

OBSERVATIONAL FORM OF MONOTONE CONVERGENCE THEOREM

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ABSTRACT. In this paper, we study the order creators for integral and by putting new conditions on them we prove the observational form of monotone convergence theorem.

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1. INTRODUCTION

The notion of the multi-dimensional observer, multi-dimensional relative topology, and entropy of compact relative topological spaces was introduced and studied in [2]. If M is a nonempty set and I is a set with the cardinality $|I|$ then a $|I|$ -dimensional observer is a mapping $\mu : M \longrightarrow \prod_{j \in I} [0, 1]$. Suppose we have a discrete dynamical system on M created by a mapping $f : M \longrightarrow M$ and E be a subset of

M . Then the multidimensional observer $m_\mu^f(E) : M \longrightarrow \prod_{j \in I} [0, 1]$ is defined by

$$(1) \quad m_\mu^f(E)(x) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1} \chi_E(f^i(x)) \mu(f^i(x)),$$

is called the relative probability measure. The notion of the new kind of integral for a mapping $\varphi : M \longrightarrow \prod_{j \in I} \mathbb{R}$ up to a relative probability measure which is denoted by $\int \varphi dm_\mu^f$ has been considered in [1]. One can refer to [3, 4, 5, 6] for more details on observers.

In this paper we find necessary and sufficient conditions to deduce the observational form of the monotone convergence theorem. In fact, we prove the following two theorems.

Theorem 1.1. *Suppose that $\mu : M \longrightarrow \prod_{\alpha \in I} [0, 1]$ is an observer, $f : M \longrightarrow M$ is a mapping and $S : M \times N \longrightarrow \mathbb{R}$ is a semi-onto order creator for integral, such that $S(x, \varphi + \psi) = S(x, \varphi) + S(x, \psi)$ for every $x \in M$ and $\varphi, \psi \in N$. Then the following statements are equivalent:*

(i) *If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $A = \bigcup_{n \in \mathbb{N}} A_n$, then*

$$\lim_{n \rightarrow \infty} m_\mu^f(A_n)(x)(j) = m_\mu^f(A)(x)(j),$$

for every $x \in M$ and $j \in I$.

(ii) *If $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $B = \bigcap_{n \in \mathbb{N}} B_n$, then*

$$\lim_{n \rightarrow \infty} m_\mu^f(B_n)(x)(j) = m_\mu^f(B)(x)(j),$$

for every $x \in M$ and $j \in I$.

(iii) *If $\varphi_1 \geq \varphi_2 \geq \varphi_3 \geq \dots \geq 0$ and $\lim_{n \rightarrow \infty} [\varphi_n] = [0]$, then*

$$\lim_{n \rightarrow \infty} \left(\int_M \varphi_n dm_\mu^f \right)(x)(j) = 0,$$

for every $x \in M$ and $j \in I$.

(iv) *If $0 \leq \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \dots$ and $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$, then*

$$\lim_{n \rightarrow \infty} \left(\int_M \varphi_n dm_\mu^f \right)(x)(j) = \left(\int_M \varphi dm_\mu^f \right)(x)(j)$$

for every $x \in M$ and $j \in I$.

(v) If $\varphi_1 \geq \varphi_2 \geq \varphi_3 \geq \dots \geq 0$ and $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$, then

$$\lim_{n \rightarrow \infty} \left(\int_M \varphi_n dm_\mu^f \right)(x)(j) = \left(\int_M \varphi dm_\mu^f \right)(x)(j),$$

for every $x \in M$ and $j \in I$.

