$.
- the value of $\limsup$ exists as $n \rightarrow \infty$ because denominator increases faster than numerator as $n \rightarrow \infty$.

Definition 2.10. Rotational entropy of IFS $\mathcal{F}$ relative to σ is defined the limit of ϵ -rotational entropy of IFS $\mathcal{F}$ relative to σ as $\epsilon \rightarrow 0$. That is $h_{r,\sigma}(\mathcal{F})$ (or $h_r(\mathcal{F}_\sigma)$) $:= \lim_{\epsilon \rightarrow 0} h_{r,\epsilon,\sigma}(\mathcal{F})$

Notation 2.11. We see that if $\epsilon_2 \leq \epsilon_1$ then $\#D_{n,\epsilon_2,\sigma}^r \geq \#D_{n,\epsilon_1,\sigma}^r$ and so $h_{r,\epsilon_2,\sigma}(\mathcal{F}) \geq h_{r,\epsilon_1,\sigma}(\mathcal{F})$.

Theorem 2.12. For given σ , two definition of the rotational entropy of IFS $\mathcal{F}$ relative to σ are equivalent. That is,

$$h_{r,\sigma}(\mathcal{F}) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log(\#E_{n,\epsilon,\sigma}^r)}{n} = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log(\#D_{n,\epsilon,\sigma}^r)}{n}$$

Proof. Let the set D be an (n, ϵ) -rotational separating set relative to σ with the maximal cardinality $\#D_{n,\epsilon,\sigma}^r$. In other words, we can't add any point to D such that this set remains (n, ϵ) -rotational separating set relative to σ . Therefore, for every $a \in A \setminus D$ and every element $d \in D$, we have the following inequalities:

- (1) $|\bar{a} - \bar{d}| < \epsilon, \quad (p(\bar{d}) = d)$
- (2) $\left| \frac{\pi_1(\bar{F}_{\sigma_j}(\bar{a})) - \pi_1(\bar{F}_{\sigma_j}(\bar{d}))}{j} \right| < \epsilon \quad \text{for every } j, j = 1, 2, \dots, n-1.$

We claim that D is an (n, ϵ) -rotational spanning set relative to σ .

Proof of the claim: Choose an arbitrary element in A like a . This element is in $A \setminus D$ or in D .

If $a \in A \setminus D$ then for every $d \in D$, the above inequalities are true by the above argument.

If $a \in D$ then we choose $d = a \in D$. The map p is locally one to one so $\bar{a} = \bar{d}$ and clearly $|\bar{a} - \bar{d}| = 0 < \epsilon$. According to the proof of Proposition 2.3, we also have the second above inequality.

Therefore we proved our claim. This gives $\#E_{n,\epsilon,\sigma}^r \leq \#D_{n,\epsilon,\sigma}^r$. We know that $\log$ is an increasing map so the previous inequality remains after taking limits. That is,

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log(\#E_{n,\epsilon,\sigma}^r)}{n} \leq \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log(\#D_{n,\epsilon,\sigma}^r)}{n}.$$

By the Notations 2.5, 2.7, 2.9 and 2.11, there exist limits for both sides, the first side of the inequality and the second side are equal to $h_{r,\sigma}(\mathcal{F})$. So the inequality is removed and the assertion is proved. $\square$

3. ROTATIONAL ENTROPY OF AN ANNULAR IFS IS A TOPOLOGICAL INVARIANT

In this section, we're going to investigate whether rotational entropy of an annular IFS relative to σ is a topological invariant property. Is this characteristic which doesn't change with the topological conjugacy? Therefore, we need a definition of the topological conjugacy for two IFS. We use the definition which is described in [7], that it's as follows:

Definition 3.1. Suppose $\mathcal{F} = \{f_\lambda, M : \lambda \in \Lambda\}$ and $\mathcal{G} = \{g_\lambda, M : \lambda \in \Lambda\}$ be two IFSs. The IFSs $\mathcal{F}$ and $\mathcal{G}$ are called topologically conjugate if there exists a homeomorphism $h : M \rightarrow M$ such that $f_\lambda \circ h = h \circ g_\lambda$ for every $\lambda \in \Lambda$.

