

HOFER-LIKE GEOMETRY AND FLUX THEORY

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ABSTRACT. This paper meticulously revisit and study the flux geometry of any compact connected oriented manifold (M, Ω) . We generalize several well-known factorization results, exhibit some orbital conditions under which flux geometry can be studied, give a proof of the discreteness of the flux group for volume-preserving diffeomorphisms, derive that any smooth isotopy in the group of all vanishing-flux volume-preserving diffeomorphisms is a vanishing-flux path, and show that the kernel of flux for volume-preserving diffeomorphisms is C^1 -closed inside the group of all volume-preserving diffeomorphisms isotopic to the identity map: We recover several well-known results from symplectic geometry. We use the above studies to construct a right-invariant metric on the group of all volume-preserving diffeomorphisms isotopic to the identity map and study the induced geometry. In the case of a symplectic volume form, the restriction of our metric to the group $Ham(N, \omega)$, of all Hamiltonian diffeomorphisms of a closed symplectic manifold (N, ω) , is controlled from above by the usual Hofer metric in general, while the Hofer-like metric control our metric in the case where the Riemannian structure is compatible with the symplectic structure (in particular, our construction implies the non-degeneracy of the Hofer and Hofer-like norms).

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1. INTRODUCTION

On a compact manifold M , given any smooth closed p -form α together with an isotopy $\Phi = (\phi_t)$, they satisfy the following equation

$$(1) \quad (\phi_t)^* \alpha - \alpha = d(\mathcal{F}_\alpha(\Phi)(t))$$

for all t , where d is the usual differential operator such that $d \circ d = 0$ (see [17]), and

$$(2) \quad \mathcal{F}_\alpha(\Phi)(t) := \int_0^t \phi_s^*(\alpha(\dot{\phi}_s)) ds.$$

Note that any smooth isotopy $\Phi = (\phi_t)$ of M gives rise to a smooth family of vector fields $(\dot{\phi}_t)$ on M defined by

$$(3) \quad \dot{\phi}_t(\phi_t(x)) := \frac{d}{dt}(\phi_t(x)),$$

for all t , and for all $x \in M$.

Equation (1) is deeply rooted in the study of flux theory on the manifold M : It had been used by Moser [12], Banyaga [1, 3], Calabi [7] and others in the study of the geometry of the group of volume-preserving diffeomorphisms. Banyaga [1], Thurston [16], and Weinstein [18] works show that if two given volume-preserving isotopies are homotopic relatively to fix endpoints, then they have the same flux.

So far, this is the most known (not to say the only) condition which guaranties the equality of the fluxes of two volume-preserving isotopies. What seems to be a bit ambiguous here is that, the flux of each isotopy is a de Rham cohomology class in some $H^*(M, \mathbb{R})$ (the $*$ -th de Rham group of M with real coefficients), while the group $H^*(M, \mathbb{R})$ only depend on the topology of M , and does not depend on any differentiable structure on M . One might expect that analyzing the orbits (in M) of a volume-preserving isotopy of (M, Ω) (where Ω is the orientation form on M) may "determine" its flux.

Of course, the difficulty to face (or the price to pay) here is the large number of orbits generated by a single isotopy on a manifold: Though it looks almost impossible to describe all of them, nothing is telling to us that such a study is not possible. The optimal cost here could reside in the way to look at the situation.

The problem we address in this paper is that of the existence of other(s) criterion (or criteria) under which one can study the flux geometry of a compact oriented manifold.

We shall see that investigating the above criteria will not only enrich the theory

of flux geometry, but also the dynamics of volume-preserving diffeomorphisms, and the theory of Hofer-like geometry of a closed oriented symplectic manifold.

Here are the main results of this paper.

Theorem 1.1. *Let (M, Ω) be a compact connected oriented manifold. Let $\alpha = (\alpha_t)$ be a smooth family of smooth closed 1-forms, let $\Phi = \{\phi_t\}$, and $\Psi = \{\psi_t\}$ be two elements in $\text{Iso}(M, \Omega)$. The following results hold.*

(1) *For each $t \in [0, 1]$, we have,*

$$\int_M \mathcal{F}_{\alpha_t}(\Phi)(t)\Omega = \langle P(\alpha_t), \tilde{S}_\Omega(\bar{\Phi}_t) \rangle + \int_{\partial M} \left(\int_0^t f_{\Phi, \alpha_t}^s \phi_s^* (\iota(\dot{\phi}_s)\Omega) ds \right),$$

where $\bar{\Phi}_t$ is the isotopy $s \mapsto \phi_{st}$, $f_{\Phi, \alpha_t}^s := \int_0^s \alpha_t(\dot{\phi}_u) \circ \phi_u du$, the map

$$\langle \cdot, \cdot \rangle : H^1(M, \mathbb{R}) \times H^{n-1}(M, \mathbb{R}) \rightarrow \mathbb{R}, ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta,$$

is the usual Poincaré pairing, and $\tilde{S}_\Omega$ is the flux homomorphism.

(2) *In particular when $\partial M = \emptyset$, then for each $t \in [0, 1]$, we have,*

$$\int_M \mathcal{F}_{\alpha_t}(\Phi)(t)\Omega = \langle P(\alpha_t), \tilde{S}_\Omega(\bar{\Phi}_t) \rangle.$$

(3) *Assume that $\partial M = \emptyset$. Let $\varphi \in (G_\Omega(M) \setminus \{id_M\})$. If $\Phi, \Psi \in \text{Iso}(\varphi)_\Omega$ and there exists a point $z_0 \in M$ for which the orbits $\mathcal{O}_{z_0}^\Psi$ and $\mathcal{O}_{z_0}^\Phi$ are homotopic relatively to fix endpoints, then $\tilde{S}_\Omega(\Phi) = \tilde{S}_\Omega(\Psi)$.*

(4) *Assume that: $\partial M = \emptyset$, and $\Omega := \alpha^l := \underbrace{\alpha \wedge \cdots \wedge \alpha}_{l\text{-times}}$, where α is a closed*

$\deg(\alpha)$ -form of degree $\deg(\alpha) := \dim(M)/l$ such that for each volume-preserving isotopy

$\Phi = \{\phi_t\}$, the $[\deg(\alpha) - 1]$ -form $\iota(\dot{\phi}_t)\alpha$ is closed for each t . Therefore,

(a) *if $\deg(\alpha)$ is even, then*

$$\tilde{S}_\Omega(\Phi) = \frac{\dim(M)}{\deg(\alpha)} P \left(\alpha \left(\left(\frac{\dim(M)}{\deg(\alpha)} \right)^{-1} \right) \wedge \int_0^1 \iota(\dot{\phi}_t) \alpha dt \right),$$

(b) *if $\deg(\alpha)$ is odd, then*

$$\tilde{S}_\Omega(\Phi) = (-1)^{\left(1+E\left(\frac{\dim(M)}{2\deg(\alpha)}\right)\right)} P \left(\alpha \left(\left(\frac{\dim(M)}{\deg(\alpha)} \right)^{-1} \right) \wedge \int_0^1 \iota(\dot{\phi}_t) \alpha dt \right),$$

where for each positive real number x , $E(x)$ is x minus its fractional part; and P is the natural projection from the space of all smooth closed $*$ -forms $\mathcal{Z}^*(M)$ onto $H^*(M, \mathbb{R})$.