Theorem 1.2. Let $\mu : M \rightarrow \prod_{\alpha \in I} [0, 1]$ be an observer, $f : M \rightarrow M$ be a mapping and $S : M \times N \rightarrow \mathbb{R}$ be a semi-onto order creator for integral, such that $S(x, \varphi + \psi) = S(x, \varphi) + S(x, \psi)$ for every $x \in M$ and $\varphi, \psi \in N$. For every constant map $w : M \rightarrow \prod_{i \in I} \mathbb{R}$ and any subset A of M , we define the mapping $\phi_{(w,A)} : M \rightarrow \prod_{i \in I} \mathbb{R}$ by $\phi_{(w,A)}(x)(i) := S(x, w)m_\mu^f(A)(x)(i)$, where $x \in M$ and $i \in I$. Then the following statements are equivalent:

- (i) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $A = \bigcup_{n \in \mathbb{N}} A_n$, then $\lim_{n \rightarrow \infty} \phi_{(w,A_n)} = \phi_{(w,A)}$, for every constant map $w : M \rightarrow \prod_{i \in I} \mathbb{R}$.
- (ii) If $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $B = \bigcap_{n \in \mathbb{N}} B_n$, then $\lim_{n \rightarrow \infty} \phi_{(w,B_n)} = \phi_{(w,B)}$, for every constant map $w : M \rightarrow \prod_{i \in I} \mathbb{R}$.
- (iii) If $\{\varphi_n\}_{n \in \mathbb{N}} \subset N$, $\varphi \in N$, $0 \leq \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \dots \leq \varphi$ and $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$, then

$$\lim_{n \rightarrow \infty} \int_M \varphi_n dm_\mu^f = \int_M \varphi dm_\mu^f.$$

- (iv) If $\{\varphi_n\}_{n \in \mathbb{N}} \subset N$, $\varphi \in N$, $\varphi_1 \geq \varphi_2 \geq \varphi_3 \geq \dots \geq 0$ and $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$, then

$$\lim_{n \rightarrow \infty} \int_M \varphi_n dm_\mu^f = \int_M \varphi dm_\mu^f.$$

2. CONVERGENCE RESPECT TO ORDER CREATOR FOR INTEGRALS

From now, we assume that M is a non-empty set, I is an index set, $f : M \rightarrow M$ is a mapping and N is the set of all functions $\psi : M \rightarrow \prod_{j \in I} \mathbb{R}$. We recall that an order creator for integral S is a map from $M \times N$ to $\mathbb{R}$ which satisfies in following conditions

- (i) If $\varphi, \psi \in N$ and $\psi(x) = \varphi(x)$, then $S(x, \psi) = S(x, \varphi)$,
- (ii) $S(x, -\psi) = -S(x, \psi)$, for every $x \in M$ and $\psi \in N$,
- (iii) If $\psi(x)(j) \leq \varphi(x)(j)$ for every $j \in I$, then $S(x, \psi) \leq S(x, \varphi)$.

Suppose that $S : M \times N \rightarrow \mathbb{R}$ is an order creator for integral. We say that φ and ψ are equivalent if and only if $S(x, \varphi) = S(x, \psi)$, for every $x \in M$, and then we denote it by $\psi \sim \varphi$. For every $\varphi \in N$, the class $[\varphi]$ is defined by $[\varphi] := \{\psi \in N \mid \psi \sim \varphi\}$

and $\tilde{N}$ is defined by $\tilde{N} := \{[\varphi] \mid \varphi \in N\}$.

Lemma 2.1. *If $\varphi \sim \psi$, then*

$$\int_M \varphi dm_\mu^f = \int_M \psi dm_\mu^f.$$

Proof. For every simple function X , $X \leq \varphi$ if and only if $X \leq \psi$. So

$$\int_M \varphi dm_\mu^f = \int_M \psi dm_\mu^f.$$

□

In the rest of this paper, we use the class of $[\varphi]$ and the set $\tilde{N}$ instead of $\varphi : M \rightarrow \prod_{\alpha \in I} \mathbb{R}$ and the set N .