So we comprehend that we should find out this matter that if an annular IFS $\mathcal{F}$ has rotational entropy relative to σ and $\mathcal{G}$ is another annular IFS which is topologically conjugate to $\mathcal{F}$. Is rotational entropy of $\mathcal{G}$ relative to σ equal to the rotational entropy of $\mathcal{F}$ relative to σ ? If the answer is positive then we deduce topological

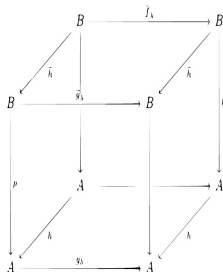


FIGURE 1. The functions and their lifts where $B = \mathbb{R} \times [0, 1]$

invariant. In the next theorem, we demonstrate that the answer to this question is positive.

Theorem 3.2. *The rotational entropy relative to σ is a topological invariant.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two annular IFSs. Suppose that $h : A \rightarrow A$ is a conjugate homeomorphism so that for every $\lambda \in \Lambda$, $g_\lambda \circ h = h \circ f_\lambda$. If $\bar{h}$, $\bar{f}_\lambda$ and $\bar{g}_\lambda$ are a lift of h , f_λ and g_λ , respectively, we have $\bar{g}_\lambda \circ \bar{h} = \bar{h} \circ \bar{f}_\lambda$, see the figure (1The functions and their lifts where $B = \mathbb{R} \times [0, 1]$ figure.1).

The function $\bar{h}$ is a lift of h so it's a $(1, 0)$ –translation invariant, see [2]. That is, for every $y \in \mathbb{R} \times [0, 1]$, $\bar{h}(y + (1, 0)) = \bar{h}(y) + (1, 0)$. Therefore, there is n_1 such that for every n , $n \geq n_1$ we have

$$\frac{|\bar{h}(z) - z_1 - (\bar{h}(w) - w_1)|}{n} < \frac{\epsilon}{2} \quad \forall z, w \in \mathbb{R} \times [0, 1], \quad z_1 = \pi_1(z), \quad w_1 = \pi_1(w).$$

Let E be an (n, ϵ) –rotational spanning set for $\mathcal{F}_\sigma$. Then for every $a \in A$ there exists $b \in E$ such that it verifies the inequalities of Definition 2.1 theorem. 2.1. Now,

we have

$$\begin{aligned}
& \left| \frac{\pi_1\left(\overline{h}\left(\overline{F}_{\sigma_j}(\overline{a})\right)\right) - \pi_1\left(\overline{h}\left(\overline{F}_{\sigma_j}(\overline{b})\right)\right)}{j} \right| \\
& \leq \frac{\left| \left(\pi_1\left(\overline{h}\left(\overline{F}_{\sigma_j}(\overline{a})\right)\right) - \pi_1\left(\overline{F}_{\sigma_j}(\overline{a})\right) \right) - \left(\pi_1\left(\overline{h}\left(\overline{F}_{\sigma_j}(\overline{b})\right)\right) - \pi_1\left(\overline{F}_{\sigma_j}(\overline{b})\right) \right) \right|}{j} \\
& + \frac{|\pi_1(\overline{F}_{\sigma_j}(\overline{a})) - \pi_1(\overline{F}_{\sigma_j}(\overline{b}))|}{j} \\
& \leq \frac{\epsilon}{2} + \epsilon = \frac{3\epsilon}{2} < 2\epsilon, \quad \text{for } j = n_1, n_1 + 1, \dots, n - 1.
\end{aligned}$$

Since $\overline{g_\lambda o h} = \overline{h o f_\lambda}$, then

$$\begin{aligned}
\overline{h o F}_{\sigma_j} &= \overline{h o f_{\lambda_j} o \dots o f_{\lambda_1}} = \overline{g_{\lambda_j} o h o f_{\lambda_{j-1}} o \dots o f_{\lambda_1}} \\
&= \dots = \overline{G}_{\sigma_j} o \overline{h}.
\end{aligned}$$

From the above relations

$$\left| \frac{\pi_1\left(\overline{G}_{\sigma_j}(\overline{h}(\overline{a}))\right) - \pi_1\left(\overline{G}_{\sigma_j}(\overline{h}(\overline{b}))\right)}{j} \right| < 2\epsilon, \quad \text{for } j = n_1, n_1 + 1, \dots, n - 1.$$

