

LYAPUNOV EXPONENTS FOR QUANTUM CHANNELS: AN ENTROPY FORMULA AND GENERIC PROPERTIES

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ABSTRACT. We denote by M_k the set of k by k matrices with complex entries. We consider quantum channels ϕ_L of the form: given a measurable function $L : M_k \rightarrow M_k$ and a measure μ on M_k we define the linear operator $\phi_L : M_k \rightarrow M_k$, by the law $\rho \rightarrow \phi_L(\rho) = \int_{M_k} L(v)\rho L(v)^\dagger d\mu(v)$.

In a previous work, the authors show that for a fixed measure μ the Φ -Erg property is generic on the function L (also irreducibility). Here we will show that the purification property is also generic on L for a fixed μ .

Given L and μ there are two related stochastic processes: one takes values on the projective space $P(\mathbb{C}^k)$ and the other on matrices in M_k . The Φ -Erg property and the purification condition are the nice hypothesis for the discrete time evolution given by the natural transition probability. In this way it will follow that generically on L , if $\int |L(v)|^2 \log |L(v)| d\mu(v) < \infty$, the Lyapunov exponents $\infty > \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_k \geq -\infty$ are well defined.

In a previous work, the concepts of entropy of a channel and Gibbs channel were presented; and also an example (associated to a stationary Markov chain) in which this definition of entropy (for a quantum channel) matches the Kolmogorov-Shanon definition of entropy. We estimate here the larger Lyapunov exponent for the mentioned example and we show that it is equal to $-\frac{1}{2}h$, where h is the entropy of the associated Markov invariant probability.

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1. INTRODUCTION

We consider quantum channels of the form $\phi_L : M_k \rightarrow M_k$, in which M_k is the set of complex k by k matrices, $\rho \rightarrow \phi_L(\rho) = \int_{M_k} L(v)\rho L(v)^\dagger d\mu(v)$, $L : M_k \rightarrow M_k$ is a measurable function and μ is a measure on M_k .

In a recent work [8], the authors considered Lyapunov exponents for this class of channels ϕ_L , when L was constant and equal to the identity matrix. The Φ -Erg property and the purification condition (see definitions on section 6) were also considered in this mentioned paper.

In a previous paper (see [11]), we show that for a fixed measure μ it is generic on the function L the Φ -Erg property (in fact we show that the irreducible condition is generic). The novelty here is that we will show that the purification condition is also generic on L for a fixed measure μ (see section 9).

The introduction of this variable L allows us to consider questions of generic nature in this type of problem. We use the C^0 topology in the set of complex matrices.

For the benefit of the reader in the Appendix Section 10 we present an outline of the results in [11] and relations of Lyapunov exponents with previous works.

Following [8], one can consider associated to L and μ two related processes: one denoted by X_n , $n \in \mathbb{N}$, takes values on the projective space $P(\mathbb{C}^k)$; and the other, denoted by ρ_n , $n \in \mathbb{N}$, takes values on D_k (where D_k is the set of density operators). The natural transition probability is defined in [8].

The Φ -Erg property and the purification property play an important role when analyzing the ergodic properties of these two processes (see Section 6).

Here we consider a concept of quantum entropy for channels in section 8 which was initially presented in [3].

For a fixed μ and a general L , it was presented in [11] a natural concept of entropy (see future section 3) for a channel in order to develop a version of Gibbs formalism in this setting. It was also presented in example 8.5 in [11] a certain

channel (related to stationary Markov Chains) in which the value obtained with this definition coincides with the classical value of entropy. This shows that the concept that was introduced is natural. This definition of entropy is a generalization of the concept described on papers [3], [5] and [4]. This particular form of defining entropy is, in a certain way, inspired by the results of [28] which considers iterated function systems.

We call the example described in example 8.5 in [11] the *Markov model in quantum information*. It is the main example to be considered in our Section 8.

The main contribution to the topic of Lyapunov exponents of quantum channels is the paper [8]. We adapt here the formalism of [8] to the case of a general L in order to estimate the Lyapunov exponents of the associated dynamical time evolution. In this part of the paper, we just outline the main points of the proof (see section 7) because it is basically the one presented in [8]. We describe the sufficient conditions for the Lyapunov exponents to be finite. The irreducibility and the purification condition are the appropriate assumptions (as mentioned in [8]). Proposition 9.1 and [11] show that those conditions are true for a fixed μ and a generic L .

Results relating entropy and Lyapunov exponents (both in the classical sense) are quite important in Ergodic Theory (see the Appendix section 10 or [7], [29], [22] and [17]).

We compute the first Lyapunov exponent (which is negative) for the *Markov model in quantum information* (see Section 8) and we show that it is equal to $-\frac{1}{2}h$, in which h is the entropy of the associated Markov probability. We also show that the second Lyapunov exponent, in this case, will be $-\infty$. Naturally, a similar general result for the class of all quantum channels is not reachable due to its inherent generality.

We point out that the definition of entropy for a (normalized) channel presented in [11] explores the use of a “kind” of Ruelle operator. This procedure uses a natural *a priori* probability and this is plausible due to the fact that the “set of preimages” can be uncountable (see [21]). The main issue on the reasoning in [11] is invariance (in time one); however, in that paper, the concept of entropy is not directly related (we mean: in an explicit form) to time evolution. On the other hand, we point out that the values of the Lyapunov exponents are of *dynamical nature*. The dynamical discrete-time evolution - on the present setting - is described by a stochastic process

taking values on the set of matrices in M_k (see sections 6 and 7). The example we consider in section 8 shows that the concept of entropy of a channel (at least in this case) presented in [11] can be linked to the natural dynamical time evolution via the main Lyapunov exponent. In the Appendix Section 10 we present the relation of our results about the Lyapunov exponents with the analogous classical concept in Dynamical Systems.

General references for Quantum Channels and Quantum Mechanics can be found at [16], [25], [1], [3], [12], [10], [18], [20], [19] and [33]. The book [32] presents several important results for the general theory of Lyapunov exponents (see [30], [14], [2], and [13]). [31] and [24] describe basic result in Ergodic Theory.

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2. BASIC RESULTS

In this section, for the benefit of the reader, we recall some well-known preliminary results on the topic. We will mention throughout the section the references where the different results are demonstrated.

We denote by M_k the set of complex k by k matrices. We denote by Id the identity matrix on M_k .

We consider the standard Borel sigma-algebra over M_k and the canonical Euclidean inner product on $\mathbb{C}^k$

According to our notation, $\dagger$ denotes the operation of taking the dual of a matrix with respect to the canonical inner product on $\mathbb{C}^k$.

Here tr denotes the trace of a matrix.

Given two matrices A and B we define the Hilbert-Schmidt product

$$\langle A, B \rangle = \text{tr} (A B^\dagger).$$

This induces a norm $\|A\| = \sqrt{\langle A, A \rangle}$ on the Hilbert space M_k which will be called the Hilbert-Schmidt norm.

Definition 2.1. *Given a linear operator Φ on M_k we denote by $\Phi^* : M_k \rightarrow M_k$ the dual linear operator in the sense of Hilbert-Schmidt, that is, if for all X, Y we get*

$$\langle \Phi(X), Y \rangle = \langle X, \Phi^*(Y) \rangle.$$

Consider a measure μ on the Borel sigma-algebra over M_k . For an integrable transformation $F : M_k \rightarrow M_k$:

$$\int_{M_k} F(v) d\mu(v) = \left(\int_{M_k} F(v)_{i,j} d\mu(v) \right)_{i,j},$$

in which $F(v)_{i,j}$ is the entry (i, j) of the matrix $F(v)$.

Definition 2.2. Given a measure μ on M_k and a measurable function $L : M_k \rightarrow M_k$, we say that μ is *L-square-integrable*, if

$$\int_{M_k} \|L(v)\|^2 d\mu(v) < \infty.$$

For a fixed L we denote by $\mathcal{M}(L)$ the set of *L-square-integrable measures*. We also denote $\mathcal{P}(L)$ the set of *L-square-integrable probabilities*.

ϕ_L is well defined for $L \in \mathcal{M}(L)$.

Proposition 2.3. Given a measurable function $L : M_k \rightarrow M_k$ and a square integrable measure μ , then, the dual transformation ϕ_L^* is given by

$$\phi_L^*(\rho) = \int_{M_k} L(v)^\dagger \rho L(v) d\mu(v).$$

Definition 2.4. Given a measure μ over M_k and a square integrable transformation $L : M_k \rightarrow M_k$ we say that L is a **stochastic square integrable transformation** if

$$\phi_L^*(Id) = \int_{M_k} L(v)^\dagger L(v) d\mu(v) = Id.$$

Definition 2.5. A linear map $\phi : M_k \rightarrow M_k$ is called **positive** if takes positive matrices to positive matrices.

Definition 2.6. A positive linear map $\phi : M_k \rightarrow M_k$ is called **completely positive**, if for any m , the linear map $\phi_m = \phi \otimes I_m : M_k \otimes M_m \rightarrow M_k \otimes M_m$ is positive, where I_m is the identity operator acting on the matrices in M_m .

Definition 2.7. If $\phi : M_k \rightarrow M_k$ is square integrable and satisfies

- (1) ϕ is completely positive;
- (2) ϕ preserves trace,

then, we say that ϕ is a **quantum channel**.

Theorem 2.8. *Given μ and L square integrable then the associated transformation ϕ_L is completely positive. Moreover, if ϕ_L is stochastic, then it preserves trace.*

For the proof see [11].

Remark 2.9. ϕ_L^* is also completely positive. We say that ϕ_L preserves unity if $\phi_L(\text{Id}_k) = \text{Id}_k$. In this case, ϕ_L^* preserves trace. If ϕ_L^* preserves the identity then ϕ_L preserves trace.

Definition 2.10 (Irreducibility). *We say that $\phi : M_k \rightarrow M_k$ is an **irreducible channel** if one of the equivalent properties is true*

- Does not exist $\lambda > 0$ and a projection p in a proper subspace of $\mathbb{C}^k$, such that, $\phi(p) \leq \lambda p$;
- For all non null $A \geq 0$, $(\mathbf{1} + \phi)^{k-1}(A) > 0$;
- For all non null $A \geq 0$ there exists $t_A > 0$, such that, $(e^{t_A \phi})(A) > 0$;
- For all pairs of non null positive matrices $A, B \in M_k$ there exists a natural number $n \in \{1, \dots, k-1\}$, such that, $\text{tr}[B\phi^n(A)] > 0$.

For the proof of the equivalences we refer the reader to [15], [27] and [33].

Definition 2.11 (Irreducibility). *Given μ , we will say (by abuse of language) that L is irreducible if the associated ϕ_L is an irreducible channel.*

Theorem 2.12 (Spectral radius of ϕ_L and ϕ_L^*). *Given a square integrable $L : M_k \rightarrow M_k$ assume that the associated ϕ_L is irreducible. Then, the spectral radius $\lambda_L > 0$ of ϕ_L and ϕ_L^* is the same and the eigenvalue is simple. We denote, respectively, by $\rho_L > 0$ and $\sigma_L > 0$, the eigenmatrices, such that, $\phi_L(\rho_L) = \lambda_L \rho_L$ and $\phi_L^*(\sigma_L) = \lambda_L \sigma_L$, where ρ_L and σ_L are the unique non null eigenmatrices (up to multiplication by scalar).*

The above theorem is the natural version of the Perron-Frobenius Theorem for the present setting.

It is natural to think that ϕ_L acts on density states and ϕ_L^* acts in selfadjoint matrices.