Definition 2.2. *Let S be an order creator for integral on M ,*

- (a) *We say that $[\varphi] \leq [\psi]$ if $S(x, \varphi) \leq S(x, \psi)$, for every $x \in M$.*
- (b) *The sequence $\{[\varphi_n]\}_{n \in \mathbb{N}}$ is called increasing sequence if $[\varphi_n] \leq [\varphi_{n+1}]$, for every $n \in \mathbb{N}$.*
- (c) *The sequence $\{[\varphi_n]\}_{n \in \mathbb{N}}$ is called decreasing sequence if $[\varphi_n] \geq [\varphi_{n+1}]$, for every $n \in \mathbb{N}$.*

Definition 2.3. *Let $S : M \times N \rightarrow \mathbb{R}$ be an order creator for integral. We say that a sequence $\{[\varphi_n]\}_{n \in \mathbb{N}}$ converges to $[\varphi]$ if $\lim_{n \rightarrow \infty} S(x, \varphi_n) = S(x, \varphi)$, for every $x \in M$.*

Lemma 2.4. *If $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$ and $\lim_{n \rightarrow \infty} [\varphi_n] = [\psi]$, then $[\varphi] = [\psi]$.*

Proof. Since $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$ and $\lim_{n \rightarrow \infty} [\varphi_n] = [\psi]$, so

$$\lim_{n \rightarrow \infty} S(x, \varphi_n) = S(x, \varphi)$$

and

$$\lim_{n \rightarrow \infty} S(x, \varphi_n) = S(x, \psi),$$

for every $x \in M$. Hence $S(x, \varphi) = S(x, \psi)$. So $\varphi \sim \psi$. □

Example 2.5. *Suppose that $M = \mathbb{R}$. let $S : M \times N \rightarrow \mathbb{R}$ be a mapping such that*

$$S(x, \varphi) = \begin{cases} 0, & x = 0 \\ \varphi(x), & x \neq 0 \end{cases}.$$

We see that S is an order creator for integral. Now, let $\varphi_n(x) = \frac{1}{n}$, for every $x \in M$ and $n \in \mathbb{N}$. so

$$S(x, \varphi_n) = \begin{cases} 0, & x = 0 \\ \frac{1}{n}, & x \neq 0 \end{cases}.$$

Hence $S(x, \varphi_n) \rightarrow 0 = S(x, 0)$ as $n \rightarrow \infty$. That means $[\varphi_n] \rightarrow [0]$ as $n \rightarrow \infty$. But, the map

$$\psi(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

is in $[0]$.

This example shows that the uniqueness of limit concerning this notion of convergence only holds under these equivalent classes.

Definition 2.6. Let $S : M \times N \rightarrow \mathbb{R}$ be an order creator for integral. We say that S is semi-onto, if $A_x : N \rightarrow \mathbb{R}$, which defines by $A_x(\varphi) := S(x, \varphi)$, is onto, for every $x \in M$.

Example 2.7. Let $M = \mathbb{R}$, $N := \{\varphi \mid \varphi : M \rightarrow \mathbb{R}\}$ and $S : M \times N \rightarrow \mathbb{R}$ be a mapping such that $S(x, \varphi) = |x| \varphi(x)$. We see that S isn't semi-onto order creator for integral. But if we change $M = \mathbb{R}$ by $M = \mathbb{R} \setminus \{0\}$, S will be a semi-onto order creator for integral.

Definition 2.8. Let $\mu : M \rightarrow \prod_{\alpha \in I} [0, 1]$ be an observer, $f : M \rightarrow M$ be a mapping and $S : M \times N \rightarrow \mathbb{R}$ be an order creator for integral. We say that the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is bounded, if there exist two maps $\varphi, \psi : M \rightarrow \prod_{\alpha \in I} \mathbb{R}$ such that $\varphi \leq \varphi_n \leq \psi$, for every $n \in \mathbb{N}$.

Definition 2.9. Let $\mu : M \rightarrow \prod_{\alpha \in I} [0, 1]$ be an observer, $f : M \rightarrow M$ be a mapping and $S : M \times N \rightarrow \mathbb{R}$ be a semi-onto order creator for integral. Also, let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a bounded sequence of N and x be a member of M . Define $\alpha_x \in \mathbb{R}$ by $\alpha_x := \liminf_{n \rightarrow \infty} S(x, \varphi_n)$, and $\beta_x \in \mathbb{R}$ by $\beta_x := \limsup_{n \rightarrow \infty} S(x, \varphi_n)$. Since $\{\varphi_n\}_{n \in \mathbb{N}}$ is bounded, so α_x and β_x are well-defined. Also, since S is semi-onto, we can find $\varphi_x, \psi_x : M \rightarrow \prod_{j \in I} \mathbb{R}$ such that $S(x, \varphi_x) = \alpha_x$ and $S(x, \psi_x) = \beta_x$. Define $\liminf_{n \rightarrow \infty} [\varphi_n]$ by $[\varphi_x]$ and $\limsup_{n \rightarrow \infty} [\varphi_n]$ by $[\psi_x]$.