So $h(E)$ is an $(n \geq n_1, 2\epsilon)$ -rotational spanning set for $\mathcal{G}_\sigma$. Now put

$$h_{r,\sigma,n_1}(\mathcal{G}) = \lim_{\epsilon \rightarrow 0} \limsup_{n \geq n_1} \frac{\log(\#(h(E))_{n \geq n_1, \epsilon, \sigma}^r)}{n}.$$

Then we obtain

$$(1) \quad h_{r,\sigma,n_1}(\mathcal{G}) \leq h_{r,\sigma}(\mathcal{F})$$

In the following, we prove

$$(2) \quad h_{r,\sigma,n_1}(\mathcal{G}) = h_{r,\sigma}(\mathcal{G})$$

For $n, n \geq n_1$, suppose that $\hat{E}$ is an (n, ϵ) -rotational spanning set relative to σ with the least numbers of elements for $\mathcal{G}$ then $\hat{E}$ will be a $(n \geq n_1, \epsilon)$ -rotational spanning set relative to σ . So $\#\hat{E}_{n,\epsilon,\sigma}^r \geq \#\hat{E}_{n \geq n_1, \epsilon, \sigma}^r$ and this inequality implies $h_{r,\sigma}(\mathcal{G}) \geq h_{r,\sigma,n_1}(\mathcal{G})$. Now, we prove the other inequality. As the functions in $\overline{\mathcal{G}}$ are continuous so they're uniformly continuous on compact space $[0, 1] \times [0, 1]$. Therefore for $\epsilon > 0$ there is $\delta < \epsilon$ so that if $|\overline{a} - \overline{b}| < \delta$ then $\left| \frac{\pi_1(\overline{G}_{\sigma_j}(\overline{a})) - \pi_1(\overline{G}_{\sigma_j}(\overline{b}))}{j} \right| < \epsilon$ for $1 \leq j < n_1$. As a result, every $(n \geq n_1, \delta)$ -rotational spanning set relative to σ is an (n, ϵ) -rotational spanning set relative to σ for $\mathcal{G}$. Therefore, we obtain

$\# \hat{E}_{n \geq n_1, \delta, \sigma}^r \geq \# \hat{E}_{n, \epsilon, \sigma}^r$. Whenever ϵ approaches to 0, δ also approaches to 0 and we have

$$\lim_{\delta \rightarrow 0} \limsup_{n \geq n_1} \frac{\log (\# \hat{E}_{n \geq n_1, \delta, \sigma}^r)}{n} \geq \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log (\# \hat{E}_{n, \epsilon, \sigma}^r)}{n}.$$

Then $h_{r, \sigma, n_1}(\mathcal{G}) \geq h_{r, \sigma}(\mathcal{G})$. Therefore the equality (2) Rotational entropy of an annular IFS is a topological invariant equation.3.2) is proved. So from the inequality (1) Rotational entropy of an annular IFS is a topological invariant equation.3.1) and the equality (2) Rotational entropy of an annular IFS is a topological invariant equation.3.2), we obtain $h_{r, \sigma}(\mathcal{G}) \leq h_{r, \sigma}(\mathcal{F})$. The other inequality arises similarly and the statement is proved. $\square$

4. ROTATIONAL ENTROPY OF AN ANNULAR IFS ON NON-WANDERING SET

We can consider an IFS as a non-autonomous system when σ is fixed. So, we apply the definition of a non-wandering point in a nonautonomous system, see [9], and review its definition for an IFS.

Definition 4.1. *A point $a \in A$ is called non-wandering point of IFS $\mathcal{F}$ relative to σ if for a neighborhood of a , $U(a)$, there exist natural numbers n and k such that $f_{\lambda_n}^k(U(a)) \cap U(a) \neq \emptyset$, where $f_{\lambda_n}^k = f_{\lambda_{n+k}} \circ f_{\lambda_{n+k-1}} \circ \dots \circ f_{\lambda_n}$.*

Let $\Omega(\mathcal{F}_\sigma)$ denotes the set of all of non-wandering points of IFS $\mathcal{F}$ relative to σ . $\Omega(\mathcal{F}_\sigma)$ is nonempty and closed, see [9].

Now, we prove that for a specific σ , the rotational entropy $\mathcal{F}_\sigma$ is equal to the rotational entropy of the restriction $\mathcal{F}_\sigma$ to $\Omega(\mathcal{F}_\sigma)$. In order to prove this theorem, we apply T. Fisher's ideas, [6]. First, notice to the following lemma.