Definition 2.13. *Given the measure μ over M_k we denote by $\mathfrak{L}(\mu)$ the set of all integrable L such that the associated ϕ_L is irreducible.*

Definition 2.14. Suppose L is in $\mathfrak{L}(\mu)$. We say that L is **normalized** if ϕ_L has spectral radius 1 and preserves trace. We denote by $\mathfrak{N}(\mu)$ the set of all normalized L .

If $L \in \mathfrak{N}(\mu)$, then, we get from Theorem 2.12 and the fact that $\phi_L^*(\text{Id}_k) = \text{Id}_k$, that $\lambda_L = 1$. That is, there exists ρ_L such that $\phi_L(\rho_L) = \rho_L$ and ρ_L is the only fixed point. Moreover, the spectral radius is equal to 1.

Theorem 2.15 (Ergodicity and temporal means). Suppose $L \in \mathfrak{N}(\mu)$. Then, for all density matrix $\rho \in M_k$ it is true that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \phi_L^n(\rho) = \rho_L,$$

in which ρ_L is the density matrix associated to L .

Proof: The proof follows from Theorem 2.12 and Corollary 6.3 in [33].

□

The above result connects irreducibility and ergodicity (the temporal means have a unique limit).

3. ENTROPY

A measure μ over M_k (which plays the role of the *a priori* probability) is fixed. Thus, given $L \in \mathfrak{L}(\mu)$, we will associate in a natural way the transformation $\phi_L : M_k \rightarrow M_k$.

Definition 3.1. We denote by $\Phi = \Phi_\mu$ the set of all L such that the associated $\phi_L : M_k \rightarrow M_k$ is irreducible and stochastic.

Suppose L is irreducible and stochastic.

Given L consider the density matrix ρ_L , which is invariant for ϕ_L (see Theorem 2.12).

Definition 3.2. We define entropy for L (or, for ϕ_L) by the expression (when finite) :

$$h(L) = h_\mu(L) := - \int_{M_k \times M_k} \text{tr}(L(v)\rho_L L(v)^\dagger) P_L(v, w) \log P_L(v, w) \, d\mu(v) d\mu(w),$$

in which

$$P_L(v, w) := \frac{\operatorname{tr}(L(w)L(v)\rho_L L(v)^\dagger L(w)^\dagger)}{\operatorname{tr}(L(v)\rho_L L(v)^\dagger)}.$$

This definition is a generalization of the analogous concept presented on the papers [3], [5] and [4].

The example presented in [11] shows that the above definition of entropy is indeed a natural generalization of the classical one in Ergodic Theory. The same example will be considered again when analyzing Lyapunov exponents (see section 8).

4. PROCESS X_n , $n \in \mathbb{N}$, TAKING VALUES ON $P(\mathbb{C}^k)$

Consider a fixed measure μ on M_k and a fixed $L : M_k \rightarrow M_k$, such that, $\int_{M_k} \|L(v)\|^2 d\mu(v) < \infty$, and, also assume that ϕ_L is irreducible and stochastic.

Note that if, for example, μ is a probability and the the function $v \rightarrow \|L(v)\|$ is bounded, we get that $\int_{M_k} \|L(v)\|^2 d\mu(v) < \infty$.

We follow the notation of [8] (and, also [11])

Denote by $P(\mathbb{C}^k)$ the projective space on $\mathbb{C}^k$ with the metric $d(\hat{x}, \hat{y}) = (1 - |\langle x, y \rangle|^2)^{1/2}$, in which x, y are representatives with norm 1 and $\langle \cdot, \cdot \rangle$ is the canonical inner product.

Take $\hat{x} \in P(\mathbb{C}^k)$ and $S \subset P(\mathbb{C}^k)$. For a stochastic ϕ_L we consider the kernel

$$(1) \quad \Pi_L(\hat{x}, S) = \int_{M_k} \mathbf{1}_S(L(v) \cdot \hat{x}) \|L(v)x\|^2 d\mu(v),$$

where the norm above is that of Hilbert-Schmidt.

The discrete-time process (described by the kernel) taking values on $P(\mathbb{C}^k)$ is determined by μ and L . If ν is a probability on the Borel σ -algebra $\mathcal{B}$ of $P(\mathbb{C}^k)$ define

$$\begin{aligned} \nu \Pi_L(S) &= \int_{P(\mathbb{C}^k)} \Pi_L(\hat{x}, S) d\nu(\hat{x}) \\ &= \int_{P(\mathbb{C}^k) \times M_k} \mathbf{1}_S(L(v) \cdot \hat{x}) \|L(v)x\|^2 d\nu(\hat{x}) d\mu(v). \end{aligned}$$

$\nu \Pi_L$ is a new probability on $P(\mathbb{C}^k)$ and Π_L is a Markov operator. The definition above of $\nu \rightarrow \nu \Pi_L$ is a simple generalization of the one in [8], in which the authors

use the L mentioned here, as the identity transformation.

The map $\nu \rightarrow \nu \Pi_L$ (acting on probabilities ν) is called the Markov operator obtained from ϕ_L in the paper [23], in which the *a priori* measure μ is a sum of Dirac probabilities. Here we consider a more general setting.

Definition 4.1. *We say that the probability ν over $P(\mathbb{C}^k)$ is invariant for Π_L , if $\nu \Pi_L = \nu$.*

See [11] (and also future Proposition 6.7) for the existence of invariant probabilities for Π_L

5. THE PROCESS ρ_n , $n \in \mathbb{N}$, TAKING VALUES ON D_k

For a fixed μ over M_k and L such ϕ_L is irreducible and stochastic, one can naturally define a process (ρ_n) on $D_k = \{\rho \in M_k : \text{tr } \rho = 1 \text{ and } \rho \geq 0\}$, which is called *quantum trajectory*.

We follow the notation of [8] (and, also [11])

Given a ρ_0 initial state, we get

$$\rho_n = \frac{L(v)\rho_{n-1}L(v)^*}{\text{tr}(L(v)\rho_{n-1}L(v)^*)}$$

with probability

$$\text{tr}(L(v)\rho_{n-1}L(v)^*)d\mu(v), \quad n \in \mathbb{N}.$$

We want to relate the invariant probabilities of the previous section with the fixed point $\rho_{inv} = \rho_{inv}^L$ of ϕ_L .

First, denote $\Omega := M_k^{\mathbb{N}}$, and for $\omega = (\omega_i)_{i \in \mathbb{N}}$, take $\phi_n(\omega) = (\omega_1, \dots, \omega_n)$.

We denote π_n the projection of ω in its first n coordinates.

We also denote by $\mathcal{M}$ the Borel sigma algebra M_k . For all, $n \in \mathbb{N}$, consider $\mathcal{O}_n$ the sigma algebra on Ω generated by the cylinder sets of size n , that is, $\mathcal{O}_n := \pi_n^{-1}(\mathcal{M}^n)$. We equip Ω with the smaller sigma algebra $\mathcal{O}$ which contains all $\mathcal{O}_n$, $n \in \mathbb{N}$.

Denote $\mathcal{J}_n := \mathcal{B} \otimes \mathcal{O}_n$ and $\mathcal{J} := \mathcal{B} \otimes \mathcal{O}$. In this way, $(P(C^k) \times \Omega, \mathcal{J})$ is an integrable space. By abuse of language, we consider $V_i : \Omega \rightarrow M_k$ as a random variable $V_i(\omega) = \omega_i$. We also introduce another random variable

$$W_n := L(V_n) \dots L(V_1), \text{ in which } W_n(\omega) = L(\omega_n) \dots L(\omega_1).$$

We point out that the symbol $\otimes$ does not represent the tensor product.

For a given a probability ν on $P(\mathbb{C}^k)$, we define for $S \in \mathcal{B}$ and $O_n \in \mathcal{O}_n$, another probability

$$\mathbb{P}_\nu(S \times O_n) := \int_{S \times O_n} \|W_n(\omega)x\|^2 d\nu(\hat{x})d\mu^{\otimes n}(\omega).$$

Denote $\mathbb{E}_\nu$ the expected value with respect to $\mathbb{P}_\nu$. Now observe that for a ν probability on $P(\mathbb{C}^k)$, if $\pi_{X_0} : \mathbb{C}^k - \{0\} \rightarrow P(\mathbb{C}^k)$ is the orthogonal projection on subspace generated by X_0 on $\mathbb{C}^k$, we have

$$\rho_\nu := \mathbb{E}_\nu(\pi_{X_0}) = \int_{P(\mathbb{C}^k)} \pi_{x_0} d\nu(x_0).$$

We call ρ_ν barycenter of ν , and it is easy to see that $\rho_\nu \in D_k$.

Proposition 5.1. *If ν is invariant for Π_L , then*

$$\rho_\nu = \mathbb{E}_\nu(\pi_{\hat{X}_0}) = \mathbb{E}_\nu(\pi_{\hat{X}_1}) = \phi_L(\rho_\nu).$$

Therefore, for an irreducible L , every invariant measure ν for Π_L has the same barycenter.

We emphasize that, in this way, we can recover ρ_{inv} , the fixed point of ϕ_L , by taking the barycenter of any invariant probability (the quantum channel ϕ_L admits only one fixed point). That is, for any invariant probability ν for Π_L , we get that $\rho_\nu = \rho_{inv}$.

Note that the previous process can be seen as $\rho_n : \Omega \rightarrow D_k$, such that, $\rho_0(\hat{x}, \omega) = \rho_\nu$ and

$$\rho_n(\omega) = \frac{W_n(\omega)\rho_0 W_n(\omega)^*}{\text{tr}(W_n(\omega)\rho_0 W_n(\omega)^*)}.$$

Using an invariant ρ we can define a Stationary Stochastic Process taking values on M_k . That is, we will define a probability $\mathbb{P}$ over $\Omega = (M_k)^\mathbb{N}$.

Take $O_n \in \mathcal{O}_n$ and define

$$\mathbb{P}^\rho(O_n) = \int_{O_n} \text{tr}(W_n(\omega)\rho W_n(\omega)^*) d\mu^{\otimes n}(\omega).$$

The probability $\mathbb{P}$ on Ω defines a Stationary Stochastic Process.

6. IRREDUCIBILITY, THE Φ -ERG PROPERTY AND THE PURIFICATION CONDITION

In this section, for the benefit of the reader, we recall the main results from [8]. We will use in this section the notation of [8].

Definition 6.1. *Given $L : M_k \rightarrow M_k$, μ on M_k and E subspace of $\mathbb{C}^k$, we say that E is (L, μ) -invariant, if $L(v)E \subset E$, for all $v \in \text{supp } \mu$.*

Definition 6.2. *Given $L : M_k \rightarrow M_k$, μ on M_k , we say that L is **Φ -Erg** for μ , if there exists a unique minimal non-trivial space E , such that, E is (L, μ) -invariant.*

In [27] it is shown that if the above space E is equal to $\mathbb{C}^k$, then L is **irreducible** for μ (or, μ -irreducible) in the sense of Definition 2.11.

The relation of $\mathbb{P}^\rho$ and $\mathbb{P}_\nu$ (described in the last sections) is described in the next result.

Proposition 6.3. *The marginal of $\mathbb{P}_\nu$ on $\mathcal{O}$ is $\mathbb{P}^{\rho_\nu}$. If Φ -Erg is true, then for any two Π -invariant probabilities ν_a and ν_b , we get $\mathbb{P}^{\rho_{\nu_a}} = \mathbb{P}^{\rho_{\nu_b}}$.*

The proof of the result above, when L is the identity, was obtained in Proposition 2.1 in [8]. The proof for the case of a general L is analogous.

Given two operators A and B we say that $A \propto B$, if there exists $\beta \in \mathbb{C}$, such that, $A = \beta B$.