Combining the first item and second item of Theorem 1.1 together, we highlight the impact of the boundary of a compact connected oriented manifold (M, Ω) on the study of flux geometry. On the other hand, the third item in Theorem 1.1 implies that if Φ and Ψ are homotopic relatively to fixed endpoints, then they have the same flux: This item improves a result found in [2]. Finally, the last item in Theorem 1.1 implies that if (M, ω) is a $2l$ -dimensional symplectic manifold oriented with the Liouville volume form $\Omega_0 := \frac{\omega^l}{l!}$, then taking $\alpha := \omega$, one has

$$\begin{aligned} \tilde{S}_{\Omega_0}(\Phi) &= \frac{l}{l!} P\left(\int_0^1 \phi_t^*(\iota(\dot{\phi}_t)\omega) dt \wedge \omega^{(l-1)}\right), \\ &= \frac{1}{(l-1)!} \tilde{S}_\omega(\Phi) \wedge P(\omega^{(l-1)}), \end{aligned} \tag{4}$$

for all symplectic isotopy Φ : The last item in Theorem 1.1 is a generalization of a factorization result found in [1, 3].

The following theorem is an unpublished result due to Thurston [16]. Here, we give a prove inspired from Theorem 1.1.

Theorem 1.2. *Let (M, Ω) be a closed oriented manifold. The group Γ_Ω is discrete.*

In particular, on any Lefschetz closed symplectic manifold (M, ω) , Theorem 1.2 implies that the Calabi group is discrete: This symplectic case was proved by Ono for any closed symplectic manifold [13].

Now that it is well known that the symplectic analogue of Theorem 1.2 is also equivalent to say that the group of all Hamiltonian diffeomorphisms is C^1 -closed in the group of all symplectic diffeomorphisms isotopic to the identity map, then in the presence of Theorem 1.2, we are tempted to establish the following volume-preserving analogue.

Lemma 1.3. *Let (M, Ω) be a closed oriented manifold. The sub-group $\ker S_\Omega$ is C^1 -closed inside the group $G_\Omega(M)$.*

Lemma 1.4. *Let (M, Ω) be a closed oriented manifold. The sub-group $\ker \tilde{S}_\Omega$ is C^0 –closed inside the group $\text{Iso}(M, \Omega)$.*

Theorem 1.5. *Let (M, Ω) be a closed oriented manifold. Then, any smooth isotopy in $\ker S_\Omega$ is a vanishing-flux path.*

Theorem 1.5 states that the sub-group $\ker S_\Omega$ is path connected through vanishing-flux paths.

Here is a rigidity result.

Theorem 1.6. *Let (M, Ω) be a closed oriented manifold. If $\{\Phi_i\}$ is a sequence of vanishing-flux isotopies which C^0 –converges to a loop γ , then any smooth orbit of γ is null homologous.*

In particular, Theorem 1.6 implies that a continuous Hamiltonian loop has the property that any of its orbits which is smooth is null homologous [6].

The following result separates orbits of a volume-preserving isotropy from minimal geodesics.

Lemma 1.7. (*Flux-Minimal geodesic*) *Let (M, Ω) be a closed oriented manifold. Let Φ be a volume-preserving isotopy out of $\ker \tilde{S}_\Omega$ with time-one map $\phi \neq \text{id}_M$. There exists $\delta_0 > 0$ which depends on Φ such that if the distance (in the C^0 –topology) between Φ and the constant path identity is less than δ_0 , then none of the orbits of Φ is a minimal geodesic between its endpoints.*

We organize the paper as follows: In Section 2 we introduce the definitions of some useful tools that we shall need. The third section is devoted to revisit the construction of the flux homomorphism of a compact connected oriented manifold based on Section 2, and generalize several factorization results found in [3, 15]: The concerned results are, Lemma 3.1, Lemma 3.2, Proposition 3.3, Proposition 3.8, and Proposition 3.9.

In Section 4, we study the displacements of closed 1–forms (Corollary 4.2, and Proposition 4.3). This includes the proofs of: Theorem 1.1 (Proposition 4.6) which gives orbital conditions under which the fluxes of two isotopies with the same endpoints are equal, Theorem 1.2 which states that the flux group for volume-preserving

diffeomorphisms is a discrete group, Lemma 1.3 which states that the sub-group of vanishing-flux volume-preserving diffeomorphisms is C^1 -closed inside the group of all volume-preserving diffeomorphisms isotopic to the identity map, Lemma 1.4 which states that the sub-group of vanishing-flux volume-preserving isotopies is C^0 -closed inside the group of all volume-preserving isotopies. Also, the proof of Lemma 1.7 which separates the orbits of a non-vanishing flux volume-preserving isotopy from minimal geodesics between their endpoints is given. Section 5 is devoted to the study of the metric geometry of the group of volume-preserving diffeomorphisms isotopic to the identity map. Here, we construct a right-invariant metric on the group volume-preserving diffeomorphisms (Proposition 5.1), and study the subsequent geometry: In the case of symplectic volume form, we prove that the restriction of our metric to the group of all Hamiltonian diffeomorphisms is controlled by the usual Hofer metric, and we show that if the Riemannian metric is compatible with the symplectic structure, then the restriction of our metric to the group of all symplectic diffeomorphisms isotopic to the identity is controlled by the usual Hofer-like metric.

2. PRELIMINARIES

Let M be an n -dimensional manifold, and let $\Omega^p(M)$ be the space of all p -forms of M . A diffeomorphism $\phi : M \rightarrow M$ preserves a p -form α , if $\phi^*(\alpha) = \alpha$. For any $\alpha \in \Omega^p(M)$, we shall denote by $Diff^\infty(M, \alpha)$ the group of all diffeomorphisms from M to M that preserve α .

2.1. Isotopies. A smooth isotopy $\Phi = (\phi_t)$ of M is a smooth map from $[0, 1]$ into $Diff^\infty(M)$ such that

$\phi_0 = id_M$ (see [1, 6, 11] for more details). Let $Diff_0^\infty(M)$ denote the set of all time-one maps of smooth isotopies in $Diff^\infty(M)$, and for each $\alpha \in \Omega^p(M)$, let $Iso(M, \alpha)$ denote the space of all smooth isotopies in $Diff^\infty(M, \alpha)$, and by $G_\alpha(M)$, we denote the set of all time-one maps of all isotopies of $Iso(M, \alpha)$. For any $\psi \in G_\alpha(M)$, we shall denote by $Iso(\psi)_\alpha$ the set $\{\Phi \in Iso(M, \alpha) | \Phi(1) = \psi\}$.

2.1.1. Volume-preserving isotopies. When M is an orientable manifold with orientation form Ω , then an isotopy $\Phi = (\phi_t)$ of (M, Ω) is said to be volume-preserving if its associated smooth family of smooth vector fields $(\dot{\phi}_t)$ consists of free divergence

vector fields i.e. for each t the $(n-1)$ -form $\iota(\dot{\phi}_t)\Omega$ is closed. This is equivalent to require that $\Phi \in Iso(M, \Omega)$.