Lemma 2.10. If a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is convergence, then the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is bounded.

Proof. Let φ be a member of N such that $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$. Then

$$\lim_{n \rightarrow \infty} S(x, \varphi_n) = S(x, \varphi),$$

for every $x \in M$. $\{S(x, \varphi_n)\}_{n \in \mathbb{N}}$ is a sequence of real number. Therefore $\{S(x, \varphi_n)\}$ is bounded. So, there exist $m_x, M_x \in \mathbb{R}$ such that $m_x < S(x, \varphi_n) < M_x$. Since S is semi-onto, we can find the maps γ_x and η_x such that $S(x, \gamma_x) = m_x$ and $S(x, \eta_x) = M_x$. Define $\gamma(x) := \gamma_x(x)$ and $\eta(x) := \eta_x(x)$. So $S(x, \gamma) = m_x$ and $S(x, \eta) = M_x$ for every $x \in M$. Therefore $S(x, \gamma) < S(x, \varphi_n) < S(x, \eta)$, for every $x \in M$ and $n \in \mathbb{N}$. So, the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is bounded. $\square$

Lemma 2.11. *Let $\mu : M \rightarrow \prod_{\alpha \in I} [0, 1]$ be an observer, $f : M \rightarrow M$ be a mapping, $S : M \times N \rightarrow \mathbb{R}$ be a semi-onto order creator for integral, $\{\varphi_n\}$ be a sequence of N and φ be a member of N . Then the followings statements are equivalent:*

- (i) $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$.
- (ii) $[\varphi] = \limsup_{n \rightarrow \infty} [\varphi_n] = \liminf_{n \rightarrow \infty} [\varphi_n]$.

Proof. Let $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$ and $x \in M$. So $\lim_{n \rightarrow \infty} S(x, \varphi_n) = S(x, \varphi)$. Since $\{S(x, \varphi_n)\}_{n \in \mathbb{N}}$ is a sequence of real numbers, there is $\alpha_x \in \mathbb{R}$ such that

$$\alpha_x = S(x, \varphi) = \lim_{n \rightarrow \infty} S(x, \varphi_n) = \limsup_{n \rightarrow \infty} S(x, \varphi_n) = \liminf_{n \rightarrow \infty} S(x, \varphi_n).$$

Hence

$$\limsup_{n \rightarrow \infty} [\varphi_n] = \liminf_{n \rightarrow \infty} [\varphi_n] = [\varphi].$$

Now, let

$$\limsup_{n \rightarrow \infty} [\varphi_n] = \liminf_{n \rightarrow \infty} [\varphi_n] = [\varphi].$$

Then

$$S(x, \varphi) = \limsup_{n \rightarrow \infty} S(x, \varphi_n) = \liminf_{n \rightarrow \infty} S(x, \varphi_n) = \lim_{n \rightarrow \infty} S(x, \varphi_n),$$

for every $x \in M$. Therefore

$$\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi].$$

$\square$

3. MAIN THEOREMS

In this section, we will present two theorems that supply necessary and sufficient conditions for convergence theorem.