Definition 6.4. *Given $L : M_k \rightarrow M_k$, μ on M_k , we say that the pair (L, μ) satisfies the **purification condition**, if an orthogonal projector π , such that, for any $n \in \mathbb{N}$*

$$\pi L(V_1)^* \dots L(V_n)^* L(V_n) \dots L(V_1) \pi \propto \pi,$$

for $\mu^{\otimes n}$ -almost all $(v_1, v_2, \dots, v_n)$, is necessarily of rank one.

Following [8], we denote $\mathbb{P}^{ch} = \mathbb{P}^{\frac{1}{k} Id}$.

We denote by Y_n , $n \in \mathbb{N}$, the matrix-valued random variable

$$Y_n = \frac{W_n^* W_n}{\text{tr}(W_n^* W_n)}, \text{ if } \text{tr}(W_n^* W_n) \neq 0,$$

that we extend the definition arbitrarily when $\text{tr}(W_n^* W_n) = 0$.

The next two propositions are of fundamental importance for the theory and they were proved in Proposition 2.2 in [8] (the same proof applies here).

Proposition 6.5. *For any probability ν over $P(\mathbb{C}^k)$ the stochastic process Y_n , $n \in \mathbb{N}$, is a martingale with respect to the sequence of sigma-algebras $\mathcal{O}_n$, $n \in \mathbb{N}$. Therefore, there exists a random variable Y_∞ which is the almost sure limit of Y_n for the probability $\mathcal{P}_\nu$ and also in the L^1 norm.*

Proposition 6.6. *For any probability ν over $P(\mathbb{C}^k)$ and $\rho \in \mathcal{D}_k$*

$$\frac{d\mathbb{P}^\rho}{d\mathbb{P}^{ch}} = k \operatorname{tr}(\rho Y_\infty).$$

Moreover, μ and L satisfy the purification condition, if and only if, Y_∞ is $\mathbb{P}_\nu$ -a.s a rank one projection for any probability ν over $P(\mathbb{C}^k)$.

Proposition 6.7. *If the pair (L, μ) satisfies the ϕ -Erg and the purification condition, then, the Markov kernel Π admits a unique invariant probability.*

$x_1 \wedge x_2 \wedge \dots \wedge x_n$, with $x_j \in \mathbb{C}^k$, denotes the classical wedge product (an alternate form on $\mathbb{C}^k$).

One can consider an inner product

$$\langle r_1 \wedge r_2 \wedge \dots \wedge r_n, s_1 \wedge s_2 \wedge \dots \wedge s_n \rangle = \det(r_i s_j)_{i,j=1,2,\dots,n},$$

and, the associated norm $|x_1 \wedge x_2 \wedge \dots \wedge x_n|$.

Given an operator $X : \mathbb{C}^k \rightarrow \mathbb{C}^k$ we define $\bigwedge^n X : \bigwedge^n \mathbb{C}^k \rightarrow \bigwedge^n \mathbb{C}^k$ by $\bigwedge^n X(x_1 \wedge x_2 \wedge \dots \wedge x_n) = X(x_1) \wedge X(x_2) \wedge \dots \wedge X(x_n)$.

Proposition 6.8. *Assume the pair (L, μ) satisfies the purification condition, then, there are two constants $C > 0$ and $\beta < 1$, such that, for each n*

$$\int_{M_k^n} |\wedge^2(L(v_n) \dots L(v_1))| d\mu^{\otimes n}(v_1, v_2, \dots, v_n) = \mathbb{E}^{ch}\left(k \frac{|\bigwedge^2 W_n|}{\operatorname{tr}(W_n^* W_n)}\right) \leq C \beta^n.$$

The proofs of the two propositions above are similar to the corresponding ones in [8].

Consider $\mathcal{B}(M_k) = \{L : M_k \rightarrow M_k \mid L \text{ is continuous and bounded}\}$ where $\|L\| = \sup_{v \in M_k} \|L(v)\|$.

Definition 6.9. *For a fixed measure μ over M_k , define*

$$\mathcal{B}_\mu(M_k) = \{L \in \mathcal{B} \mid L \text{ is } \mu\text{-irreducible}\},$$

and

$$\mathcal{B}_\mu^\Phi(M_k) = \{L \in \mathcal{B} \mid L \text{ is } \Phi\text{-Erg for } \mu\}.$$

Proposition 6.10. *Given μ over M_k with $\#\text{supp } \mu > 1$, $\mathcal{B}_\mu^\Phi(M_k)$ is open and dense on $\mathcal{B}(M_k)$.*

Proof: See [11]. □

In section 9 we will prove:

Proposition 6.11. *Given μ over M_k with $\#\text{supp } \mu > 1$, the set of L satisfying the purification condition is generic in $\mathcal{B}(M_k)$.*

7. LYAPUNOV EXPONENTS FOR QUANTUM CHANNELS

In this section, we will consider a discrete-time process taking values on M_k .

Take μ over M_k and $L : M_k \rightarrow M_k$ in such way that the associated channel Φ defines a Φ -Erg stochastic map. We assume in this section that $\rho \in D_k$ is such that $\Phi(\rho) = \rho$. Such ρ plays the role of the initial vector of probability (in the analogy with the theory of Markov Chains).

We follow the notation of [8].

Take $\Omega = M_k^{\mathbb{N}}$, and for $n \in \mathbb{N}$ let $\mathcal{O}_n$ be the σ -algebra on Ω generated by the n -cylinder sets (as in Section 5).

An element on Ω is denoted by $(\omega_1, \omega_2, \dots, \omega_n, \dots)$. Following section 5 we denote $W_n(\omega) = L(\omega_n) \dots L(\omega_1)$.

Taking $O_n \in \mathcal{O}_n$ we define

$$\mathbb{P}(O_n) = \int_{O_n} \text{tr}(W_n(\omega)\rho W_n(\omega)^*) d\mu^{\otimes n}(\omega).$$

If $\mathcal{O}$ is the smallest σ -algebra of Ω that contains all $\mathcal{O}_n$ we can extend the action of $\mathbb{P}$ to this σ -algebra.

The probability $\mathbb{P}$ on Ω defines a Stationary Stochastic Process.

Theorem 7.1. *$(\Omega, \mathbb{P}, \theta)$ is ergodic where θ is the shift map.*

The above theorem has been proved in Lemma 4.2 in [8].

Theorem 7.2. *Suppose the pair (L, μ) satisfies irreducibility, the ϕ -Erg and the purification condition. Assume also that $\int |L(v)|^2 \log |L(v)| \, d\mu(v) < \infty$, then, there exists numbers*

$$\infty > \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_k \geq -\infty,$$

such that, for any probability ν over $P(\mathbb{C}^k)$ and any $p \in \{1, 2, \dots, k\}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \bigwedge^p W_n \right| = \sum_{j=1}^p \gamma_j,$$

$\mathbb{P}_\nu$ -a.s.

The above theorem was proved in [8] and the same proof works in our current setting. We point out that a key element in this proof (see (35) in [8]) is the fact that if (L, μ) is ϕ -Erg and irreducible, then, $\rho_{inv} > 0$, and for any $\rho \in \mathcal{D}_k$

$$\mathbb{P}^\rho \ll \mathbb{P}^{\rho_{inv}}.$$

Proposition 6.3 is also used in the proof (corresponds to proposition 2.1 in [8]).

The numbers

$$\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_k,$$

are called the **Lyapunov exponents**.

Theorem 7.3. *Suppose that L is generic for μ and $\int |L(v)|^2 \log |L(v)| \, d\mu(v) < \infty$, then, the Lyapunov exponents*

$$\infty > \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_k \geq -\infty$$

are well defined.

(a) $\gamma_2 - \gamma_1 < 0$, where $\gamma_2 - \gamma_1$ is the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{|\bigwedge^2 W_n|}{|W_n|^2},$$

whenever $\gamma_1 = -\infty$

(b) For $\mathbb{P}_\nu$ -almost sure x we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} (\log |W_n(x)| - \log |W_n|) = 0.$$

proof:

The proof of (a) is similar to the one in [8].

Proof of (b): We start with

$$\begin{aligned} \frac{\|W_n x\|}{\|W_n\|} &= \frac{[\operatorname{tr}(W_n \pi_x W_n^\dagger)]^{1/2}}{[\operatorname{tr}(W_n W_n^\dagger)]^{1/2}} = \left[\frac{\operatorname{tr}(W_n^\dagger W_n \pi_x)}{\operatorname{tr}(W_n W_n^\dagger)} \right]^{1/2} \\ &= \left[\operatorname{tr} \left(\frac{W_n^\dagger W_n}{\operatorname{tr}(W_n W_n^\dagger)} \pi_x \right) \right]^{1/2} = [\operatorname{tr}(M_n \pi_x)]^{1/2}. \end{aligned}$$

Those calculations are valid for all ω such that $W_n W_n^\dagger(\omega) \neq 0$. Since $\mathbb{P}_\nu(W_n = 0) = 0$, no extra work is required. Now we use proposition 2.2 in [8] which says that M_n converges $\mathbb{P}_\nu - a.s.$ and in L^1 norm to a $\mathcal{O}$ -measurable random variable M_∞ . By continuity of the trace and square root, we have

$$\lim_{n \rightarrow \infty} [\operatorname{tr}(M_n \pi_x)]^{1/2} = [\operatorname{tr}(M_\infty \pi_x)]^{1/2}, \text{ for } \mathbb{P}_\nu - a.s. \ x \in P(\mathbb{C}^k).$$

The proof is similar to the one in [8].

□

8. THE MAIN EXAMPLE - LYAPUNOV EXPONENTS AND ENTROPY

Now we will present an example (see example 8.5 in [11]) in which we can estimate the Lyapunov exponents and show a relation with entropy. We will call it the *Markov model in quantum information*.

A concept of entropy for Channels was introduced in [3] and this concept is quite natural because extends the concept of entropy for Markov Chains. Given a stochastic matrix $P = (p_{ij})$ we can associate to it an invariant Markov probability measure.

We will consider a particular case (see example 8.5 in [11]) where P is a 2×2 matrix. Our main result in this section is for the first Lyapunov exponent γ_1 :

$$(2) \quad \gamma_1 = \frac{1}{2} \sum_{i,j \in \{0,1\}} \pi_j p_{ij} \log(p_{ij}) = -\frac{1}{2} h,$$

in which h is the entropy of the Markov invariant measure associated to the 2×2 matrix P .

Let $V_{ij} = \sqrt{p_{ij}}|i\rangle\langle j|$ where $P = (p_{ij})$ is a irreducible (in the classical sense for a Markov chain) k by k column stochastic matrix, $\mu = \sum_{ij} \delta_{V_{ij}}$ and $L = I$. In this case, we get that

$$V_{ij}^* V_{ij} = p_{ij} |j\rangle\langle j|,$$

is a diagonal matrix and, if $A = (a_{ij})$,

$$V_{ij}^* A V_{ij} = p_{ij} a_{ii} |j\rangle\langle j|.$$

Therefore, when $\omega = (V_{i_n j_n})$, we have

$$W_2(\omega)^* W_2(\omega) = p_{i_1 j_1} p_{i_2 j_2} |j_2\rangle\langle j_2| \delta_{i_2 j_1},$$

in which $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$. By induction,

$$W_n(\omega)^* W_n(\omega) = \left(\prod_{k=1}^n p_{i_k j_k} \right) \left(\prod_{k=1}^{n-1} \delta_{i_{k+1} j_k} \right) |j_n\rangle\langle j_n|.$$

Thus, $W_n(\omega)^* W_n(\omega)$ is 0 or a diagonal matrix with a unique entry different from 0. This entry is exactly $(\prod_{k=1}^n p_{i_k j_k})$ which implies that

$$\|W_n(\omega)^* W_n(\omega)\| = \left(\prod_{k=1}^n p_{i_k j_k} \right) = p_{11}^{X_{11,n}(\omega)} p_{12}^{X_{12,n}(\omega)} p_{21}^{X_{21,n}(\omega)} p_{22}^{X_{22,n}(\omega)},$$

in which

$$X_{ij,n}(\omega) = \sum_{k=0}^{n-1} 1_{[V_{ij}]} \circ \theta^k(\omega),$$

with $1_{[V_{ij}]}$ being the characteristic function of cylinder $[V_{ij}]$.