Lemma 4.2. *Suppose U is an open cover for $\Omega(\mathcal{F}_\sigma)$. Then there is a positive β uniformly such that*

$$(3) \quad f_{\lambda_k}^m(B_\beta(b)) \cap B_\beta(b) = \emptyset, \quad \forall m, k \in \mathbb{N} \text{ and } \forall b \in U^c.$$

Proof. U^c is compact so there are the open sets $V_1, V_2, \dots, V_{\hat{l}}$ such that $U^c = \bigcup_{j=1}^{\hat{l}} V_j$, ($V_j \subset U^c$). For $b \in U^c$, there exists j , $1 \leq j \leq \hat{l}$, such that $b \in V_j$. Since $b \in U^c$ is wandering point then there is a neighborhood $V(b)$ of b such that for all $m, k \in \mathbb{N}$, we have $f_{\lambda_k}^m(V(b)) \cap V(b) = \emptyset$. Set $W(b) = V(b) \cap V_j$. Clearly $W(b)$ is a subset of U^c and it's an open set of b so there exists a positive number $\beta(b)$ with $B_{\beta(b)}(b) \subset W(b)$. We can assume that $\beta(b) \in [0, 1]$. Put $\beta = \inf\{\beta(b) : b \in U^c\}$. The infimum value

exists and it's belong to this set because the interval $[0, 1]$ is compact in $\mathbb{R}$. Therefore $\beta > 0$ and we have

$$(4) \quad f_{\lambda_k}^m(B_\beta(b)) \cap B_\beta(b) = \emptyset, \quad \forall m, k \in \mathbb{N} \text{ and } \forall b \in U^c.$$

Since for $b \in U^c$, we have $B_\beta(b) \subset W(b) \subset V(b) \subset U^c$. $\square$

Theorem 4.3. *Let l be an arbitrary natural number. Suppose that E is an (n, ϵ) -rotational spanning set relative to σ for $\Omega(\mathcal{F}_\sigma)$ with the least numbers of elements. Let $U = \left\{ a \in A \mid |\bar{a} - \bar{b}| < \epsilon \text{ \& } \left| \frac{\pi_1(\bar{F}_{\sigma_j}(\bar{a})) - \pi_1(\bar{F}_{\sigma_j}(\bar{b}))}{j} \right| < \epsilon, \forall j, j = 1, 2, \dots, n-1, \text{ for some } b \in E \right\}$. Assume that B is an (n, β) -rotational spanning set relative to σ for U^c where β is the same value of the previous lemma. If given σ has the following condition:*

If $f_{\lambda_{kl}}^{kn}(a) \in U^c$ with $b \in B$ then $f_{\lambda_{kl}}^{ml}(a) \in U^c$ with $b \in B$ where $k = 0, 1, \dots, l-1$, $m = 0, \dots, k$ and $a \in A$. Then $h_{r,\sigma}(\mathcal{F}) = h_{r,\sigma}(\mathcal{F} \mid_{\Omega(\mathcal{F}_\sigma)})$.

Proof. Using the definition of (n, ϵ) -rotational spanning set relative to σ , we have $h_{r,\sigma}(\mathcal{F}) \geq h_{r,\sigma}(\mathcal{F} \mid_{\Omega(\mathcal{F}_\sigma)})$.

In the following, we will prove the other inequality. We define the function d_n from $[0, 1] \times [0, 1]$ to $\mathbb{R}^+$ with the following criterion:

$$d_n(\bar{a}, \bar{b}) = \max \left\{ |\bar{a} - \bar{b}|, \left| \frac{\pi_1(\bar{F}_{\sigma_j}(\bar{a})) - \pi_1(\bar{F}_{\sigma_j}(\bar{b}))}{j} \right|, j = 1, 2, \dots, n-1 \right\}.$$

Clearly, d_n is well-known and metric.