2.1.2. Concatenation operations for isotopies. Let $t \in [0, 1]$ (fixed). It is always possible to construct a smooth increasing function $f_t : [0, t] \rightarrow [0, t]$ such that its restriction to an interval $[0, \delta]$ vanishes while its restriction to $[(t - \delta), t]$ is the constant function t with $0 < \delta \leq \frac{t}{8}$. Assume this done, and consider the smooth functions $\lambda(s) := f_t(2s)$ for all $0 \leq s \leq \frac{t}{2}$, and $\tau(s) := f_t(2s - t)$ for all $\frac{t}{2} \leq s \leq t$. If $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are two given isotopies, then one defines an isotopy $(\Phi *_r^t \Psi)$ with time-one map $\phi_t \circ \psi_t$ by setting:

$$(\Phi *_r^t \Psi)(s) = \begin{cases} \phi_{\lambda(s)}, & \text{if } 0 \leq s \leq \frac{t}{2}, \\ \phi_t \circ \psi_{\tau(s)}, & \text{if } \frac{t}{2} \leq s \leq t. \end{cases}$$

For each fixed $t \in [0, 1]$, the isotopy $s \mapsto (\Phi *_r^t \Psi)(s)$ constructed above is called the t -right concatenation of the isotopy $\Phi^t : u \mapsto \phi_{ut}$ by the isotopy $\Psi^t : u \mapsto \psi_{ut}$. Similarly, we can define the t -left concatenation of the isotopy $\Phi^t : u \mapsto \phi_{ut}$ by the isotopy $\Psi^t : u \mapsto \psi_{ut}$, denoted $(\Psi *_l^t \Phi)$, as map $\psi_t \circ \phi_t$ by setting:

$$(\Psi *_l^t \Phi)(s) = \begin{cases} \phi_{\lambda(s)}, & \text{if } 0 \leq s \leq \frac{t}{2}, \\ \psi_{\tau(s)} \circ \phi_t, & \text{if } \frac{t}{2} \leq s \leq t. \end{cases}$$

When $t = 1$, the t -right concatenation (resp. t -left concatenation) is called right concatenation (resp. left concatenation), and denoted $(\Psi *_r \Phi)$ (resp. $(\Psi *_l \Phi)$). An orbit of a point $p \in M$ under $(\Phi *_r \Psi)$ (resp. under $(\Psi *_l \Phi)$) is the orbit of p under Φ "glued" with the image under ϕ_1 of the orbit of p under Ψ (resp. is the orbit of p under Φ "glued" with the orbit of $\phi_1(p)$ under Ψ). Throughout all the paper, for each isotopy Φ and each point $p \in M$, we shall denote by $\mathcal{O}_p^\Phi$ the orbit of p under the action of Φ , while $-\mathcal{O}_p^\Phi$ will represent the orbit of p under the action of Φ^{-1} .

Remark 2.1. Let $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ be two given isotopies. For each $t \in [0, 1]$ (fixed), and for each $z \in M$, the orbit $\mathcal{O}_z^{\Phi *_l^t \Psi}$ is exactly the curve $\mathcal{O}_z^{\Psi^t} \sqcup \mathcal{O}_{\psi_t(z)}^{\Phi^t}$, where for each t (fixed), the isotopies Φ^t and Ψ^t are defined as $\Phi^t : u \mapsto \phi_{ut}$, and $\Psi^t : u \mapsto \psi_{ut}$. This observation tells us that the set

$$\blacksquare(\Phi, \Psi)_z := \sqcup_{t \in [0, 1]} \mathcal{O}_z^{\Phi *_l^t \Psi},$$

forms a smooth 2-chain in M whose boundary consists of the curve $\mathcal{O}_z^{\Phi *_l \Psi} := \mathcal{O}_z^{\Phi *_l^1 \Psi}$, and the curve $\sqcup_{t \in [0,1]} \phi_t(\psi_t(z)) =: \mathcal{O}_z^{\Phi \circ \Psi}$.

2.2. Properties of the smooth function $\mathcal{F}_\alpha(\Phi)$. We assume that the manifold M is compact and connected.

Definition 2.2. Let $\Phi = (\phi_t)$ be an isotopy of M , and let α be any closed 1-form of M . For each $t \in [0, 1]$ (fixed), consider the reparameterized path $\Phi^t : u \mapsto \phi_{ut}$. Then, we define the smooth function $\mathcal{F}_\alpha(\Phi)(t) : M \rightarrow \mathbb{R}$ as: $\mathcal{F}_\alpha(\Phi)(t)(x) := \int_{\mathcal{O}_x^{\Phi^t}} \alpha$.

We have the following facts.

Lemma 2.3. Let $\Phi = (\phi_t)$ be a given isotopy. For each closed 1-form α , we have

- (1) $\phi_t^*(\alpha) - \alpha = d(\mathcal{F}_\alpha(\Phi)(t))$, for each t [12], [4].
- (2) If $\Psi = (\psi_t)$ is another isotopy, then

$$\begin{aligned}
 \mathcal{F}_\alpha(\Phi *_r^t \Psi)(p) &:= \int_{\mathcal{O}_p^{\Phi *_r^t \Psi}} \alpha \\
 &= \int_{\mathcal{O}_p^{\Phi^t}} \alpha + \int_{\phi_t(\mathcal{O}_p^{\Psi^t})} \alpha, \\
 &= \mathcal{F}_\alpha(\Phi)(t)(p) + \int_{\mathcal{O}_p^{\Psi^t}} \phi_t^*(\alpha) \\
 &= \mathcal{F}_\alpha(\Phi)(t)(p) + \int_{\mathcal{O}_p^{\Psi^t}} (\alpha + d(\mathcal{F}_\alpha(\Phi)(t))) \\
 &= \mathcal{F}_\alpha(\Phi)(t)(p) + \int_{\mathcal{O}_p^{\Psi^t}} \alpha + \int_{\mathcal{O}_p^{\Psi^t}} (d(\mathcal{F}_\alpha(\Phi)(t))) \\
 &= \mathcal{F}_\alpha(\Psi)(t)(p) + \mathcal{F}_\alpha(\Phi)(t) \circ \Psi(t)(p),
 \end{aligned}$$

for each $p \in M$. Thus,

$$\mathcal{F}_\alpha(\Phi *_r^t \Psi)(p) = \mathcal{F}_\alpha(\Psi)(t) + \mathcal{F}_\alpha(\Phi)(t) \circ \Psi(t),$$

for each closed 1-form α , and for each $p \in M$.

- (3) Similarly, we have

$$(5) \quad \mathcal{F}_\alpha(\Phi *_l^t \Psi)(p) = \mathcal{F}_\alpha(\Psi)(t) + \mathcal{F}_\alpha(\Phi)(t) \circ \Psi(t),$$

for each closed 1-form α .

- (4) Since the isotopies $\Phi *_l^t \Psi$, and $\Phi^t \circ \Psi^t$ have the same endpoints (which are id_M , and $\phi_t \circ \psi_t$), then we derive from formula (1) (or from item 1) that

$$(6) \quad \mathcal{F}_\alpha(\Phi \circ \Psi)(t) = \mathcal{F}_\alpha(\Psi)(t) + \mathcal{F}_\alpha(\Phi)(t) \circ \Psi(t) + \delta_t(\Phi, \Psi, \alpha),$$

where $\delta_t(\Phi, \Psi, \alpha)$ is a constant which depends on Φ , Ψ , α and t .

Lemma 2.4. *Let M be a compact and connected manifold, and let α be a closed 1-form. If Φ and Ψ are two isotopies, then we have $\delta_t(\Phi, \Psi, \alpha) = 0$, for all t , i.e.,*

$$(7) \quad \mathcal{F}_\alpha(\Phi \circ \Psi)(t) = \mathcal{F}_\alpha(\Psi)(t) + \mathcal{F}_\alpha(\Phi)(t) \circ \Psi(t),$$

for all t .