Note that under the ergodicity hypothesis we get the property: for any i, j and $\mathbb{P}$ -almost sure ω , we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[V_{ij}]} \circ \theta^k(\omega) = \mathbb{P}([V_{ij}]).$$

It follows from the arguments of example 8.5 in [11] that the pair (L, μ) satisfies the purification condition, the ϕ -Erg conditions, and also irreducibility.

Remember that for given matrix $A \in M_k(\mathbb{C})$, $a_1(A) \geq a_2(A) \geq \dots \geq a_k(A)$ are the singular values of A , i.e., the square roots of eigenvalues of A^*A , labeled in

decreasing order. From Lemma III.5.3 in [9] we have

$$\left\| \bigwedge^p W_n(\omega) \right\| = a_1(W_n) \dots a_p(W_n).$$

Therefore, $\left\| \bigwedge^1 W_n(\omega) \right\| = a_1(W_n) = \|W_n^* W_n\|^{\frac{1}{2}}.$

Following Proposition 7.2 (which corresponds to Proposition 4.3 in [8]) we can obtain the greater Lyapunov exponent γ_1 taking the limit

$$\begin{aligned} \gamma_1 &:= \lim_{n \rightarrow \infty} \frac{1}{n} \left\| \bigwedge^1 W_n(\omega) \right\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log(\|W_n(\omega)^* W_n(\omega)\|^{\frac{1}{2}}) \\ &= \frac{1}{2} \sum_{ij} \lim_{n \rightarrow \infty} \frac{X_{ij,n}(\omega)}{n} \log(p_{ij}) \\ &= \sum_{ij} \frac{\mathbb{P}([V_{ij}])}{2} \log(p_{ij}). \end{aligned}$$

We know from [11] that if Φ is the channel defined for such μ and L , then, Φ is Φ -Erg. Assume, $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ is the unique invariant probability vector for the stochastic matrix P . Then, the unique ρ , such that, $\Phi(\rho) = \rho$, is exactly the diagonal matrix ρ with entries $\pi_1, \pi_2, \dots, \pi_k$. We also know, in this case, that the entropy (see example 8.5 in [11]) for a channel as defined in [11] is equal to the classical Shannon-Kolmogorov entropy of the stationary Markov Process associated to the column stochastic matrix $P = (p_{ij})$ (see formula in [26]).

Now, we can estimate

$$\begin{aligned} \mathbb{P}([V_{ij}]) &= \int_{[V_{ij}]} \text{tr}(v \rho v^*) d\mu(v) = \text{tr}(V_{ij}^* V_{ij} \rho) = \text{tr}(p_{ij} |j\rangle \langle j| \rho) = p_{ij} \langle j | \rho | j \rangle \\ &= p_{ij} \pi_j. \end{aligned}$$

Therefore,

$$(3) \quad \gamma_1 = \frac{1}{2} \sum_{i,j \in \{0,1\}} \pi_j p_{ij} \log(p_{ij}) = -\frac{1}{2} h,$$

in which h is the entropy of the Markov invariant measure associated to the matrix P .

The value $\frac{1}{2}$ which multiplies the entropy on the above expression is due to the fact that we considered the norm $\|A\| = \langle A, A \rangle^{1/2}$.

Now we estimate the second Lyapunov exponent γ_2 .

We showed that $W_n(\omega)^* W_n(\omega) = (\prod_{k=1}^n p_{i_k j_k}) \left(\prod_{k=1}^{n-1} \delta_{i_{k+1} j_k} \right) |j_n\rangle \langle j_n|$, which implies that the second eigenvalue is 0 and therefore $a_2(W_n(\omega)) = 0$.

Now, we can get γ_2 , in fact,

$$\gamma_1 + \gamma_2 = \lim_n \frac{1}{n} \log(a_1(W_n(\omega))a_2(W_n(\omega))) = \lim_n \frac{1}{n} \log(0),$$

which implies that $\gamma_2 = -\infty$.

9. THE PURIFICATION CONDITION IS GENERIC

The measure μ is fixed from now on.

Our main goal in this section is to show:

Proposition 9.1. *Given μ over M_k with $\# \text{supp } \mu > 1$, the set of L satisfying the purification condition is generic in $\mathcal{B}(M_k)$.*

This will follow from Lemma 9.14.

Definition 9.2. *We say that the projection π n -purifies $L : M_k \rightarrow M_k$, in which $\text{rank } \pi \geq 2$, if there exists $E \in \mathcal{O}_n$, with $\mu^{\otimes n}(E) > 0$, such that,*

$$\pi W_n(\omega) W_n(\omega) \pi \not\propto \pi,$$

for all $\omega \in E$.

In order to prove that a certain L satisfies the purification condition, we have to consider all possible projections π (see definition 6.4).

Observe that if Q is a unitary matrix, π has rank great or equal to 2 and n -purifies L for $E \in \mathcal{O}_n$, then

$$Q\pi Q^* Q W_n(\omega)^* W_n(\omega) Q^* Q \pi Q^* \not\propto Q\pi Q^*.$$

Besides that, $W_n(\omega) = L(\omega_n) \dots L(\omega_1)$, so, as $Q^* Q = \text{Id}_k$, we have

$$Q W_n(\omega)^* W_n(\omega) Q^* =$$

$$QL(\omega_1)^* Q^* Q \dots Q^* QL(\omega_n)^* Q^* QL(\omega_n) Q^* Q \dots Q^* QL(\omega_1) Q^* Q.$$

From this follows:

Proposition 9.3. *If $L_Q(v) := QL(v)Q^*$, then for a projection π , such that, $\text{rank } \pi \geq 2$ and an unitary matrix Q , it's true that*

$$\pi \text{ } n\text{-purifies } L \iff Q\pi Q^* \text{ } n\text{-purifies } L_Q.$$

Definition 9.4. *For an orthogonal projection π and $n \in \mathbb{N}$ we define*

$$Pur_\pi^n = \{L \in \mathcal{B}(M_k) \mid \pi \text{ } n\text{-purifies } L\}.$$

Note that if

$$Pur = \{L \in \mathcal{B}(M_k) \mid \Phi_L \text{ satisfies (Pur) condition } \},$$

and if we denote

$$P_2 = \{\pi \text{ orthogonal projection} \mid \text{rank } \pi \geq 2\},$$

it follows that

$$Pur = \bigcap_{\pi \in P_2} \bigcup_{n \in \mathbb{N}} Pur_\pi^n.$$

Proposition 9.5. *For any $\pi \in P_2$ and $n \in \mathbb{N}$, Pur_π^n is open.*

proof: Take $\pi \in P_2$ with $\text{rank } \pi = l$, Q an unitary matrix that diagonalizes π . Suppose that

$$\tilde{\pi} := Q\pi Q^* = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & \dots & 0 \end{bmatrix}.$$

So, if $L \in Pur_\pi^n$, we know that $\tilde{\pi}$ n -purifies L_Q .

If

$$(s_{ij}^L(\omega)) := QW_n(\omega)^* W_n(\omega) Q^*$$

we know that s_{ij}^L are continuous functions. Moreover, there exists $\omega_0 \in \text{supp } \mu^{\otimes n}$ such that at least one of following conditions occurs:

1. There exists $i \neq j$, such that, $s_{ij}^L(\omega_0) \neq 0$, or there exists $j = i > \text{rank } \tilde{\pi}$, such that, $s_{ii}^L(\omega_0) \neq 0$.

In this case, we define the matrix $s^L(\omega) := s_{ij}^L(\omega)$;

2. There exists $i \neq j$, such that, $s_{ii}^L(\omega_0) - s_{jj}^L(\omega_0) \neq 0$.

In this case, we define $s^L(\omega) := s_{ii}^L(\omega) - s_{jj}^L(\omega)$.

It is clear that for $F \in \mathcal{B}(M_k)$ with $\|L - F\| \leq \varepsilon$, we have, for ε small enough, that $s^F(\omega_0) \neq 0$, because $s^F(\omega_0)$ has a continuous dependence on F . Furthermore, s^F is continuous, so there is a open set B with $\omega_0 \in B$, such that, $s^F(\omega) \neq 0$, for $\omega \in B$. Moreover, $\omega_0 \in \text{supp } \mu^{\otimes n}$, which implies that $\mu^{\otimes n}(B) > 0$, and therefore $F \in \text{Pur}_\pi^n$.

□

We point out that $s^L(\omega_0) \neq 0$ is the good condition for purification.

Proposition 9.6. *For any $\pi \in P_2$, Pur_π^1 is dense.*

proof: Take $L \notin \text{Pur}_\pi^1$ and Q unitary matrix that diagonalizes π as above. If $L \notin \text{Pur}_\pi^1$, defining (s_{ij}^L) as in the previous proposition, we know that $s_{11}(v) = s_{22}(v)$, for almost every v . If $D = |1\rangle\langle 1|$, and $\varepsilon > 0$, we consider $L_Q^\varepsilon(v) = L_Q(v) + \varepsilon D$. So, we have

$$\begin{aligned} L_Q^\varepsilon(v)^* L_Q^\varepsilon(v) &= (L_Q(v) + \varepsilon D)^* (L_Q(v) + \varepsilon D) \\ &= L_Q(v)^* L_Q(v) + \varepsilon L_Q^*(v) D + \varepsilon D L_Q(v) + \varepsilon^2 D. \end{aligned}$$

This perturbation changes the values $s_{11}^L(v)$ but not $s_{22}^L(v)$, which implies that $Q\pi Q^*$ n -purifies L_Q^ε . Now, we only need to take $L^\varepsilon(v) = L(v) + \varepsilon Q^* D Q$ and, as $\|L - L^\varepsilon\|$ is small, it will follow the density property at once.

□

Definition 9.7. *For any $\pi \in P_2$, we define $\text{Pur}_\pi = \bigcup_{n \in \mathbb{N}} \text{Pur}_\pi^n$.*

Note that by the two propositions above given a fixed π the set Pur_π is open and dense on $\mathcal{B}(M_k)$. Note also that for the purification condition it is necessary to consider all possible projections π (see definition 6.4).

Lemma 9.8. *If $\pi_1, \pi_2 \in P_2$ has the same rank, $E_i := Im \pi_i$, and $\{x_i\}$ is an orthonormal basis for E_i , then, $\{\pi_2 x_i\}$ is a basis for E_2 if π_1 and π_2 are close enough.*

proof: The proof will be by contradiction. Suppose $rank \pi_1 = l$ and $y_i := \pi_2 x_i$ are linearly dependent. So, the dimension generated by $\{y_i\}$ is at most $l - 1$. Then, there exists a vector $\hat{y} \in E_2$, which has norm 1 and it is orthogonal to the subspace generated by $\{y_i\}$. Therefore, $\langle \hat{y}, y_i \rangle = 0$, for all i . This implies that $\langle \hat{y}, \pi_2 x_i \rangle = \langle \pi_2 \hat{y}, x_i \rangle = \langle \hat{y}, x_i \rangle = 0$ and, moreover, $\pi_1 \hat{y} = 0$. Finally, we get $\|\pi_1 - \pi_2\| \geq \|\pi_1 \hat{y} - \pi_2 \hat{y}\| = \|\hat{y}\| = 1$.