By the theorem's assumptions, we see that for an element a in $\Omega(\mathcal{F}_\sigma)$ there exists an element b in E such that $|\bar{a} - \bar{b}| < \epsilon$ and $\left| \frac{\pi_1(\bar{F}_{\sigma_j}(\bar{a})) - \pi_1(\bar{F}_{\sigma_j}(\bar{b}))}{j} \right| < \epsilon$, for every $j, j = 1, 2, \dots, n-1$. First, we show that U is an open cover for $\Omega(\mathcal{F}_\sigma)$. Because by the definition of the set U , we have $\Omega(\mathcal{F}_\sigma) \subset U$ and the other hand U is an open set. Because suppose $u \in U$, by the definition U , there exists $b \in E$ such that $d_n(\bar{u}, \bar{b}) < \epsilon$ ($p(\bar{u}) = u$ and $p(\bar{b}) = b$). On the other words, $\bar{u} \in B_\epsilon(\bar{b})$, therefore $u = p(\bar{u}) \in p(B_\epsilon(\bar{b}))$. Now, we show $p(B_\epsilon(\bar{b})) \subset U$. Assume that $a = p(\bar{a}) \in p(B_\epsilon(\bar{b}))$, so $\bar{a} \in B_\epsilon(\bar{b})$, that is, $d_n(\bar{a}, \bar{b}) < \epsilon$. In result by the definition U , $a \in U$. The function p is locally invertible, so we can choose $\delta > 0$ such that $B_\delta(u) \subset p(B_\epsilon(\bar{b}))$. Then we find an open set including u such that it's subset of U . Therefore U is an open set. We result that the set $U^c = A \setminus U$ is compact and all of its points are wandering

because $\Omega(\mathcal{F}_\sigma) \subset U$. For given β , if $\beta \leq \epsilon$ then this value is β needed, if $\beta > \epsilon$ then we choose the value for β that it's equal or less than ϵ . So we have

$$(5) \quad f_{\lambda_k}^m(B_\beta(b)) \cap B_\beta(b) = \emptyset, \quad \forall m, k \in \mathbb{N} \text{ and } \forall b \in U^c.$$

Put $C = E \cup B$. We claim that C is an (n, ϵ) -rotational spanning set relative to σ for A . In order to prove the claim, suppose $a \in A$. It's clear $a \in U$ or $a \in U^c$. If $a \in U$ then, by the definition of the set U , there is a $b \in E$ such that it satisfies the inequalities of the definition of the rotational spanning set (Definition 2.1 theorem 2.1). As $E \subset C$, this element of the set E , b , is belong to C . So we found the element of the set C such that it satisfies the inequalities of the definition of the rotational spanning set. If $a \in U^c$ then in accord with being (n, β) -rotational spanning set B relative to σ for U^c , there exists a $b \in B$ such that $|\bar{a} - \bar{b}| < \beta$ and $\left| \frac{\pi_1(\overline{F_{\sigma_j}(a)}) - \pi_1(\overline{F_{\sigma_j}(b)})}{j} \right| < \beta$. As $\beta \leq \epsilon$ and $B \subset C$ so we find $b \in C$ that it satisfies the inequalities of the definition of the rotational spanning set and the our claim is proved.

Now, we define the function $\varphi_l : A \rightarrow C^l$ with the criterion $\varphi_l(a) = (b_0, \dots, b_{l-2}, b_{l-1})$ where b_i , $i = 0, 1, \dots, l-1$, is chosen by utilizing the following procedure:

- If $f_{\lambda_{il}}^{in}(a)$ is belong to U , then b_i in E is chosen such that $d_n(\overline{f_{\lambda_{il}}^{in}(a)}, \bar{b}_i) < \epsilon$ where $p(\overline{f_{\lambda_{il}}^{in}(a)}) = f_{\lambda_{il}}^{in}(a)$.
- If $f_{\lambda_{il}}^{in}(a)$ is belong to U^c , then b_i in B is chosen such that $d_n(\overline{f_{\lambda_{il}}^{in}(a)}, \bar{b}_i) < \beta$.

Notice that for $i = 0$, we consider that the function $f_{\lambda_0}^0$ be equal to identity. We also see that there is b_i for both cases because C is an (n, ϵ) -rotational spanning set relative to σ for A . In the following, we have three claims and their proofs.

First claim: If $\varphi_l(a) = (b_0, \dots, b_{l-1})$ then $b_i \in B$ is not repeated in ordered l -tuple.