Proof. Let α be a closed 1-form. For each $z \in M$, we derive from Remark 2.1 that $\int_{\blacksquare(\Phi, \Psi)_z} d\alpha = 0$, and hence it follows from Stokes' theorem that

$$0 = \int_{\partial \blacksquare(\Phi, \Psi)_z} \alpha = \int_{\mathcal{O}_z^{\Phi * \iota \Psi}} \alpha - \int_{\mathcal{O}_z^{\Phi \circ \Psi}} \alpha.$$

That is, $\mathcal{F}_\alpha(\Phi \circ \Psi)(t) = \mathcal{F}_\alpha(\Phi * \iota \Psi)(t)$, and this together with (5) yields the desired result. $\square$

From now on, we assume that (M, Ω) is a closed oriented manifold, with orientation form Ω .

3. THE FLUX HOMOMORPHISM REVISITED

According to Lemma 2.4, we have

$$(8) \quad \mathcal{F}_\alpha(\Phi \circ \Psi)(t) = \mathcal{F}_\alpha(\Psi)(t) + \mathcal{F}_\alpha(\Phi)(t) \circ \Psi(t),$$

for all t . The above equality tells us that for each fixed closed 1-form α , the map $\Phi \mapsto \int_M \mathcal{F}_\alpha(\Phi)(1)\Omega$, induces a group homomorphism from the space $Iso(M, \Omega)$ into $\mathbb{R}$.

We have the following facts.

Lemma 3.1. *For each fixed closed 1-form α , and for each volume-preserving isotopy Φ , the integral $\int_M \mathcal{F}_\alpha(\Phi)(1)\Omega$ is independent of the choice of any representative β in the de Rham cohomology class $[\alpha]$.*

Proof. Let $\beta \in [\alpha]$, i.e., $\alpha - \beta = df$ for some smooth function $f : M \rightarrow \mathbb{R}$. Assume that $\Phi = \{\phi_t\}$, and derive from a direct computation that,

$0 = \iota(\dot{\phi}_t)[(\alpha - \beta) \wedge \Omega] = \iota(\dot{\phi}_t)[(\alpha - \beta)]\Omega - (\alpha - \beta) \wedge \iota(\dot{\phi}_t)\Omega$. Since the $(n-1)$ -form $\iota(\dot{\phi}_t)\Omega$ is closed and $\alpha - \beta = df$, then $(\alpha - \beta) \wedge \iota(\dot{\phi}_t)\Omega = d(f\iota(\dot{\phi}_t)\Omega)$. Hence, we have $-(\iota(\dot{\phi}_t)\alpha)\Omega + (\iota(\dot{\phi}_t)\beta)\Omega = d(f\iota(\dot{\phi}_t)\Omega)$, and integrating the above relation over M yields, $-\int_M (\iota(\dot{\phi}_t)\alpha)\Omega + \int_M (\iota(\dot{\phi}_t)\beta)\Omega = \int_M d(f\iota(\dot{\phi}_t)\Omega)$, where the right hand side vanishes because of Stokes' Theorem since M is closed. $\square$

Lemma 3.2. *If Φ and Ψ are two isotopies homotopic relatively to fix endpoints, then $\mathcal{F}_\alpha(\Phi)(1) = \mathcal{F}_\alpha(\Psi)(1)$, for each closed 1-form α .*

Proof. Let α be a closed 1-form; set $\Psi = (\psi_t)$, and $\Phi = (\phi_t)$. We have to prove that $\mathcal{F}_\alpha(\Phi)(1)(x) = \mathcal{F}_\alpha(\Psi)(1)(x)$, for all $x \in M$. To that end, we recall that a homotopy between Ψ and Φ is a smooth map $H : I \times I \rightarrow \text{Diff}_0^\infty(M)$ such that $H(0, t) = \psi_t$, $H(1, t) = \phi_t$ for all t ; $H(s, 0) = \text{id}_M$ and $H(s, 1) = \phi_1 = \psi_1$ for all s . Thus, for each $x \in M$, the homotopy H induces a homotopy $\bar{H}_x$ between the orbits $\mathcal{O}_x^\Phi : t \mapsto \phi_t(x)$ and $\mathcal{O}_x^\Psi : t \mapsto \psi_t(x)$; i.e., $\mathcal{O}_x^\Phi$ and $\mathcal{O}_x^\Psi$ constitute the boundary of a 2-chain in M . Therefore, Stokes' theorem implies that $\int_{\mathcal{O}_x^\Phi} \alpha = \int_{\mathcal{O}_x^\Psi} \alpha$, because α is closed. To conclude, observe that for each $x \in M$, we also have $\mathcal{F}_\alpha(\Phi)(1)(x) = \int_{\mathcal{O}_x^\Phi} \alpha$, and $\mathcal{F}_\alpha(\Psi)(1)(x) = \int_{\mathcal{O}_x^\Psi} \alpha$. $\square$

3.1. Flux homomorphism $\tilde{S}_\Omega$. Lemma 2.4 and Lemma 3.1 suggest that: Each volume-preserving isotopy Φ induces an element $\tilde{S}_\Omega(\Phi) \in \text{Hom}(H^1(M, \mathbb{R}), \mathbb{R}) \cong H^{n-1}(M, \mathbb{R})$, defined as follows: To any

$[\alpha] \in H^1(M, \mathbb{R})$ one assigns $\tilde{S}_\Omega(\Phi)([\alpha]) := \int_M \mathcal{F}_\alpha(\Phi)(1)\Omega$.

Thus, we have a group homomorphism $\tilde{S}_\Omega : \text{Iso}(M, \Omega) \rightarrow H^{n-1}(M, \mathbb{R})$. The investigation of an explicit formula for the flux homomorphism motivated the following factorization result that generalizes Proposition 2.4–[15].

Proposition 3.3. (Factorization I) *Let (M, Ω) be a closed oriented manifold. Let $\alpha = (\alpha_t)$ be a smooth family of smooth closed 1-forms, and let $\Phi = \{\phi_t\} \in \text{Iso}(M, \Omega)$. Then, for each $t \in [0, 1]$, we have,*

$$\int_M \mathcal{F}_{\alpha_t}(\Phi)(t)\Omega = \langle [\alpha_t], [\int_0^t \phi_s^* (\iota(\dot{\phi}_s)\Omega) ds] \rangle,$$

where $\bar{\Phi}_t$ is the isotopy $s \mapsto \phi_{st}$, and the map

$$\langle \cdot, \cdot \rangle : H^1(M, \mathbb{R}) \times H^{n-1}(M, \mathbb{R}) \rightarrow \mathbb{R}, ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta,$$

is the usual Poincaré pairing.