If we assume that $\|\pi_1 - \pi_2\| < 1$, we are done.

□

Observe that, for $i \neq j$ and $\|\pi_2 - \pi_1\| < \varepsilon$,

$$|\langle y_i, y_j \rangle| = |\langle y_i, x_j \rangle| = |\langle y_i - x_i, x_j \rangle| \leq \|y_i - x_i\| \|x_j\| = \|y_i - x_i\|$$

$$(4) \quad \leq \|\pi_2 - \pi_1\| < \varepsilon$$

The set of y_i is not an orthonormal basis.

We would like to get an orthonormal basis close to the orthogonal basis $x_1, \dots, x_n$. Our aim is to prove corollary 9.12 which claims that, given ε there exists an orthonormal basis (u_i) for E_2 with $\|u_i - x_i\| < C\varepsilon$, for some constant $C > 0$. In this direction, we will perform a Gram-Schmidt normalization procedure.

Denote $u_1 := \frac{y_1}{\|y_1\|}$, $N_i := y_i - \sum_{j=1}^{i-1} \langle y_i, u_j \rangle u_j$ and $u_i := \frac{N_i}{\|N_i\|}$, for $i > 1$. Then, we have

$$\begin{aligned}
\|u_1 - x_1\| &= \left\| \|\pi_2 x_1\|^{-1} \pi_2 x_1 - \|\pi_2 x_1\|^{-1} x_1 + \|\pi_2 x_1\|^{-1} x_1 - x_1 \right\| \\
&\leq \|\pi_2 x_1\|^{-1} \|\pi_2 x_1 - \pi_1 x_1\| + \|x_1\| \left| \|\pi_2 x_1\|^{-1} - 1 \right| \\
&\leq \|\pi_2 x_1\|^{-1} \varepsilon + \left| \|\pi_2 x_1\|^{-1} - 1 \right| \\
&< \varepsilon(1 - \varepsilon)^{-1} + (1 - \varepsilon)^{-1} \varepsilon \\
&< 4\varepsilon,
\end{aligned}$$

and

$$|\|\pi_2 x_i\| - \|x_i\|| \leq \|\pi_2 x_i - \pi_1 x_i\| < \varepsilon \implies 1 - \varepsilon < \|\pi_2 x_i\| < 1 + \varepsilon.$$

Furthermore,

$$\begin{aligned}
\|y_2 - \langle y_2, u_1 \rangle u_1\| &= \left\| y_2 - \|\pi_2 x_i\|^{-2} \langle y_2, y_1 \rangle y_1 \right\| \\
&\leq \|x_2\| + (1 - \varepsilon)^{-2} \varepsilon \|x_1\| \\
&< 1 + 4\varepsilon.
\end{aligned}$$

Proposition 9.9. *For any $i \in \{1, \dots, n\}$, $j < i$, there is $C_{ij} > 0$, such that, $|\langle y_i, u_j \rangle| < C_{ij}\varepsilon$.*

proof:

Take $N := \min \|N_i\| > 0$.

Observe that $|\langle y_2, u_1 \rangle| \leq N^{-1}\varepsilon$ (this follows from a similar procedure described in (4) and the Cauchy-Schwartz inequality).

Then suppose, for all $l < i$, $|\langle y_l, u_j \rangle| \leq C_{lj}\varepsilon$, and for all $j < l$, with $C_{lj} > 0$. If $j < i$, we have

$$\begin{aligned}
|\langle y_i, u_j \rangle| &\leq N^{-1} |\langle y_i, y_j \rangle| + N^{-1} \sum_{k=1}^{j-1} |\langle y_j, u_k \rangle| \\
&\leq N^{-1}\varepsilon + N^{-1} \sum_{k=1}^{j-1} C_{jk}\varepsilon \\
&= \left(1 + \sum_{k=1}^{j-1} C_{jk}\right) N^{-1}\varepsilon.
\end{aligned}$$

Taking $C_{ij} = \left(1 + \sum_{k=1}^{j-1} C_{jk}\right) N^{-1}$ the claim follows by induction.

□

Proposition 9.10. *For all i , $|\|N_i\| - 1| < K\varepsilon$, for some $K > 0$.*

proof:

$$\begin{aligned} \|N_i\| &\leq 1 + \|y_i - x_i\| + \sum_{j=1}^{i-1} |\langle y_i, u_j \rangle| \\ &\leq 1 + \varepsilon + \left(\sum_{j=1}^{i-1} C_{ij} \sum \right) \varepsilon \\ &= 1 + \left(1 + \left(\sum_{j=1}^{i-1} C_{ij} \right) \right) \varepsilon. \end{aligned}$$

Taking $K = \left| 1 + \sum_{j=1}^{i-1} C_{ij} \right|$ the proof is completed.

□

Proposition 9.11. *For all i , we have $\|u_i - x_i\| < C_i\varepsilon$.*

proof:

$$\begin{aligned} \|u_i - x_i\| &\leq N^{-1} \|N_i - x_i\| + \|\|N_i\| x_i - x_i\| \\ &\leq N^{-1} \|y_i - x_i\| + N^{-1} \sum_{j=0}^{i-1} |\langle y_i, u_j \rangle| + |\|N_i\| - 1| \\ &\leq N^{-1} \varepsilon + N^{-1} \sum_{j=1}^{i-1} C_{ij} + K\varepsilon \\ &= \left(N^{-1} + N^{-1} \sum_{j=1}^{i-1} C_{ij} + K \right) \varepsilon. \end{aligned}$$

Define $C_i := N^{-1} + N^{-1} \sum_{j=1}^{i-1} C_{ij} + K$ and the statement has been proved.

□

Corollary 9.12. *There exists $C > 0$, such that, for all $\varepsilon > 0$, there exists an orthonormal basis (u_i) for E_2 with $\|u_i - x_i\| < C\varepsilon$.*

If we repeat the process, but now for $E_1^\perp$ and $E_2^\perp$, we get another constant C_2 , and in a similar way, we obtain new vectors (u_i) from the (x_i) . These u_i define an orthonormal basis for $\mathbb{C}^k$ with $\|u_i - x_i\| < C_2\varepsilon$.

Now we define Q_1, Q_2 , such that $Q_1 x_i = e_i$ and $Q_2 u_i = e_i$, in which (e_i) is a canonical basis for $\mathbb{C}^k$. Then, $Q_i \pi_i Q_i^*$ is a diagonalization for π_i . Observe that

$$\|Q_2 Q_1^* - I\| = \|Q_2 Q_1^* - Q_2 Q_2^*\| < C_3 \varepsilon,$$

the map $A \rightarrow Q_2 Q_1^* A (Q_2 Q_1^*)^*$ is continuous and, in addition, this map is close to the identity map.

We know that if $L \in \text{Pur}_{\pi_1}^n$, then, we take s_1^L from the π_1, Q_1 and L , as in proposition 9.5. In the same way, there exists $\omega_0 \in \text{supp } \mu$ with $s_1^L(\omega_0) \neq 0$. Observe that $s_1^L(\omega_0)$ depends only on the coordinates of $Q_1 W_n^L(\omega_0)^* W_n^L(\omega_0) Q_1^*$. Now, applying $Q_2 Q_1^* (\cdot) (Q_2 Q_1^*)^*$ we get $Q_2 W_n^L(\omega_0)^* W_n^L(\omega_0) Q_2^*$. Note that this is the same as considering π_2, Q_2, L and the associated $s_2^L(\omega_0)$. If ε is small enough, $s_2^L(\omega_0) \neq 0$ and we can repeat the argument used in proposition 9.5 in order to obtain an open set B , such that, $\omega_0 \in B$ and, moreover, if $\omega \in B$ then $s_2^L(\omega) \neq 0$. Therefore, $L \in \text{Pur}_{\pi_2}^n$.

The previous arguments prove the following lemma.

Lemma 9.13. *If $L \in \text{Pur}_{\pi_1}$ and π_1, π_2 are close enough, then $L \in \text{Pur}_{\pi_2}$.*

Lemma 9.14. *Take K_2 as a countable dense subset of P_2 . Then,*

$$\text{Pur} = \bigcap_{\pi \in K_2} \text{Pur}_{\pi}.$$

proof: We will use the classical Baire Theorem.

Suppose that $L \in \bigcap_{\pi \in K_2} \text{Pur}_{\pi}$, then for lemma 9.13, for every $\pi \in K_2$, as $L \in \text{Pur}_{\pi}$, there exists $\varepsilon(\pi)$, such that, if $\hat{\pi} \in P_2$ and $\|\pi - \hat{\pi}\| < \varepsilon(\pi)$, then $L \in \text{Pur}_{\hat{\pi}}$. Furthermore, if we define $B(\pi) = \{\hat{\pi} \in P_2 \mid \|\hat{\pi} - \pi\| < \varepsilon(\pi)\}$, then $\bigcup_{\pi \in K_2} B(\pi)$ covers P_2 . Therefore, for any $\hat{\pi} \in P_2$, there is $\pi \in K_2$, such that, $\hat{\pi} \in B(\pi)$ and thus $L \in \text{Pur}_{\hat{\pi}}$. This implies that $L \in \bigcap_{\pi \in P_2} \text{Pur}_{\pi} = \text{Pur}$.

□

10. APPENDIX - RELATIONS WITH PREVIOUS WORKS AND WITH CLASSICAL DYNAMICAL SYSTEMS

In [11] it was we introduced a concept of quantum entropy and pressure which was used here. In the classical dynamical setting the entropy is obtained via the

limit of finite dynamical partitions. The definition of quantum entropy considered in [11] is not inspired in this procedure. The version considered in [11] (and also in here) is more on the spirit of Rokhlin Formula (see [31]). This is in accordance with the point of view of considering *a priori* probabilities in the definition of entropy (see [21]). Although at first glance it is not transparent, the classical concept of entropy is associated with an *a priori* measure (not an *a priori* probability according to the discussion in [21] and [6]).

In order to make more clear the need - in the Dynamical Systems setting - of an *a priori* probability consider the shift σ acting on the symbolic space $K^{\mathbb{N}}$, where K is a compact metric space (not necessarily countable as in [6] where K is the unit circle). In this case, entropy can not be defined in a natural way by the classical procedure obtained via the limit of dynamical partitions. Each point $x \in K^{\mathbb{N}}$ may have an uncountable number of preimages by the shift. We consider a probability μ on K which will be called the *a priori* probability. Entropy can be defined however, via the Ruelle operator, which in turn requires an *a priori* probability to be defined (in the dynamical setting this is described in [21]). Entropy and the Ruelle operator are connected in a natural and fundamental way. If $K = \{1, 2, \dots, d\}$ is natural to consider μ as the counting measure (or the normalized counting probability) and this will give rise to the classical Ruelle operator (as described in [24]). The dual of the Ruelle operator plays a very important role in thermodynamic formalism (see [24]).

The time evolution considered in section 6 requires a quantum entropy with a dynamical flavor. In the present setting, the probability μ on M_k of Section 2 will play the role of an *a priori probability*. To emphasize the differences of the various concepts of entropy we denote in this appendix the entropy of the channel ϕ_L , when L is stochastic, by $\mathfrak{h}(\phi_L)$.

Our point of view for defining the concept of quantum entropy is in consonance with [3], [4], [5] and [28]. A confirmation that the entropy considered here is in fact a concept that describes valuable information, from a dynamic point of view, is its relationship with Lyapunov exponents in the case of the *Markov model in quantum information* (see expression (2)).

Section 9 in [11] describes several distinct concepts of quantum entropy which are well known in the literature in Quantum Information (but different from ours). As mentioned there the von Neumann entropy is not exactly of dynamical nature.