Example 3.1. Let $M = [0, 1]$, $\mu : M \longrightarrow M$ be an observer such that $\mu \equiv 1$, $f : M \longrightarrow M$ be a mapping such that $f(x) = 1 - x$ and $S : M \times N \longrightarrow \mathbb{R}$ be an order creator for integral with $S(x, \varphi) = \varphi(x)$, for every $x \in M$. Also, let $E = [0, \frac{1}{2})$. Therefore $m_\mu^f(E)(x) = \begin{cases} \frac{1}{2} & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$ and $m_\mu^f(E^c)(x) = \begin{cases} \frac{1}{2} & x \neq \frac{1}{2} \\ 1 & x = \frac{1}{2} \end{cases}$. Consider the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ such that $\varphi_n := \chi_E$, when n is even and $\varphi_n := \chi_{E^c}$, when n is odd. If n is even, then

$$\left(\int_M \varphi_n dm_\mu^f\right)(x) = S(x, 1)m_\mu^f(E)(x) + S(x, 0)m_\mu^f(E^c)(x) = \begin{cases} \frac{1}{2} & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases},$$

and if n is odd, then

$$\int_M \varphi_n dm_\mu^f = S(x, 0)m_\mu^f(E) + S(x, 1)m_\mu^f(E^c) = \begin{cases} \frac{1}{2} & x \neq \frac{1}{2} \\ 1 & x = \frac{1}{2} \end{cases}.$$

Hence

$$\liminf_{n \rightarrow \infty} \int_M \varphi_n dm_\mu^f = \begin{cases} \frac{1}{2} & x \neq \frac{1}{2} \\ 1 & x = \frac{1}{2} \end{cases}.$$

Since $\liminf_{n \rightarrow \infty} \varphi_n = 0$, so

$$\int_M (\liminf_{n \rightarrow \infty} \varphi_n) dm_\mu^f = 0.$$

Therefore, in this case,

$$\int_M (\liminf_{n \rightarrow \infty} \varphi_n) dm_\mu^f \neq \liminf_{n \rightarrow \infty} \left(\int_M \varphi_n dm_\mu^f\right).$$

Proof of Theorem 1.1: (i) $\implies$ (ii) :

Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $B := \bigcap_{n \in \mathbb{N}} B_n$. Define $A_n := M \setminus B_n$. So $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and

$$\bigcup_{n=1}^{\infty} A_n = M \setminus \bigcap_{n=1}^{\infty} B_n.$$

Therefore

$$\begin{aligned}
\lim_{n \rightarrow \infty} m_\mu^f(B_n)(x)(j) &= \lim_{n \rightarrow \infty} m_\mu^f(M \setminus A_n)(x)(j) \\
&= m_\mu^f(M)(x)(j) - \lim_{n \rightarrow \infty} m_\mu^f(A_n)(x)(j) \\
&= m_\mu^f(M)(x)(j) - m_\mu^f\left(\bigcup_{n=1}^{\infty} A_n\right)(x)(j) \\
&= m_\mu^f\left(M - \bigcup_{n=1}^{\infty} A_n\right)(x)(j) \\
&= m_\mu^f\left(\bigcap_{n=1}^{\infty} B_n\right)(x)(j) \\
&= m_\mu^f(B)(x)(j).
\end{aligned}$$

(ii) $\implies$ (i) :

In a similar way, we can prove this part.

(i), (ii) $\implies$ (iii) :

Since $S(x, \varphi + \psi) = S(x, \varphi) + S(x, \psi)$, by theorem (3.3) of [1], we have

$$(2) \quad \int_M (\varphi + \psi) dm_\mu^f = \int_M \varphi dm_\mu^f + \int_M \psi dm_\mu^f.$$

Let $\varepsilon > 0$, $x \in M$ and $j \in I$. Define $A_n := \{x \in M : S(x, \varphi_n) < \varepsilon\}$ and $B_n := M \setminus A_n$. So $\bigcup_{n=1}^{\infty} A_n = M$ and $\bigcap_{n=1}^{\infty} B_n = \emptyset$. By (ii), $\lim_{n \rightarrow \infty} m_\mu^f(B_n)(x)(j) = 0$.