Proof of the first claim. We have $b_i \in B$ and the set B is (n, β) -rotational spanning set relative to σ for U^c so $B \subset U^c$ and in result $b_i \in U^c$. By the construction $B_\beta(b_i)$, we have $B_\beta(b_i) \subset U^c$, this means that all of the points $B_\beta(b_i)$ are wandering therefore the relation (5 Rotational entropy of an annular IFS on non-wandering set equation 4.5) is true for b_i and $B_\beta(b_i)$. As $b_i \in B$, by the definition of the function φ_l , we have $f_{\lambda_{il}}^{in}(a) \in U^c$ with $b_i \in B$ and by the theorem's assumption, we see that for every k , $k = 0, 1, \dots, i$, $f_{\lambda_{il}}^{kn}(a) \in U^c$ and, $d_n(\overline{f_{\lambda_{il}}^{kn}(a)}, \bar{b}_i) < \beta$. The recent relation means that for every k , $k = 0, 1, \dots, i$, $\overline{f_{\lambda_{il}}^{kn}(a)}$ is belong to a ball of radius β centered at point $\bar{b}_i$ with the meter d_n , $B_\beta(\bar{b}_i)$. By reductio ad absurdum, assume

that there is $j > i$ such that $b_j = b_i$ is repeated in ordered l -tuple. As $b_j = b_i \in B$, by the definition φ_l , $f_{\lambda_{jl}}^{jn}(a) \in U^c$ with $b_j \in B$ and by the theorem's assumption, we see that for every k , $k = 0, 1, \dots, j$, $f_{\lambda_{jl}}^{kn}(a) \in U^c$ and $d_n(\overline{f_{\lambda_{jl}}^{kn}(a)}, \overline{b_j}) < \beta$. So for every k , $k = 0, 1, \dots, j$, $\overline{f_{\lambda_{jl}}^{kn}(a)} \in B_\beta(\overline{b_i})$. We know that there is one to one correspondence between a circle and the interval $[0, 1)$. Therefore we can find a value $\delta < \beta$ such that $f_{\lambda_{il}}^{k_1 n}(a)$ and $f_{\lambda_{jl}}^{k_2 n}(a)$ are belong to $B_\delta(b_i)$ for every $k_1 = 0, 1, \dots, i$ and $k_2 = 0, 1, \dots, j$. Clearly $B_\delta(b_i) \subset B_\beta(b_i)$. Since $j > i$, we can write

$$\begin{aligned} f_{\lambda_{jl}}^{jn}(a) &= f_{\lambda_{jl+jn}} \circ f_{\lambda_{jl+jn-1}} \circ \dots \circ f_{\lambda_{jl+in+1}} \circ f_{\lambda_{jl+in}} \circ \dots \circ f_{\lambda_{jl}}(a) \\ &= (f_{\lambda_{jl+jn}} \circ f_{\lambda_{jl+jn-1}} \circ \dots \circ f_{\lambda_{jl+in+1}})(f_{\lambda_{jl}}^{in}(a)) \\ &= f_{\lambda_{jl+in+1}}^{jn-in-1}(f_{\lambda_{jl}}^{in}(a)). \end{aligned}$$

Since $i < j$, we have $f_{\lambda_{jl}}^{in}(a) \in B_\beta(b_i)$. The first side of the above equality is belong to $B_\beta(b_i)$ and the other side is belong to $f_{\lambda_{jl+in+1}}^{jn-in-1}(B_\beta(b_i))$. Therefore $f_{\lambda_{jl+in+1}}^{jn-in-1}(B_\beta(b_i)) \cap B_\beta(b_i) \neq \emptyset$ and this is contradiction. So there isn't any j more than i such that b_i repeated in that component and our claim is proved.

Second claim: The map φ_l is one to one on an $(n, 2\epsilon)$ -rotational separating set relative to σ .

Proof of the second claim. We apply demonstration by contradiction. Suppose that a_1 and a_2 are belong to A such that $d_n(\overline{a_1}, \overline{a_2}) \geq 2\epsilon$ with $\varphi_l(a_1) = \varphi_l(a_2)$. So if $\varphi_l(a_1) = (b_{01}, b_{11}, \dots, b_{(l-1)1})$ and $\varphi_l(a_2) = (b_{02}, b_{12}, \dots, b_{(l-1)2})$ then $b_{i1} = b_{i2}$ for every i , $i = 0, 1, \dots, l-1$. By considering one to one correspondence between a circle and the interval $[0, 1)$, we have $\overline{b_{i1}} = \overline{b_{i2}}$, $i = 0, 1, \dots, l-1$.