Proof. Fix an arbitrary t in $[0, 1]$. Since the form $\alpha_t \wedge \Omega$ is of degree $(n+1)$ on an n -dimensional manifold, then it is trivial. This implies that $\alpha_t(\dot{\phi}_s)\Omega - \alpha_t \wedge \iota(\dot{\phi}_s)\Omega = 0$, for each $s \in [0, t]$. Composing each member of the above equality by ϕ_s^* gives

$$(9) \quad \phi_s^* (\alpha_t(\dot{\phi}_s)) \Omega - \phi_s^* (\alpha_t) \wedge \phi_s^* (\iota(\dot{\phi}_s)\Omega) = 0,$$

for each $s \in [0, t]$. Using (1), one deduces that for each $s \in [0, t]$, we have

$$(10) \quad \phi_s^*(\alpha_t) = \alpha_t + df_{\{\phi_t\}, \alpha_t}^s,$$

where $f_{\{\phi_t\}, \alpha_t}^s := \int_0^s \alpha_t(\dot{\phi}_u) \circ \phi_u du$. Observe that (9) and (10) together implies that,

$$(11) \quad \phi_s^* \left(\alpha_t(\dot{\phi}_s) \right) \Omega = \alpha_t \wedge \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) + df_{\{\phi_t\}, \alpha_t}^s \wedge \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right),$$

for each $s \in [0, t]$. Thus,

$$(12) \quad \begin{aligned} \int_M \left(\int_0^t \phi_s^* \left(\alpha_t(\dot{\phi}_s) \right) ds \right) \Omega &= \int_M \alpha_t \wedge \left(\int_0^t \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds \right) \\ &+ \int_M \left(\int_0^t \left(\phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) \wedge df_{\{\phi_t\}, \alpha_t}^s \right) ds \right). \end{aligned}$$

On the other hand, since the form $\phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right)$ is closed, then

$$(13) \quad \int_M \int_0^t \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) \wedge df_{\{\phi_t\}, \alpha_t}^s ds = \int_M d \left[\int_0^t f_{\{\phi_t\}, \alpha_t}^s \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds \right],$$

and since M is without boundary, then Stokes' theorem implies

$$(14) \quad \int_M d \left[\int_0^t f_{\{\phi_t\}, \alpha_t}^s \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds \right] = \int_{\partial M} \int_0^t f_{\{\phi_t\}, \alpha_t}^s \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds = 0,$$

for each t . Combining (12), (13), and (14) together yields

$$(15) \quad \int_M \left(\int_0^t \phi_s^* \left(\alpha_t(\dot{\phi}_s) \right) ds \right) \Omega = \int_M \alpha_t \wedge \left(\int_0^t \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds \right).$$

For each t , (15) becomes $\int_M \mathcal{F}_{\alpha_t}(\Phi)(t) \Omega = \langle [\alpha_t], [\int_0^t \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds] \rangle$, for all t . $\square$

From Proposition 3.3, we derive the following explicit definition of the flux homomorphism.

Definition 3.4. *We have a group homomorphism*

$$\tilde{S}_\Omega : Iso(M, \Omega) \rightarrow H^{n-1}(M, \mathbb{R}),$$

$$\Phi := \{\phi_t\} \mapsto \left[\int_0^1 \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds \right],$$

called the flux homomorphism.

3.1.1. *Surjectivity of flux homomorphism.* Let us mention that for each closed 1-form α (fixed), we have a surjective mapping $\Phi \mapsto \int_M \mathcal{F}_\alpha(\Phi)(1)\Omega$, from $Iso(M, \Omega)$ onto $H^{n-1}(M, \mathbb{R})$. Let r be a real number: If $r = 0$, then take Φ to be the constant path identity. Suppose $r \neq 0$: This means that there exists at least $\Phi = \{\phi^t\} \in Iso(M, \Omega)$ such that $\int_M \mathcal{F}_\alpha(\Phi)(1)\Omega \neq 0$. With Proposition 3.3, this is equivalent to say that $\int_0^1 \langle P(\alpha), P(\iota(\dot{\phi}^t)\Omega) \rangle dt \neq 0$, where $P : Z^1(M) \longrightarrow H^1(M, \mathbb{R}), \alpha \mapsto [\alpha]$ i.e., there exists a $t_0 \in [0, 1]$ such that $\langle P(\alpha), P(\iota(X)\Omega) \rangle \neq 0$, with $X := \dot{\phi}^{t_0}$. Let Ψ be the flow generated by X , and define another conservative vector field by setting: $Y := rX / \langle P(\alpha), P(\iota(X)\Omega) \rangle$. If Θ stands for the flow generated by Y , then one has:

$$\begin{aligned}
 \int_M \mathcal{F}_\alpha(\Theta)(1)\Omega &= \langle P(\alpha), \tilde{S}_\Omega(\Theta) \rangle \\
 &= \langle P(\alpha), P(\iota(Y)\Omega) \rangle, \\
 (16) \quad &= r \langle P(\alpha), P(\iota(X)\Omega) \rangle / \langle P(\alpha), P(\iota(X)\Omega) \rangle \\
 &= r.
 \end{aligned}$$

Furthermore, for each non-exact closed 1-form α (fixed), since the linear map

$$L_\alpha : H^{(n-1)}(M, \mathbb{R}) \ni \beta \mapsto \langle P(\alpha), \beta \rangle \in \mathbb{R},$$

is non-trivial, hence the latter is surjective. If we denote by H_α the surjective mapping $\Phi \mapsto \int_M \mathcal{F}_\alpha(\Phi)(1)\Omega$, then by Proposition 3.3 we have $H_\alpha = L_\alpha \circ \tilde{S}_\Omega$, which is surjective. The flux homomorphism is as well a surjective map:

This comes from the fact that the map which assigns to each vector field X the $(n-1)$ -form $\iota_X \Omega$ is onto : In fact, let β be a $(n-1)$ -form, we have to find Y such that $\iota_Y \Omega = \beta$. Consider any chart (U, ϕ) of M such that for each $x \in U$, the vectors $\frac{\partial}{\partial \phi_1}|_x, \dots, \frac{\partial}{\partial \phi_n}|_x$, forms a basis of the tangent space $T_x M$ at x so that $d\phi_j(\frac{\partial}{\partial \phi_i}) = \delta_i^j$, the kronecker symbol. Write Y in the form $Y|_U = \sum_{i=1}^n Y_i \frac{\partial}{\partial \phi_i}$. Our main task is to determine Y_i with respect to the chart (U, ϕ) . Let α be any closed 1-form, and derive that $\alpha(Y)\Omega = \alpha \wedge \iota_Y \Omega$. This together with the condition $\iota_Y \Omega = \beta$ implies that $\alpha(Y)\Omega = \alpha \wedge \beta$. In particular, for $\alpha = d\phi_i$, we have that $Y_i \Omega = d\phi_i(Y)\Omega = d\phi_i \wedge \beta$. Since Ω is a volume form on M , then for each $z \in U$, there exist vector fields $X_1, \dots, X_n$ on M such that $\Omega(X_1, \dots, X_n)(z) \neq 0$. That is,

$$Y_i(z) := \frac{(d\phi_i \wedge \beta)(X_1, \dots, X_n)(z)}{\Omega(X_1, \dots, X_n)(z)}.$$

In particular, when β is a closed form, then the vector field Y determined as above is conservative, and so its generating flow Φ_Y is a volume-preserving isotopy so that $\tilde{S}_\Omega(\Phi_Y) = [\beta]$.