Thus, there exists $n \in N$ such that $x \in A_n$ and

$$(3) \quad 0 \leq \left(\int_M \varphi_n \chi_{B_n} dm_\mu^f \right)(x)(j) < \varepsilon.$$

On the other hand, since S is a semi-onto order creator for integral, there is a map λ_ε such that $S(y, \lambda_\varepsilon) = \varepsilon$, for every $y \in M$. Therefore $S(x, \varphi_n \chi_{A_n}) < \varepsilon = S(x, \lambda_\varepsilon)$, for every $x \in M$. So

$$(4) \quad \left(\int_M \varphi_n \chi_{A_n} dm_\mu^f \right)(x)(j) < \left(\int_M \lambda_\varepsilon dm_\mu^f \right)(x)(j) < \varepsilon.$$

So, by 2, 3 and 4,

$$\left(\int_M \varphi_n dm_\mu^f \right)(x)(j) = \left(\int_M \varphi_n \chi_{A_n} dm_\mu^f \right)(x)(j) + \left(\int_M \varphi_n \chi_{B_n} dm_\mu^f \right)(x)(j) \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Therefore

$$\lim_{n \rightarrow \infty} \left(\int_M \varphi_n dm_\mu^f \right)(x)(j) = 0.$$

(iii) $\implies$ (iv) :

Let $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \dots \leq \varphi$ and $\varphi_n \rightarrow \varphi$ as $n \rightarrow \infty$. So

$$\lim_{n \rightarrow \infty} (\varphi - \varphi_n) = 0.$$

Therefore, by (iii),

$$\lim_{n \rightarrow \infty} \left(\int_M (\varphi - \varphi_n) dm_\mu^f(x)(j) \right) = 0,$$

for every $x \in M$ and $j \in I$. Hence, by theorem (3.3) of [1],

$$\lim_{n \rightarrow \infty} \left(\left(\int_M (\varphi) dm_\mu^f(x)(j) \right) - \left(\int_M (\varphi_n) dm_\mu^f(x)(j) \right) \right) = 0.$$

So

$$\lim_{n \rightarrow \infty} \left(\int_M (\varphi_n) dm_\mu^f(x)(j) \right) = \left(\int_M (\varphi) dm_\mu^f(x)(j) \right).$$

(iii) $\implies$ (v) : The proof of this part is similar to proof of previous part.

(v) $\implies$ (ii) :

Assume that $B_1 \supset B_2 \supset B_3 \supset \dots$ and $B := \bigcap_{n \in \mathbb{N}} B_n$. let $x \in M$. Since S is semi-onto, there is a map $\psi : M \rightarrow \prod_{j \in I} \mathbb{R}$ such that $S(x, \psi) = 1$. Now, we define the constant map λ by $\lambda(y) := \psi(x)$, for every $y \in M$. Hence, we have $S(x, \lambda) = S(x, \psi) = 1$. Define $\varphi_n = \lambda \chi_{B_n}$ and $\varphi = \lambda \chi_B$. So

$$\lim_{n \rightarrow \infty} \varphi_n = \varphi.$$

Therefore, by (v),

$$\lim_{n \rightarrow \infty} \left(\int_M \varphi_n dm_\mu^f(x)(j) \right) = \left(\int_M \varphi dm_\mu^f(x)(j) \right).$$

But

$$\left(\int_M (\varphi_n) dm_\mu^f(x)(j) \right) = S(x, \lambda) m_\mu^f(B_n)(x)(j) = m_\mu^f(B_n)(x)(j)$$

and

$$\left(\int_M (\varphi) dm_\mu^f(x)(j) \right) = S(x, \lambda) m_\mu^f(B)(x)(j) = m_\mu^f(B)(x)(j).$$

Therefore

$$\lim_{n \rightarrow \infty} m_\mu^f(B_n)(x)(j) = m_\mu^f(B)(x)(j).$$

(iv) $\implies$ (v) :

Let $\varphi_1 \geq \varphi_2 \geq \varphi_3 \geq \dots \geq 0$, and $\lim_{n \rightarrow \infty} [\varphi_n] = [\varphi]$. Define $\psi_i = \varphi_i - \varphi$. So $0 \leq \psi_1 \leq \psi_2 \leq \dots$ and $\lim_{n \rightarrow \infty} [\psi_i] = 0$. Therefore, by (iv),

$$\lim_{n \rightarrow \infty} \left(\int_M \psi_i dm_\mu^f(x)(j) \right) = 0,$$

for every $x \in M$ and $j \in I$. So

$$\lim_{n \rightarrow \infty} \left(\int_M (\varphi_i - \varphi) dm_\mu^f \right)(x)(j) = 0,$$

for every $x \in M$ and $j \in I$. Since S is semi-onto, then

$$\lim_{n \rightarrow \infty} \left(\int_M \varphi_i dm_\mu^f \right)(x)(j) = \left(\int_M \varphi dm_\mu^f \right)(x)(j),$$

for every $x \in M$ and $j \in I$.