Consider $H : M_k \rightarrow M_k$ we defined the Ruelle operator $\phi_H : M_k \rightarrow M_k$:

$$\rho \rightarrow \phi_H(\rho) = \int_{M_k} H(v)\rho H(v)^\dagger d\mu(v),$$

where μ is a probability on M_k . H plays the role of an Hamiltonian.

In [11] it was proved a version of the Ruelle Theorem: a variational principle of pressure related to an eigenvalue problem for the Ruelle operator. Entropy and the Ruelle operator are also connected in a natural and fundamental way in our setting (see section 4 in [11]).

In cite [11], given an Hamiltonian $H : M_k \rightarrow M_k$, it is introduced the concepts of Gibbs channel and equilibrium channel for H .

We present in this appendix three tables where the reader can compare different settings. The first table, with three columns, describes the set of correspondences (definitions and results) on the topics: in the left side of the table we consider Markov Process (with values on a finite set), in the middle we outline results in Thermodynamic Formalism for a symbolic space with symbols in a metric space, and on the right side of the table we present the setting of Quantum Information. This table presents an outline of the sequence of results of [11]. We believe this will help the reader to understand our reasoning. For example: the Perron Theorem, the Ruelle Theorem, and the Evans-Høegh-Krohn Theorem are on the same line, each one in one distinct column. They play similar roles in each one of the settings.

In the second table we outline the main points describing the *Markov model in quantum information* considered in Section 8.

In the third table we make a comparison of the results on Lyapunov exponents of the present paper with the ones in Markov Chains and Thermodynamic Formalism.

In the tables [BFPP] corresponds to results in [8].

In order to make a comparison of our main result with the classical study of Lyapunov exponents in Ergodic Theory, we briefly describe below some important theorems in this long-established theory.

Given a $C^{1+\alpha}$ expanding d to 1 transformation $T : S^1 \rightarrow S^1$, there exist a unique ergodic probability measure $d\rho(x)$ (abs. con. with respect to Lebesgue measure)

for T . The Kolmogorov entropy of this probability satisfies

$$(5) \quad h(\rho) = \int \log |T'(x)| d\rho(x).$$

This is a nice relation between entropy and Lyapunov exponent in a smooth case (see [26]).

The Oseledec Theorem describes the ergodic properties of the product of random matrices (see [32]).

Now, we will describe a more complex example concerning the smooth case: consider M a compact manifold of dimension 2 and a C^∞ Anosov diffeomorphism $F : M \rightarrow M$ which is volume-preserving. We denote $dF(x)$ the derivative of F in x , which is defined for tangent vectors $v \in TM_x$. The tangent plane TM_x is decomposed in the sum of two bundles $E_s(x) \oplus E_u(x)$, where the norm of the vectors v on $E_u(x)$ contracts by the action of DF^{-1} along orbits by F . The bundle $E_u(x)$ is called the unstable bundle.

We denote the Lebesgue probability by m , which is known to be ergodic for such diffeomorphism F . We consider the function $x \in M \rightarrow DF^{-1}(x)$ and then the random products of the inverse of the derivative DF^{-1} along orbits.

A version of Oseledec Theorem claims in this case that there exist $\gamma_1 > 0 > \gamma_2$, s. t

$$\lim_n \frac{1}{n} \log |[(DF^{-1}(F^{n-1}(x))) \dots DF^{-1}(x)](v)| = \gamma_1 \text{ or } \gamma_2,$$

for Lebesgue a.e.w. x , and $\forall v \in TM_x$. Moreover, $\gamma_1 + \gamma_2 = 0$.

The value $\gamma_2 < 0$ describes the mean value of contraction of the norm of vectors $v \in E_u(x)$ by DF^{-n} , in the logarithm scale, of a random $x \in M$ under the iteration by F^n .

An important result in the theory is the following:

Pesin Theorem: Consider a compact manifold M of dimension 2 and a C^∞ Anosov diffeomorphism $F : M \rightarrow M$, which is volume preserving. Then,

$$(6) \quad \gamma_2 = -h(m),$$

where $h(m)$ is the Kolmogorov entropy of the Lebesgue probability m .

We point out that Sections 13.1 and 13.2 in [1] consider results related to quantum versions of Ruelle inequality and Pesin Theorem but for a different setting.

We will show that our result - for the Lyapunov exponent of a Markov model in Quantum Information - is analogous to the Pesin Theorem. We will elaborate on

this claim following the notation of [8]. Thus, the reader will have a synthetic view of the chain of results in [8] which were used in our work.

Take $\Omega = M_k^{\mathbb{N}}$, and for $n \in \mathbb{N}$ let $\mathcal{O}_n$ be the σ -algebra on Ω generated by the n -cylinder sets. An element on Ω is of the form $(\omega_1, \omega_2, \dots, \omega_n, \dots)$.

We also denote $W_n(\omega) = L(\omega_n) \dots L(\omega_1)$.

Taking $O_n \in \mathcal{O}_n$ it was defined

$$\mathbb{P}(O_n) = \int_{O_n} \text{tr}(W_n(\omega) \rho W_n(\omega)^*) d\mu^{\otimes n}(\omega).$$

If $\mathcal{O}$ is the smallest σ -algebra of Ω that contains all $\mathcal{O}_n$ we can extend the action of $\mathbb{P}$ to this σ -algebra.

The probability $\mathbb{P}$ on Ω defines a Stationary Stochastic Process.

Theorem $(\Omega, \mathbb{P}, \theta)$ is ergodic where θ is the shift map.

The above theorem has been proved when $L = Id$ in Lemma 4.2 in [8].

Now, we will briefly describe the results that are necessary for the study of Lyapunov exponents in the setting of quantum information (adapted from [8]).

$x_1 \wedge x_2 \wedge \dots \wedge x_n$, with $x_j \in \mathbb{C}^k$, denotes the classical wedge product (an alternate form on $\mathbb{C}^k$).

One can consider an inner product

$$\langle r_1 \wedge r_2 \wedge \dots \wedge r_n, s_1 \wedge s_2 \wedge \dots \wedge s_n \rangle = \det(r_i s_j)_{i,j=1,2,\dots,n},$$

and, the associated norm $|x_1 \wedge x_2 \wedge \dots \wedge x_n|$.

Given an operator $X : \mathbb{C}^k \rightarrow \mathbb{C}^k$ we define $\bigwedge^n X : \bigwedge^n \mathbb{C}^k \rightarrow \bigwedge^n \mathbb{C}^k$ by $\bigwedge^n X(x_1 \wedge x_2 \wedge \dots \wedge x_n) = X(x_1) \wedge X(x_2) \wedge \dots \wedge X(x_n)$.

The next theorem is an easy adaptation of the reasoning which was considered in [8] for the case $L = Id$. Below we consider the case of dimension 2.

Theorem. Suppose the pair (L, μ) satisfies irreducibility, the ϕ -Erg and the purification condition. Assume also that $\int |L(v)|^2 \log |L(v)| d\mu(v) < \infty$, then, there exists numbers

$$\infty > \gamma_1 \geq \gamma_2 \geq -\infty,$$

such that, for any probability ν over $P(\mathbb{C}^k)$ and any $p \in \{1, 2\}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \bigwedge^p W_n \right| = \sum_{j=1}^p \gamma_j,$$

$\mathbb{P}_\nu$ -a.s.

Consider a 2×2 stochastic matrix P and the associated Markov invariant probability in $\{0, 1\}^{\mathbb{N}}$.

In [11] (see also [3], [4] and [5]) it was shown that

$$\mathfrak{H}(\phi_{Id}) = - \sum_{i,j=0}^1 \pi_j p_{ij} \log(p_{ij}) = h \geq 0,$$

where h is the entropy of the Markov invariant measure associated to the matrix P and ϕ_{Id} is the channel (the Markov model in quantum information) described in our section 8.

One of our main results here is for the Markov model in quantum information:

$$(7) \quad \gamma_1 = -\frac{1}{2} \mathfrak{H}(\phi_{Id}),$$

and also that $\gamma_2 = -\infty$.

The above result is a quantum information version of (6).