3.1.2. *C^1 continuity of the flux homomorphism.* Let $\{\Phi_i = (\phi_i^t)\} \subset Iso(M, \Omega)$ be a sequence that converges in the C^1 -topology to

$\Phi = \{\phi^t\} \in Iso(M, \Omega)$, i.e., $\Phi_i \circ \Phi^{-1} \xrightarrow{C^1} Id$. Then, it is not hard derive with the help of Lemma 3.10-[15] that,

$$\lim_i \left| \int_M \mathcal{F}_\alpha(\Phi_i \circ \Phi^{-1})(1)\Omega \right| \leq 4|\alpha|_0 Vol_\Omega(M) d_{C^1}(\Phi_i \circ \Phi^{-1}, Id)\Omega \rightarrow 0, i \rightarrow \infty.$$

Thus, by Proposition 3.3, this means that for each closed 1-form α , we have $\lim_i \langle P(\alpha), \tilde{S}_\Omega(\Phi_i \circ \Phi^{-1}) \rangle = 0$, i.e. $\langle P(\alpha), \lim_i \tilde{S}_\Omega(\Phi_i \circ \Phi^{-1}) \rangle = 0$, for all closed 1-form α . Hence, $\lim_i \tilde{S}_\Omega(\Phi_i) = \tilde{S}_\Omega(\Phi)$: The map $\tilde{S}_\Omega$ is a continuous group homomorphism. $\blacklozenge$

The homomorphism $\tilde{S}_\Omega$ have been studied by Weinstein [18] and Thurston [16]. Set $\Gamma_\Omega = \tilde{S}_\Omega(\pi_1(G_\Omega(M)))$, and derive from elementary group theory that the map $\tilde{S}_\Omega$ descends to the quotient into a map S_Ω so that the following diagram commutes:

$$(17) \quad \begin{array}{ccc} \tilde{G}_\Omega(M) & \xrightarrow{\tilde{S}_\Omega} & H^{(n-1)}(M, \mathbb{R}) \\ P_1 \downarrow & & \downarrow P_2 \\ G_\Omega(M) & \xrightarrow{S_\Omega} & H^{(n-1)}(M, \mathbb{R})/\Gamma_\Omega, \end{array}$$

where $\tilde{G}_\Omega(M)$ represents the quotient space of $Iso(M, \Omega)$ with respect to the equivalent relation "homotopic relatively to fixed endpoints"; P_i are projection maps.

The following facts are well-known:

- The sub-group Γ_Ω is discrete, (unpublished result of Thurston): We shall give the proof of this result later on.
- The group $\ker S_\Omega$ has the fragmentation property, [16].
- As a consequence of Moser's theorem, the group $G_\Omega(M)$ is locally connected by smooth arcs, [3].
- Boothby proved that the group $G_\Omega(M)$ is p -transitive, [9].

Proposition 3.3 gives a short proof of Lemma 2.4: (6) implies that

$$\int_M \mathcal{F}_\alpha(\Phi \circ \Psi)(t)\Omega = \int_M \mathcal{F}_\alpha(\Psi)(t)\Omega + \int_M \mathcal{F}_\alpha(\Phi)(t) \circ \Psi(t)\Omega + \delta_t(\Phi, \Psi, \alpha) \int_M \Omega,$$

for all t , and according to Proposition 3.3, we have

$$\int_M \mathcal{F}_\alpha(\Phi \circ \Psi)(t)\Omega = \int_M \mathcal{F}_\alpha(\Psi)(t)\Omega + \int_M \mathcal{F}_\alpha(\Phi)(t)\Omega.$$

So, we see that $\delta_t(\Phi, \Psi, \alpha) \int_M \Omega = 0$, for all t ; this implies that $\delta_t(\Phi, \Psi, \alpha) = 0$, for all t . $\blacklozenge$

Here are some consequences of Proposition 3.3.

Lemma 3.5. *Let $\Phi \in \text{Iso}(M, \Omega)$ be a loop at the identity map. For each closed 1-form α , the smooth function $x \mapsto \int_{\mathcal{O}_x^\Phi} \alpha$ is constant and agrees with*

$$\frac{1}{\text{Vol}_\Omega(M)} \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle,$$

where $P : Z^1(M) \longrightarrow H^1(M, \mathbb{R}), \alpha \mapsto [\alpha]$.

Proof. Let α be a closed 1-form. If $P(\alpha) = 0$, then $\int_{\mathcal{O}_x^\Phi} \alpha = 0$, for all $x \in M$ since $\mathcal{O}_x^\Phi$ is a closed curve. On the other hand, suppose $P(\alpha) \neq 0$, and set $\mathcal{H} = \mathcal{S}(P(\alpha))$, where $\mathcal{S}$ is any linear section of the projection P . Since $\Phi(0) = \Phi(1) = \text{id}_M$, by formula (1) the map $x \mapsto \int_{\mathcal{O}_x^\Phi} \alpha = \mathcal{F}_\alpha(\Psi)(1)(x)$ is constant. Thus, Proposition 3.3 implies that

$$\begin{aligned} \text{Vol}_\Omega(M) \int_{\mathcal{O}_x^\Phi} \alpha &= \text{Vol}_\Omega(M) \int_{\mathcal{O}_x^\Phi} \mathcal{H} \\ (18) \qquad \qquad \qquad &= \int_M \mathcal{F}_\alpha(\Psi)(1)\Omega \\ &= \langle P(\mathcal{H}), \tilde{S}_\Omega(\Phi) \rangle, \end{aligned}$$

for all $x \in M$. $\square$

Remark 3.6. *Geometrically, Lemma 3.5 seems to suggest that on a closed oriented manifold (M, Ω) , the set $\pi_1(G_\Omega(M)) \setminus (\ker \tilde{S}_\Omega \cap \pi_1(G_\Omega(M)))$ measures the obstruction that impeaches the orbits generated by all the elements of $\pi_1(G_\Omega(M))$ to be all null homologous in M . $\blacklozenge$*

Lemma 3.7. *Let (M, Ω) be a closed oriented manifold. Let $\Phi \in \text{Iso}(M, \Omega)$ be a loop at the identity. Then, $\tilde{S}_\Omega(\Phi) = 0$, if and only if, there exists $z \in M$ such that the orbit $\mathcal{O}_z^\Phi$ is null homologous.*

Proof. As in the proof of Lemma 3.5, we have $\frac{1}{\text{Vol}_\Omega(M)} \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle = \int_{\mathcal{O}_x^\Phi} \alpha$, for each closed 1-form α : If $z \in M$ is such that the orbit $\mathcal{O}_z^\Phi$ is null homologous, then Stokes' theorem implies that $\int_{\mathcal{O}_z^\Phi} \alpha = 0$ for each closed 1-form α ; and this imposes

that $\langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle = 0$, for all closed 1-form α , i.e. $\tilde{S}_\Omega(\Phi) = 0$. Conversely, if $\tilde{S}_\Omega(\Phi) = 0$, then we immediately see that $0 = \int_{\mathcal{O}_z^\Phi} \alpha$, for all closed 1-form α , and for all $z \in M$. This implies that the orbit $\mathcal{O}_z^\Phi$ must be null homologous. $\square$

Proposition 3.8. (Factorization II) *Let (M, Ω) be a compact connected oriented manifold with boundary. Let $\alpha = (\alpha_t)$ be a smooth family of smooth closed 1-forms, and let $\Phi = \{\phi_t\} \in \text{Iso}(M, \Omega)$. Then, for each $t \in [0, 1]$, we have,*

$$\int_M \mathcal{F}_{\alpha_t}(\Phi)(t) \Omega = \langle P(\alpha_t), \tilde{S}_\Omega(\bar{\Phi}_t) \rangle + \int_{\partial M} \left(\int_0^t f_{\Phi, \alpha_t}^s \phi_s^* (\iota(\dot{\phi}_s) \Omega) ds \right),$$

where $\bar{\Phi}_t$ is the isotopy $s \mapsto \phi_{st}$, and $f_{\Phi, \alpha_t}^s := \int_0^s \alpha_t(\dot{\phi}_u) \circ \phi_u du$.