Proof of Theorem 1.2:

(i) $\implies$ (ii) :

Assume that $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $B = \bigcap_{n=1}^\infty B_n$. Define $A_n := M - B_n$ and $A := M - B$. Then $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $A = \bigcup_{n \in \mathbb{N}} A_n$. So, by (i), $\lim_{n \rightarrow \infty} \phi_{(w, A_n)} = \phi_{(w, A)}$, for every constant map $w : M \longrightarrow \prod_{i \in I} \mathbb{R}$. Since

$$\phi_{(w, A_n)} = \phi_{(w, M - B_n)} = \phi_{(w, M)} - \phi_{(w, B_n)}$$

and

$$\phi_{(w, A)} = \phi_{(w, M - B)} = \phi_{(w, M)} - \phi_{(w, B)},$$

so

$$\lim_{n \rightarrow \infty} \phi_{(w, B_n)} = \phi_{(w, B)},$$

for every constant map $w : M \longrightarrow \prod_{i \in I} \mathbb{R}$.

(ii) $\implies$ (i) :

The proof of this part is similar to the proof of previous part.

(i), (ii) $\implies$ (iii) :

Let $e : M \longrightarrow \prod_{j \in I} \mathbb{R}$, be a mapping such that $e(x)(j) := 1$ for every $x \in M$ and $j \in I$. Assume that $\varepsilon > 0$ is a rational number. Also, let $\lambda_\varepsilon : M \longrightarrow \prod_{j \in I} \mathbb{R}$ be a function such that $S(x, \lambda_\varepsilon) = \varepsilon$, for every $x \in M$. Define $A_n := \{x \in M : S(x, \varphi) - S(x, \varphi_n) \leq \varepsilon\}$ and $B_n := M \setminus A_n$ for every $n \in M$. Now, let $X = \sum_{i=1}^k w_i \chi_{C_i}$ be a simple function and $0 \leq S(x, X) \leq S(x, \lambda_\varepsilon) = \varepsilon$. Then

$$\begin{aligned} \left(\int_M X dm_\mu^f \right)(x)(j) &= \sum_{i=1}^k S(x, w_i) m_\mu^f(C_i)(x)(j) \\ &\leq \sum_{i=1}^k \varepsilon m_\mu^f(C_i)(x)(j) \\ &= \varepsilon m_\mu^f(M)(x)(j) \leq \varepsilon. \end{aligned}$$

Therefor

$$(\int_M \lambda_\varepsilon dm_\mu^f)(x)(j) = \sup\{(\int_M X dm_\mu^f)(x)(j) : 0 \leq X \leq \lambda_\varepsilon\} \leq \varepsilon.$$

Hence $S(x, \int_M \lambda_\varepsilon dm_\mu^f) \leq S(x, \varepsilon e)$, for every $x \in M$. Since $S(x, \varphi + \psi) = S(x, \varphi) + S(x, \psi)$, so

$$S(x, \int_M \lambda_\varepsilon dm_\mu^f) \leq S(x, \varepsilon e) \leq \varepsilon S(x, e).$$