Therm. Form. for Markov probabilities	Therm. Form. with symbols on comp. met. space K	Quantum Channel Formalism
state space S is $S = \{1, 2, \dots, d\}$	state space $S = K$ is a compact metric space	state space $S = M_k$ is the set of $k \times k$ complex matrices M_k
consider a fixed <i>a priori</i> measure $\tilde{\mu} = \sum_{n=1}^d \delta_n$	consider a fixed a priori probability μ on Borel K ; Special case $K = \{1, 2, \dots, d\}$, $\mu = \frac{1}{d} \sum_{n=1}^d \delta_n = \frac{1}{d} \tilde{\mu}$	consider a fixed a priori probability μ on Borel M_k
shift $\sigma : \Omega = \{1, 2, \dots, d\}^{\mathbb{N}} \hookrightarrow$	shift $\sigma : \Omega = K^{\mathbb{N}} \hookrightarrow$	shift $\sigma : \Omega = M_k^{\mathbb{N}} \hookrightarrow$, on $\Omega = D_k^{\mathbb{N}}$, or on $\Omega = S_A^{\mathbb{N}}$, with D_k = density matrices, or $S_A k$ = self adjoint matrices
function $f : \{1, 2, \dots, d\} \rightarrow \mathbb{R}$, $f = (f_1, f_2, \dots, f_d)$, $f = \sum_{j=1}^d f_j x_j$ depends on the first variable x_1	Continuous function $f : K^{\mathbb{N}} \rightarrow \mathbb{R}$ (observable) $f(x) = f(x_1, x_2, x_3, \dots)$, $x_j \in K$, $j \in \mathbb{N}$	$L : M_k \rightarrow M_k$, square integrable transformation, or continuous, and from L we define the operator $M \rightarrow \phi_L^*(M) = \int_{M_k} L(B)^{\dagger} M L(B) \mu(B)$
Potential $\tilde{A} : \{1, 2, \dots, d\}^{\mathbb{N}} \rightarrow \mathbb{R}$ depen. on first 2 var., $\tilde{A}(x_1, x_2, x_3, \dots) = \tilde{A}(x_1, x_2)$ constant in cylinders x_1, x_2 . Such $\tilde{A} : \{1, 2, \dots, d\}^{\mathbb{N}} \rightarrow \mathbb{R}$ is of Holder class.	Continuous Potential $A : K^{\mathbb{N}} \rightarrow \mathbb{R}$; $A(x) = A(x_1, x_2, x_3, \dots)$, $x_j \in K$; interaction in the lattice $\mathbb{N}$	a square integrable $H : M_k \rightarrow M_k$, called Hamiltonian, defines the Hamiltonian operator ϕ_H^* (acting in $S_A k$): $M \in M_k \rightarrow \phi_H^*(M) = \int_{M_k} H(B)^{\dagger} M H(B) \mu(B)$
Duality: line vector $l \leftrightarrow$ column vector r	Duality: functions $\leftrightarrow$ mesures	Duality in M_k : self adjoint matrices $S_A k \leftrightarrow D_k$ density matrices
Matrix Operation $\tilde{A}$ on line vector $f = (f_1, \dots, f_d) \in \mathbb{R}^d$, left vectors are functions. Action on the left: $f \rightarrow g = f \tilde{A}$, $g(j) = \mathcal{L}_{\log \tilde{A}}(f)(j) = \sum_{i=1}^d e^{i \log \tilde{A}(j)} f(i) = \sum_{i=1}^d \tilde{A}_{ij} f(i) \in \mathbb{R}^d$ matrix $\tilde{A} = (\tilde{A}_{ij})_{i,j=1}^d$ acts on right vectors in $\mathbb{R}^d$, right vectors are measures π on $\{1, 2, \dots, d\}$, $\pi_1 \rightarrow \tilde{A} \pi_1 = \pi_2$, action on the right.	Ruelle for $A : K^{\mathbb{N}} \rightarrow \mathbb{R}$ Holder: acts on continuous functions $f : K^{\mathbb{N}} \rightarrow \mathbb{R}$, via $g(x) = \mathcal{L}_A^*(f)(x) = \mathcal{L}_A(f)(x) = \int e^{A(x)} f(\sigma x) d\mu(x)$ $A : \Omega \rightarrow \mathbb{R}$ will induce an operation on measures: take $\mathcal{L}_A^*$ Riesz dual of $\mathcal{L}_A$, $\mathcal{L}_A^*(\mu_1) = \mu_2$, if $\int g d\mu_2 = \int \mathcal{L}_A(f) d\mu_1$, $\forall$ cont. g .	"Quantum Ruelle operator" ^o acts on M_k (L looks like $(e^{A(x)})^{1/2}$), $M \rightarrow \phi_L^*(M) = \int_{M_k} L(B)^{\dagger} M L(B) \mu(B)$ M self adjoint $\Rightarrow \phi_L^*(M)$ self adjoint $L : M_k \hookrightarrow$ also defines: $M \rightarrow \phi_L(M) = \int_{M_k} L(B) M L(B)^{\dagger} \mu(B)$, $\rho \in M_k$ positive matrix $\Rightarrow \phi_L(\rho)$ positive matrix
Duality: line vector $l \leftrightarrow$ column vector r	Duality: functions $\leftrightarrow$ mesures	Duality of operators: $\phi_L^* : M_k \rightarrow M_k \leftrightarrow \phi_L : M_k \rightarrow M_k$, Quantum Ruelle operator $\leftrightarrow$ Quantum dual Ruelle operator
column stochastic matrix $P = (P_{ij})_{i,j=1}^d$ = transition kernel, action: $\pi_1 \rightarrow \pi_2 = P \pi_1$, P preserves prob. on $\{1, \dots, d\}$	$A = \log P$ is called normalized if $\mathcal{L}_{\log J}^*(1) = \mathcal{L}_{\log J}(1) = 1$, if μ_1 probability on Ω , then $\mathcal{L}^*(\mu_1) = \mu_2$ is a probability on Ω	L stochastic if $\phi_L^*(1) = 1_{M_k}$. (this L looks like $(e^{i \log J(x)})^{1/2}$), if $\rho_1 \in M_k$ is a density matrix $\Rightarrow \rho_2 = \phi_L(\rho_1)$ is a density matrix. We denote $\Phi = \{\text{all stochastic } L : M_k \rightarrow M_k\}$
Given $J_m(i, j, \dots) = P_{ij}$, then, $\mathcal{L}_{\log J_m}(1) = 1$, μ_1 prob on Ω , then $\mu_2 = \mathcal{L}_{\log J_m}^*(\mu_1)$ prob. on Ω (*) initial prob. π on $\{1, 2, \dots, d\}$, s. t. $P \pi = \pi$, defines σ -inv prob. m , s. t. $m(\bar{a}_1, \dots, \bar{a}_n) = \pi_{a_1} \pi_{a_2} \dots \pi_{a_n}$, $a_i \in \{1, \dots, d\}$ P column stoc., $\log J = \log P_j$, all $P_j > 0$, $\mathcal{L}_{\log J}(1) = 1$, $\Leftrightarrow (1, 1, \dots, 1) P = (1, 1, \dots, 1)$ Stoch. Process X_n , transition P_{ij} const. on cylinders $\bar{t}_j$.	If $\log J$ is normalized: $\mathcal{L}_{\log J}(1) = 1 \Rightarrow$ if μ_1 prob. on Ω , then $\mu_2 = \mathcal{L}_{\log J}^*(\mu_1)$ is prob. on Ω If $\log J$ is normal, m is Gibbs for $J = J_m$, called Jacobian of m , if $\mathcal{L}_{\log J}^*(m) = m$. Prob. m is σ -inv. There is a bijection $m \leftrightarrow J_m$, Jacobian $J > 0$, $J : \Omega \rightarrow \mathbb{R}$, $\mathcal{L}_{\log J}(1) = 1$, Stoch. Process X_n , $n \in \mathbb{N}$, with transition $J(x_1, x_2, \dots, x_n, \dots)$	If $L \in \Phi$ is then, ϕ_L^* is a quantum channel, indeed $\rho_1 \rightarrow \phi_L(\rho_1)$ is completely positive and preserves trace if $L \in \Phi$, then $\rho_L \in D_k$ is called Gibbs density matrix for L , in the case $\phi_L^*(1) = \rho_L$ (ρ_L is GDM for L , for short)
m on Ω of Definition (*) is σ -invariant. Shannon-Kolmogorov entropy $h(m) = - \int \log J_m dm = - \sum_{i,j=1}^d \pi P_{ij} \log(P_{ij}) > 0$ l^A is left eigenvalue: if $l^A \in \mathbb{R}^d$, $\lambda^A > 0$, s. t. $l^A \tilde{A} = \lambda^A l^A$, If $\tilde{A} = P$, then $\lambda^A = 1$ and $l^A = (1, 1, \dots, 1)$.	entropy $-\infty < H(m) = H_\mu(m) = - \int \log J_m dm \leq 0$, in case $M = \{1, \dots, d\}$, $\mu = \frac{1}{d} \tilde{\mu}$, $H_\mu(m) = h(m) - \log d \leq 0$ $f^A : \Omega \rightarrow \mathbb{R}$ eigenfunction for $\mathcal{L}_A$ if $\mathcal{L}_A(f^A) = \lambda^A f^A$, If $A = \log J$, then $f^A = 1$, $\lambda^A = 1$.	If $L \in \Phi$ is irreducible, then ϕ_L^* is irreducible $\Rightarrow$ transition for a Stoch. Proc. X_n , $n \in \mathbb{N}$, the transition is $P_L(v, w) = \frac{\text{tr}(L(v) \phi_L(L(v)) L(v)^{\dagger} L(w))}{\text{tr}(L(v) \rho_L L(v))}$ If $L \in \Phi$ is irreducible, define the entropy of ϕ_L by $\mathcal{H}(\phi_L) = - \int \text{tr}(L(v) \rho_L L(v)^{\dagger}) P_L(v, w) \log P_L(v, w) dv dw$ $\sigma \in S_A k$ is eigenmatrix for ϕ_L^* if exists $\lambda > 0$, s. t. $\phi_L^*(\sigma) = \lambda \sigma$. If ϕ_L stochastic, then, $\phi_L^*(1) = 1 \Rightarrow \lambda = 1$ and $\sigma = 1d$
r^A is right eigenvalue: if $r^A \in \mathbb{R}^d$, $\lambda^A > 0$, s. t. $\tilde{A} r^A = \lambda^A r^A$, If $\tilde{A} = P$ there exists π such that $P \pi = \pi$, $\lambda = 1$	q^A on Ω is eigenprobability for $\mathcal{L}_A^*$, if $\mathcal{L}_A^*(q^A) = \lambda^A q^A$. In case of normalized $A = \log J$, then, $\lambda = 1$, $\mathcal{L}_{\log J_m}(m) = m$; Bijection relation $\log J_m \leftrightarrow m$	$\rho \in D_k$ is eigenmatrix for L if $\phi_L(\rho) = \lambda \rho$, for some $\lambda > 0$. If L stochastic, there exist GDM ρ_L s. t. $\phi_L(\rho_L) = \rho_L$ and $\lambda = 1$
Perron Theorem: given the matrix $\tilde{A}$ with positive entries there exist $\lambda^A > 0$, positive l^A , $r^A \in \mathbb{R}^d$, such that, $l^A \tilde{A} = \lambda^A l^A$, $\tilde{A} r^A = \lambda^A r^A$	Ruelle Theorem: for $A : \Omega \rightarrow \mathbb{R}$ Holder there exists $\lambda^A > 0$, probability q^A on Ω , and function $f^A > 0$, s. t. $\mathcal{L}_A(f^A) = \lambda^A f^A$ and $\mathcal{L}_A^*(q^A) = \lambda^A q^A$	Evans-Hoegh-Krohn: given L , there exist $\lambda_L > 0$, $\rho_L, \sigma_L \in M_k$, $\phi_L(\rho_L) = \lambda_L \rho_L$, $\phi_L^*(\sigma_L) = \lambda_L \sigma_L$; $\rho_L \in D_k$, $\sigma_L \in S_A$ SA positive
$\tilde{A}$ matrix; $x = (x_1, x_2, x_3, \dots) \rightarrow \log \tilde{A}(x) = \log \tilde{A}_{x_1 x_2} \in \mathbb{R}$	$x = (x_1, x_2, x_3, \dots) \rightarrow A(x) \in \mathbb{R}$	for $H : M_k \hookrightarrow \Rightarrow \log H(M) = \log \left(\text{tr} \left(\sigma_H H(M) \rho_H H(M)^{\dagger} \right) \right) \in \mathbb{R}$
Pressure for $\log \tilde{A}$ is $\Psi(\log \tilde{A}) = \sup_{p \rightarrow \text{inv } m} (h(m) + \int \log \tilde{A} dm)$	Pressure of A is $\Psi(A) = \sup_{p \rightarrow \text{inv } m} (H(m) + \int A dm)$	$\Psi(H) := \sup_{L \in \Phi} \mathcal{H}(\phi_L) + \int U_H(v) \text{tr}(L(v) \rho_L L(v)^{\dagger}) dv(v)$
$\tilde{A}(x) = \tilde{A}(x_1, x_2)$, Ruelle Theorem: $\Psi(\log \tilde{A}) = \log \lambda^A$	$A : \Omega \rightarrow \mathbb{R}$, Holder, Ruelle Theorem $\Psi(A) = \log \lambda^A$	Our theorem: $\Psi(H) = \log \lambda_H$, where λ_H comes from E-H Theorem
m^A on Ω is called equilibrium probability for $\log \tilde{A}$ if $\Psi(\log \tilde{A}) = h(m^A) + \int \log \tilde{A} dm^A$, m^A is unique because $\tilde{A}$ is Holder.	m^A on Ω is called equilibrium probability for A if $\Psi(A) = H(m^A) + \int A dm^A$, A Holder $\Rightarrow m^A$ unique. If A just continuous maybe not unique.	$L \in \Phi$ equil. mat. for H if $\Psi(H) = \mathcal{H}(\phi_L) + \int U_H \text{tr}(L \rho_L L^{\dagger}) dv$. Call ϕ_L the equil. quantum channel for the Hamiltonian H .
init. vec. of prob. π^A in $\{1, 2, \dots, d\}$, s. t. $\pi^A = l^A r^A$, that is $\pi_j^A = l_j^A r_j^A$, $j = 1, 2, \dots, d$, then $m^A(j) = l_j^A r_j^A$	m^A equil. for A : then, $\forall j \in \{1, \dots, d\}$, $m^A(j) = \int \tilde{A} dm^A$	If ρ_L, σ_L of Evans-Hoegh-Krohn Theorem are s. t. $\rho_L, \sigma_L \in D_k$, we call $\rho_L, \sigma_L \in D_k$ the equilibrium density matrix for L .
$m^A(\bar{a}_1, \dots, \bar{a}_n) = \pi_{a_1}^A \pi_{a_2}^A \dots \pi_{a_n}^A$, where P depends on $\tilde{A}$: $\log P_{ij} = \tilde{A}_{ij} + \log l_j^A - \log l_i^A - \log \lambda^A$, σ , in an equivalent form $P_{ij} = \frac{e^{\tilde{A}_{ij}}}{\pi_{a_i}^A \pi_{a_j}^A}$	m^A equil. for $A \Rightarrow m^A = f^A q^A$. Moreover, $m^A = m^{\text{inv } J_A}$, $m^A \Leftrightarrow \log J_A = A + \log f^A - (\log f^A \circ \sigma) - \log \lambda^A$, or, in an equivalent form $J = \frac{e^{A f^A}}{\pi_{a_i}^A \pi_{a_j}^A}$.	Given L , then, $\tilde{L} \in \Phi$, is called equilibrium stochastic transformation for L , if $\tilde{L}$ is such that $M \rightarrow \tilde{L}(M) = \frac{1}{\lambda^A} \phi_L^{1/2}(L(M)) \phi_L^{-1/2}$.
$P \pi = \pi$, all entries P positive, then $m = m^P$. m of Definition (*) is σ invariant and $\mathcal{L}_{\log P}^*(m) = m$ m is ergodic for $\sigma \Leftrightarrow$ the Process X_n is ergodic $\Leftrightarrow$ given Borel B , if $\sigma^{-1}(B) = B \Rightarrow m(B) = 0$ or $m(B) = 1$ generically all $P_{ij} > 0 \Rightarrow$ a unique π s. t. $P \pi = \pi$	$\mathcal{L}_{\log J_m}^*(m) = m$ and a natural association $m \Leftrightarrow \log J_m$. m is σ invariant. m is ergodic for $\sigma \Leftrightarrow$ given Borel B , if $\sigma^{-1}(B) = B \Rightarrow m(B) = 0$ or $m(B) = 1$ C^0 -generically on A the equilibrium probability m^A is ergodic	For $L \in \Phi$, a natural association is $L \Leftrightarrow \phi_L$ stochastic operator. ρ_L the Gibbs dens. matrix for L is, such that, $\phi_L(\rho_L) = \rho_L$. concept: $L \in \Phi$ is Φ -Ergodic for μ . see definition on the paper
		Our theorem: for fixed μ , C^0 -generically L is Φ -Ergodic for μ