$$\begin{aligned} d_n(\overline{f_{\lambda_{il}}^{in}(a_1)}, \overline{f_{\lambda_{il}}^{in}(a_2)}) &\leq d_n(\overline{f_{\lambda_{il}}^{in}(a_1)}, \overline{b_{i1}}) + d_n(\overline{b_{i1}}, \overline{f_{\lambda_{il}}^{in}(a_2)}) \\ &\leq d_n(\overline{f_{\lambda_{il}}^{in}(a_1)}, \overline{b_{i1}}) + d_n(\overline{b_{i12}}, \overline{f_{\lambda_{il}}^{in}(a_2)}) \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

So $d_n(\overline{f_{\lambda_{il}}^{in}(a_1)}, \overline{f_{\lambda_{il}}^{in}(a_2)}) < 2\epsilon$. This relation is true for $i = 0, 1, \dots, l-1$. If $i = 0$ then $d_n(\overline{a_1}, \overline{a_2}) < 2\epsilon$. This is a contradiction and the second claim is proved.

Third claim: We know that B is (n, β) -rotational spanning set relative to σ for U^c and E is (n, ϵ) -rotational spanning set relative to σ for $\Omega(\mathcal{F}_\sigma)$. Suppose $q = \#B_{n,\beta,\sigma}^r$, $p = \#E_{n,\epsilon,\sigma}^r$ and, $D \subset A$ be an $(n, 2\epsilon)$ -rotational separating set relative to σ with a maximum number of elements. Then $\#(\varphi_l(D)) \leq (q+1)!q^l p^l$.

Proof of the third claim. Let I_j be a subset including all of ordered l -tuples in $\varphi_l(D)$ which exactly they have j components in B . By the first claim, $b_i \in B$ isn't

repeated in $\varphi_l(a)$ so $j \leq q$. Now, we're going to enumerate the element the number of elements in I_j . j -selection of elements from a q set is $\binom{q}{j}$ and for every one of these selections, $\frac{l!}{(l-j)!}$ number of ways to choose j position in a l -tuple ordered exist and in order to fill vacant positions, the number of ways are $(\#E_{n,\epsilon,\sigma}^r)^{l-j} = p^{l-j}$ with maximum value p^l . Therefore we obtain the following relation:

$$\#(I_j) \leq \binom{q}{j} \frac{l!}{(l-j)!} p^l.$$

So

$$\#(\varphi_l(D)) = \sum_{j=0}^q \#(I_j) \leq \sum_{j=0}^q \binom{q}{j} \frac{l!}{(l-j)!} p^l.$$

We have

$$\frac{l!}{(l-j)!} = \frac{l(l-1) \dots (l-j+1)(l-j)!}{(l-j)!} \leq l^j \leq l^q.$$

and $\binom{q}{j} \leq q!$. Therefore

$$\#(\varphi_l(D)) \leq \sum_{j=0}^q q! l^q p^l = (q+1)q! l^q p^l = (q+1)! l^q p^l.$$

The third claim is proved.

Now, we continue the proof of the theorem. By the second and third claims and the definition φ_l we have

$$\begin{aligned} h_{r,2\epsilon,\sigma}(\mathcal{F}) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log (\#D_{n,2\epsilon,\sigma}^r) \\ &\leq \limsup_{l \rightarrow \infty} \frac{1}{n(l-1)} \log (\#(\varphi_l(D))) \\ &\leq \limsup_{l \rightarrow \infty} \frac{1}{n(l-1)} \log ((q+1)! l^q p^l) \\ &\leq \limsup_{l \rightarrow \infty} \frac{1}{n(l-1)} \left(\log(q+1)! + q \log l + l \log p \right) \\ &\leq \frac{\log p}{n} = \frac{1}{n} \log \#E_{n,\epsilon,\sigma}^r. \end{aligned}$$

Since E is an (n, ϵ) -rotational spanning set relative to σ for $\Omega(\mathcal{F}_\sigma)$, so as $n \rightarrow \infty$, we have $h_{r,2\epsilon,\sigma}(\mathcal{F}) \leq h_{r,\epsilon,\sigma}(\mathcal{F} \mid_{\Omega(\mathcal{F}_\sigma)})$. Then as ϵ approaches to 0, we have $h_{r,\sigma}(\mathcal{F}) \leq h_{r,\sigma}(\mathcal{F} \mid_{\Omega(\mathcal{F}_\sigma)})$. $\square$

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