Since $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ and $\bigcap_{n=0}^\infty B_n = \emptyset$, by (ii),

$$\lim_{n \rightarrow \infty} \phi_{(w, B_n)} = \phi_{(w, \emptyset)}.$$

Therefore

$$(5) \quad \lim_{n \rightarrow \infty} S(x, \phi_{(w, B_n)}) = 0.$$

Consider $y \in M$. Define $w : M \longrightarrow \prod_{i \in I} \mathbb{R}$, such that $w(x) = \varphi(y)$, for every $x \in M$. So, by 5, there is $n \in N$ such that

$$(6) \quad S(y, \phi_{(w, B_n)}) < \varepsilon.$$

If X is a simple function such that $0 \leq X \leq \varphi \chi_{B_n}$ and $X = \sum_{i=1}^m v_i \chi_{E_i \cap B_n}$, then

$$\begin{aligned} (\int_M X dm_\mu^f)(y)(j) &= \sum_{i=1}^m S(y, v_i) m_\mu^f(E_i \cap B_n)(y)(j) \\ &\leq S(y, w) m_\mu^f(B_n)(y)(j) \\ &= \phi_{(w, B_n)}(y)(j). \end{aligned}$$

Hence

$$(\int_M \varphi \chi_{B_n} dm_\mu^f)(y)(j) \leq \phi_{(w, B_n)}(y)(j).$$

Therefore, by 6,

$$S(y, \int_M \varphi \chi_{B_n} dm_\mu^f) \leq S(y, \phi_{(w, B_n)}) \leq \varepsilon.$$

So

$$\begin{aligned} S(y, \int_M \varphi dm_\mu^f) &\leq S(y, \int_M \varphi_n dm_\mu^f) + S(y, \int_M \varphi \chi_{B_n} dm_\mu^f) + S(y, \int_M \lambda_\varepsilon dm_\mu^f) \\ &\leq S(y, \int_M \varphi_n dm_\mu^f) + \varepsilon + \varepsilon S(y, e). \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} S(y, \int_M \varphi_n dm_\mu^f) = S(y, \int_M \varphi dm_\mu^f).$$

So

$$\lim_{n \rightarrow \infty} \int_M \varphi_n dm_\mu^f = \int_M \varphi dm_\mu^f.$$

(iii) $\implies$ (iv) :

Let $\varphi_1 \geq \varphi_2 \geq \varphi_3 \geq \dots \geq \varphi \geq 0$ and $\lim_{n \rightarrow \infty} \varphi_n = \varphi$. Define $\psi_n := \varphi_1 - \varphi_n$ and $\psi_0 = \varphi_1 - \varphi$, for every $n \in \mathbb{N}$. Therefore $0 \leq \psi_1 \leq \psi_2 \leq \psi_3 \leq \dots \leq \psi_0$ and $\lim_{n \rightarrow \infty} \psi_n = \psi_0$. Thus, by (iii),

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_M \psi_n dm_\mu^f &= \int_M \psi_0 dm_\mu^f \implies \lim_{n \rightarrow \infty} \int_M (\varphi_1 - \varphi_n) dm_\mu^f = \int_M \psi_0 dm_\mu^f \\ &\implies \lim_{n \rightarrow \infty} \int_M (\varphi_1 - \varphi_n) dm_\mu^f = \int_M \varphi_1 dm_\mu^f - \int_M \varphi dm_\mu^f \\ &\implies \int_M \varphi_1 dm_\mu^f - \lim_{n \rightarrow \infty} \int_M \varphi_n dm_\mu^f = \int_M \varphi_1 dm_\mu^f - \int_M \varphi dm_\mu^f \\ &\implies \lim_{n \rightarrow \infty} \int_M \varphi_n dm_\mu^f = \int_M \varphi dm_\mu^f. \end{aligned}$$

(iv) $\implies$ (ii) :

Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$, $B = \bigcap_{n=1}^\infty B_n$ and $w : M \rightarrow \prod_{j \in I} \mathbb{R}$ be a constant map. Define $\varphi_n := w\chi_{B_n}$ and $\varphi := w\chi_B$. So $\lim_{n \rightarrow \infty} \varphi_n = \varphi$ and $\varphi_1 \geq \varphi_2 \geq \dots \geq \varphi \geq 0$. Thus, by (iv),

$$S(x, w)m_\mu^f(B_n) = \lim_{n \rightarrow \infty} \int_M (\varphi_n) dm_\mu^f = \int_M (\varphi) dm_\mu^f = S(x, w)m_\mu^f(B).$$

Therefore

$$\lim_{n \rightarrow \infty} \phi(w, B_n) = \phi(w, B).$$

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