Proposition 3.9. (Factorization III) *Let (M, Ω) be a closed oriented manifold with $\Omega := \alpha^l := \underbrace{\alpha \wedge \cdots \wedge \alpha}_{l\text{-times}}$, where α is a closed $\deg(\alpha)$ -form of degree $\deg(\alpha) := \dim(M)/l$ such that for each volume-preserving isotopy $\Phi = \{\phi_t\}$, the $[\deg(\alpha) - 1]$ -form $\iota(\dot{\phi}_t)\alpha$ is closed for each t . Therefore,*

(1) *if $\deg(\alpha)$ is even, then*

$$\tilde{S}_\Omega(\Phi) = \frac{\dim(M)}{\deg(\alpha)} P \left(\alpha \left(\left(\frac{\dim(M)}{\deg(\alpha)} \right)^{-1} \right) \wedge \int_0^1 \iota(\dot{\phi}_t) \alpha dt \right),$$

(2) *if $\deg(\alpha)$ is odd, then*

$$\tilde{S}_\Omega(\Phi) = (-1)^{\left(1+E\left(\frac{\dim(M)}{2\deg(\alpha)}\right)\right)} P \left(\alpha \left(\left(\frac{\dim(M)}{\deg(\alpha)} \right)^{-1} \right) \wedge \int_0^1 \iota(\dot{\phi}_t) \alpha dt \right),$$

where for each positive real number x , $E(x)$ is x minus its fractional part; and P is the natural projection $P : \mathcal{Z}^*(M) \rightarrow H^*(M, \mathbb{R})$.

Proof. Assume α is a closed $\deg(\alpha)$ -form of even degree, and let β be any closed 1-form. From $\beta \wedge \Omega = 0$, we derive as in the proof of Proposition 3.3 that

$$\begin{aligned} \phi_s^* (\beta(\dot{\phi}_s)) \Omega &= \phi_s^* (\beta \wedge \iota(\dot{\phi}_s) \Omega), \\ &= \beta \wedge \phi_s^* (\iota(\dot{\phi}_s) \Omega) + df_{\Phi, \beta}^s \wedge \phi_s^* (\iota(\dot{\phi}_s) \Omega), \\ &= l\beta \wedge \phi_s^* (\iota(\dot{\phi}_s) \alpha) \wedge \phi_s^* (\alpha^{l-1}) + df_{\Phi, \beta}^s \wedge \phi_s^* (\iota(\dot{\phi}_s) \Omega), \end{aligned} \tag{19}$$

for each $s \in [0, 1]$, with $l = \dim(M)/\deg(\alpha)$. Thus,

$$(20) \quad \int_M \left(\int_0^1 \phi_s^* \left(\beta(\dot{\phi}_s) \right) ds \right) \Omega = l \int_M \beta \wedge \left(\int_0^1 \phi_s^* \left(\iota(\dot{\phi}_s) \alpha \wedge \alpha^{l-1} \right) ds \right) \\ + \int_{\partial M} \int_0^1 \left(f_{\Phi, \alpha_t}^s \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right) ds \right).$$

By the means of Proposition 3.8, we derive that,

$$\langle P(\beta), \tilde{S}_\Omega(\Phi) \rangle = \int_M \left(\int_0^1 \phi_s^* \left(\beta(\dot{\phi}_s) \right) ds \right) \Omega = \langle P(\beta), lP \left(\alpha^{l-1} \wedge \int_0^1 \iota(\dot{\phi}_s) \alpha ds \right) \rangle,$$

for all closed 1-form β . Therefore, the non-degeneracy of $\langle \cdot, \cdot \rangle$ implies that

$$\tilde{S}_\Omega(\Phi) = lP \left(\alpha^{l-1} \wedge \int_0^1 \iota(\dot{\phi}_s) \alpha ds \right).$$

For the second item, assume that α is a closed $\deg(\alpha)$ -form of odd degree, and let β be any closed 1-form. Similar calculations lead to

$$(21) \quad \phi_s^* \left(\beta(\dot{\phi}_s) \right) \Omega = (-1)^\epsilon \phi_s^*(\beta) \wedge \phi_s^* \left(\iota(\dot{\phi}_s) \alpha \right) \wedge \phi_s^* \left(\alpha^{l-1} \right) + df_{\Phi, \beta}^s \wedge \phi_s^* \left(\iota(\dot{\phi}_s) \Omega \right),$$

for each $s \in [0, 1]$, with $\epsilon := \left(1 + E \left(\frac{\dim(M)}{2 \deg(\alpha)} \right) \right)$, where for each positive real number x , $E(x)$ is x minus its fractional part. $\square$

4. THE GEOMETRY OF DISPLACEMENTS OF CLOSED 1-FORMS

In this section, we assume that $\partial M = \emptyset$. Let $\psi \in \text{Diff}_0^\infty(M)$, for each fixed $p \in M$ (such a point is called a base point), and for all $z \in M$, let ξ_z denote a geodesic from p to z . Then, for each closed 1-form α , the following smooth function

$$\nu_p^{\psi, \alpha} : M \rightarrow \mathbb{R}, z \mapsto \int_{\xi_z} (\psi^* \alpha - \alpha),$$

is well-defined since it does not depend on the choice of any curve from p to z . The real number $\int_{\xi} (\psi^* \alpha - \alpha)$ can be viewed (in a certain sense) as the displacement of α under the action of ψ along a curve ξ .

For further investigations, consider the following function defined on $\text{Diff}_0^\infty(M)$ as follows: For each $p \in M$ and $\alpha \in \mathcal{Z}^1$ (fixed), we assign to each $\psi \in \text{Diff}_0^\infty(M)$, the quantity $\Delta(\psi, \alpha)_p$ defined by:

$$(22) \quad \Delta(\psi, \alpha)_p := \begin{cases} 0, & \text{if } \alpha = 0 \\ \frac{1}{\|\alpha\|_{L^2}} \int_M \nu_p^{\psi, \alpha} \Omega, & \text{if } \alpha \in (\mathcal{Z}^1(M) \setminus \{0\}), \end{cases}$$

where $\|\alpha\|_{L^2}$ is the L^2 –Hodge norm of α . The above formula is well-defined for each $p \in M$ and all $\alpha \in \mathcal{Z}^1(M)$.

Remark 4.1. *Note that given a Riemannian metric g on an oriented manifold M , let $dVol_g$ denote the volume element induced by the Riemannian metric, and by $\flat$ denote the isomorphism induced by the metric g from the space of vector fields to that of all 1-forms so that $\flat(X) := \iota_X g$. The inverse mapping of $\flat$ will be denoted $\sharp$. So, one can equip the space of 1-forms with a metric tensor $\tilde{g}$ defined by:*

$$\tilde{g}(\alpha_1, \alpha_2) := g(\sharp\alpha_1, \sharp\alpha_2),$$

for all differential 1-forms α_1, α_2 . Thus, the L^2 –Hodge norm of α is defined by:

$$(23) \quad \|\alpha\|_{L^2}^2 := \int_M g(\sharp\alpha, \sharp\alpha) dVol_g.$$

Corollary 4.2. *Fix a point $x \in M$, and $\alpha \in (\mathcal{Z}^1(M) \setminus \{0\})$. For each $\psi \in G_\Omega(M)$, we have*

$$\Delta(\psi, \alpha)_x = \frac{1}{\|\alpha\|_{L^2}} \langle P(\alpha), \tilde{S}_\Omega(\Psi) \rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \int_{\mathcal{O}_x^\Psi} \alpha,$$

and the real number $\Delta(\psi, \alpha)_x$ does not depend on any choice of an isotopy $\Psi \in Iso(\psi)_\Omega$.