TABLE 1. An outline of [11] and comparison with the dynamical setting

$\tilde{\mu} = \sum_{j=0}^1 \delta_j$ and $P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$	$\tilde{\mu} = \sum_{j=0}^1 \delta_j$ and $P = J$	$V_1 = \begin{pmatrix} \sqrt{p_{00}} & 0 \\ 0 & 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 & \sqrt{p_{01}} \\ 0 & 0 \end{pmatrix}$,
$\begin{pmatrix} 1 & 1 \end{pmatrix} P = \begin{pmatrix} 1 & 1 \end{pmatrix}$	$\mathcal{L}_{\log J}(1) = 1$	$V_3 = \begin{pmatrix} 0 & 0 \\ \sqrt{p_{10}} & 0 \end{pmatrix}$, $V_4 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p_{11}} \end{pmatrix}$.
$P \pi = \pi$	$\mathcal{L}_{\log J}^*(m) = m$	$\mu = \sum_{i=1}^4 \delta_{V_i}$,
$r_1 \rightarrow P r_1 = r_2$ and $l_1 \rightarrow l_1 P = l_2$	$\varphi_1 \rightarrow \varphi_2 = \mathcal{L}_{\log J}(\varphi_1)$	$\phi_{Id}(\rho) = \sum_{i=1}^4 V_i \rho V_i^{\dagger}$ and $\phi_{\sigma_d}^*(M) = \sum_{i=1}^4 V_i^{\dagger} M V_i$
$\pi = \left(\frac{p_{01}}{1 - p_{00} + p_{01}}, \frac{1 - p_{00}}{1 - p_{00} + p_{01}} \right)$	$\mu_1 \rightarrow \mu_2 = \mathcal{L}_{\log J}^*(\mu_1)$	$\rho = \begin{pmatrix} \frac{p_{01}}{1 - p_{00} + p_{01}} & 0 \\ 0 & \frac{1 - p_{00}}{1 - p_{00} + p_{01}} \end{pmatrix}$ is fixed for ϕ_L
$h(m) = - \sum_{i,j=0}^1 \pi_j p_{ij} \log(p_{ij}) \geq 0$	$H_{\tilde{\mu}}(m) = - \int \log J_m dm$	$\mathcal{H}(\phi_{Id}) = - \sum_{i,j=0}^1 \pi_j p_{ij} \log(p_{ij}) \geq 0$

TABLE 2. The Markov model in quantum information

shift $\sigma : \Omega = \{1, 2, \dots, d\}^{\mathbb{N}} \circ$	shift $\sigma : \Omega = K^{\mathbb{N}} \rightarrow K^{\mathbb{N}}$	Schrodinger; shift $\sigma : P(\mathbb{C}^k)^{\mathbb{N}} \rightarrow P(\mathbb{C}^k)^{\mathbb{N}}$, $P(\mathbb{C}^k)$ is the projective space in $\mathbb{C}^k$. Heisenberg; shift $\sigma : \Omega = M_k^{\mathbb{N}} \rightarrow M_k^{\mathbb{N}}$
$P(X_1 = a_1, X_2 = a_2, \dots, X_n = a_n)$ $a_1, a_2, \dots, a_n \in \{1, 2, \dots, d\}$	$P(X_1 \in B_1, X_2 \in B_2, \dots, X_n \in B_n) =$ with $B_1 \subset K, \dots, B_n \subset K$	$P(X_1 \in B_1, X_2 \in B_2, \dots, X_n \in B_n)$, $B_1 \subset M_k, \dots, B_n \subset M_k$, or $B_1 \subset P(\mathbb{C}^k), \dots, B_n \subset P(\mathbb{C}^k)$
transition $P(X_2 = a_2 P(X_1 = a_1)) = P_{a_1 a_2}$	Continuous transition function $J : K^{\mathbb{N}} \rightarrow \mathbb{R}$ $\log J(x) = A(x_1, x_2, x_3, \dots), x_j \in K$	$L \in \Phi$, then trans. $\rho_{n-1} \rightarrow \rho_n = \frac{L(v)\rho_{n-1}L(v)^*}{\text{tr}(L(v)\rho_{n-1}L(v)^*)}$, with probability $\text{tr}(L(v)\rho_{n-1}L(v)^*)d\mu(v)$.
Markov Kernel P_{ij} , $i, j = 1, 2, \dots, d$,	Holder Jacobian $J > 0$	$\hat{x} \in P(\mathbb{C}^k)$, $S \subset P(\mathbb{C}^k)$. Given $L \in M_k$, s. t. ϕ_L is stochastic; Markov Kernel $\Pi_L(\hat{x}, S) =$ $\int_{M_k} \mathbf{1}_S(L(v) \cdot \hat{x}) \ L(v)x\ ^2 d\mu(v)$,
All $P_{ij} > 0$, $i, j = 1, 2, \dots, d$, unique π such that $P(\pi) = \pi$	Holder Jacobian $J > 0$, unique m such that $L^*(m) = m$	BFPP Theo: ϕ -Erg and purification condit. for $L \in \Phi \Rightarrow \Pi_L$ has a unique invariant prob.
		our theorem: for $L \in \Phi$ the purification condition is C^0 -generic.
$\omega = (\omega_1, \dots, \omega_n, \dots) \rightarrow X_i(\omega) \in \mathbb{R}$, $i \in \mathbb{N}$ example of multiplicative random variable: $n \in \mathbb{N}$, $W_n = e^{X_1 + X_2 + \dots + X_n}$,	$\omega = (\omega_1, \dots, \omega_n, \dots) \rightarrow X(\omega) \in$ real matrices M_m , random product of matrices $X(\sigma^{n-1}(\omega)) \dots X(\sigma^2(\omega)) X(\sigma(\omega)) X(\omega)$,	$V_i : \Omega \rightarrow M_k$ the random var. $V_i(\omega) = \omega_i$, and $W_n := L(V_n) \dots L(V_1)$, or, $W_n(\omega) = L(\omega_n) \dots L(\omega_1)$.
$L_j, j = 1, \dots, d$, invert. 2×2 real matrices, independent probability $(p_1, \dots, p_d)$, $\omega \rightarrow L^n = L_{\omega_n} L_{\omega_{n-1}} \dots L_{\omega_1}$	smooth case: M manifold $F : M \rightarrow M$ Anosov, $x \in M \rightarrow DF^{-1}(x)$. Take matrix product, $n \in \mathbb{N}$, $DF^{-1}(F^{n-1}(x)) \dots DF^{-1}(F(x)) DF^{-1}(x)$	$x_1 \wedge x_2 \wedge \dots \wedge x_n$, with $x_j \in \mathbb{C}^k$, wedge product. Given $X \in M_k$, take $\bigwedge^n X(x_1 \wedge \dots \wedge x_n) = X(x_1) \wedge \dots \wedge X(x_n)$.
π_0 initial conditon of Markov Process. π_0 prob. on $\{1, 2, \dots, d\}$. If $\pi_0 = P(\pi_0) \Rightarrow$ π_0 invariant for stoch. process $X_i(\omega) = \omega_i$.	smooth case: $F : M \rightarrow M$, F Anosov. The physical measure: μ_{SBR} is invariant for F . μ_{SBR} is ergodic and abs. cont. Lebesgue	stoc. process $\rho_n(\omega) = \frac{W_n(\omega)\rho_0 W_n(\omega)^*}{\text{tr}(W_n(\omega)\rho_0 W_n(\omega)^*)}$. If $\rho_0 \in \mathcal{D}_k$ satisfies $\phi_L(\rho_0) = \rho_0$, then, ρ_0 is invariant for the stoch. process ρ_n
expansion of vector $v \in \mathbb{C}^2$ by the random product on orbits $\omega \in \Omega$, $ L^n(v) $	$\dim M = 2$, F Anosov volume preserving, $\mu_{SBR} =$ Lebesgue. Expansion of vector $v \in TM_x$, $ DF^{-1}(F^{n-1}(x)) \dots DF^{-1}(F(x)) DF^{-1}(x)(v) $	$L \in M_2$, $x \in P(\mathbb{C}^2)$, $p \in \{1, 2\}$, expansion running the Stochas. Process: $ \bigwedge^p W_n $
Furstenberg-Kesten Theorem: exist γ^+ , such that P a.e.w. $\omega \in \Omega$, an all $v \in \mathbb{R}^2$: $\lim_{n \rightarrow \infty} \frac{1}{n} \log L^n(v) = \gamma^+$	Osedelet Theorem: there exist $\gamma_1 > 0 > \gamma_2$, s. t. $\lim_n \frac{1}{n} \log [(DF^{-1}(F^{n-1}(x)) \dots DF^{-1}(x)](v) $ $= \gamma_1$ or γ_2 , for μ_{SBR} a.e.w. x, and $\forall v \in TM_x$.	BFPP Th: ϕ -Erg and purification cond. for L , $\Rightarrow$ exist $\gamma_1 > 0 > \gamma_2$, s. t. there exists $\lim_{n \rightarrow \infty} \frac{1}{n} \log \bigwedge^2 W_n = \sum_{j=1}^2 \gamma_j$.
	Pesin Theorem: Anosov vol. pres. $\dim M = 2$: $\gamma_2 =$ the minimal Lyapunov exponent = $-h(\mu_{SBR}) = -$ entropy of μ_{SBR}	our Theorem for Markov: $\gamma_2 = -\infty$, and $\lim_n \frac{1}{n} \log \ \bigwedge^1 W_n(\omega)\ = \gamma_1 = -\frac{1}{2} \mathfrak{H}(\phi_L) =$ $-\frac{1}{2}$ entropy Markov $= \frac{1}{2} \sum_{i,j} \pi_j p_{ij} \log(p_{ij})$,

TABLE 3. An outline of the present paper and comparison with the dynamical setting

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