Proof. Fix a point $x \in M$. Formula (1) implies that if $\Phi, \Psi \in Iso(\psi)_\Omega$, then $\mathcal{F}_\alpha(\Psi)(1)(y) = \mathcal{F}_\alpha(\Phi)(1)(y) + cte$, for all $y \in M$, i.e.

$$\mathcal{F}_\alpha(\Psi)(1)(y) - \mathcal{F}_\alpha(\Psi)(1)(x) = \mathcal{F}_\alpha(\Phi)(1)(y) - \mathcal{F}_\alpha(\Phi)(1)(x),$$

for all $y \in M$. On the other hand, if ξ_y denotes any curve from x to y , then it follows that

$$\nu_x^{\psi, \alpha}(y) = \mathcal{F}_\alpha(\Psi)(1)(y) - \mathcal{F}_\alpha(\Psi)(1)(x),$$

for all $y \in M$, i.e.

$$\mathcal{F}_\alpha(\Psi)(1)(y) - \mathcal{F}_\alpha(\Psi)(1)(x) = \nu_x^{\psi, \alpha}(y) = \mathcal{F}_\alpha(\Phi)(1)(y) - \mathcal{F}_\alpha(\Phi)(1)(x),$$

for all $y \in M$. Hence, integrating the above relation over M yields,

$$\int_M \mathcal{F}_\alpha(\Psi)(1) \Omega - \mathcal{F}_\alpha(\Psi)(1)(x) \int_M \Omega = \int_M \nu_x^{\psi, \alpha} \Omega$$

$$= \int_M \mathcal{F}_\alpha(\Phi)(1)\Omega - \mathcal{F}_\alpha(\Phi)(1)(x) \int_M \Omega.$$

Therefore, multiplying the above equality on both sides by $1/\|\alpha\|_{L^2}$ yields

$$(24) \quad \frac{1}{\|\alpha\|_{L^2}} \int_M \mathcal{F}_\alpha(\Psi)(1)\Omega - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \mathcal{F}_\alpha(\Psi)(1)(x) = \frac{1}{\|\alpha\|_{L^2}} \int_M \nu_x^{\psi, \alpha} \Omega,$$

$$= \frac{1}{\|\alpha\|_{L^2}} \int_M \mathcal{F}_\alpha(\Phi)(1)\Omega - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \mathcal{F}_\alpha(\Phi)(1)(x),$$

for all $\Phi, \Psi \in Iso(\psi)_\Omega$. Proposition 3.3 implies that, $\int_M \mathcal{F}_\alpha(\Phi)(1)\Omega = \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle$. $\square$

Let Φ be an isotopy, for each positive integer l , we shall write Φ^l to mean the l -fold of the isotopy Φ , i.e., $\Phi^l = \underbrace{\Phi \circ \dots \circ \Phi}_{l\text{-times}}$. Similarly, for each diffeomorphism ϕ and for each positive integer k , we shall write ϕ^k to mean $\phi^k = \underbrace{\phi \circ \dots \circ \phi}_{k\text{-times}}$.

Proposition 4.3. (Fragmentation) *For each $x \in M$ (fixed), and for each $\alpha \in \mathcal{Z}^1(M) \setminus \{0\}$, the mapping $\Delta(\cdot, \alpha)_x : G_\Omega(M) \rightarrow \mathbb{R}, \psi \mapsto \Delta(\psi, \alpha)_x$, has the following properties:*

(1) *For all $\psi^1, \dots, \psi^k \in G_\Omega(M)$, we have*

$$\Delta(\psi^1 \circ \dots \circ \psi^k, \alpha)_x = \Delta(\psi^k, \alpha)_x + \sum_{1 \leq i \leq k-1} \Delta(\psi^i, \alpha)_{(\psi^{i+1} \circ \dots \circ \psi^k)(x)}.$$

(2) *In particular,*

$$\Delta(\psi^1 \circ \psi^2, \alpha)_x = \Delta(\psi^1, \alpha)_x + \Delta(\psi^2, \alpha)_x + \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \left(\int_{\mathcal{O}_x^\Phi} \alpha - \int_{\mathcal{O}_{\psi^2(x)}^\Phi} \alpha \right),$$

for all $\Phi \in Iso(\psi^1)_\Omega$.

(3) *For all $l \in \mathbb{Z}^*$, and for each $\psi \in G_\Omega(M)$, we have*

$$\Delta(\psi^l, \alpha)_x = l\Delta(\psi, \alpha)_x + \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \left(l \int_{\mathcal{O}_x^{\Phi^\epsilon}} \alpha - \sum_{i=0}^{l-1} \int_{\mathcal{O}_{\psi^{\epsilon^i}(x)}^{\Phi^\epsilon}} \alpha \right),$$

where $\Phi \in Iso(\psi)_\Omega$, and $\epsilon := \text{sign}(l)$, and $\psi^{\epsilon^i} = \underbrace{\psi^\epsilon \circ \dots \circ \psi^\epsilon}_{i\text{-times}}$.

Proof.

For the first item, we proceed by induction: For $k = 2$, we derive from Corollary 4.2 that

$$\Delta(\psi^1 \circ \psi^2, \alpha)_x = \frac{1}{\|\alpha\|_{L^2}} \langle P(\alpha), \tilde{S}_\Omega(\Psi *_l \Phi) \rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \int_{\mathcal{O}_x^{\Psi *_l \Phi}} \alpha,$$

where $\Phi *_l \Psi \in Iso(\psi^1 \circ \psi^2)_\Omega$ is the left concatenation of $\Psi \in Iso(\psi^2)_\Omega$ by $\Phi \in Iso(\psi^1)_\Omega$. Observe that $\tilde{S}_\Omega(\Psi *_l \Phi) = \tilde{S}_\Omega(\Psi) + \tilde{S}_\Omega(\Phi)$; and $\mathcal{O}_x^{\Phi *_l \Psi} = \mathcal{O}_x^\Psi \sqcup \mathcal{O}_{\psi^2(x)}^\Phi$, and deduce that $\Delta(\psi^1 \circ \psi^2, \alpha)_x = \Delta(\psi^2, \alpha)_x + \Delta(\psi^1, \alpha)_{\psi^2(x)}$. Let $k > 2$: Assume that

$$\Delta(\psi^1 \circ \dots \circ \psi^{k-1}, \alpha)_x = \Delta(\psi^{k-1}, \alpha)_x + \sum_{k-2 \geq i \geq 1} \Delta(\psi^i, \alpha)_{(\psi^{i+1} \circ \dots \circ \psi^{k-1})(x)},$$

set $\phi := \psi^1 \circ \dots \circ \psi^{k-1}$, and compute

$$\begin{aligned} \Delta(\phi \circ \psi^k, \alpha)_x &= \Delta(\psi^k, \alpha)_x + \Delta(\phi, \alpha)_{\psi^k(x)} \\ &= \Delta(\psi^k, \alpha)_x + \Delta(\psi^{k-1}, \alpha)_{(\psi^k(x))} + \sum_{k-2 \geq i \geq 1} \Delta(\psi^i, \alpha)_{(\psi^{i+1} \circ \dots \circ \psi^{k-1})(x)}, \\ &= \Delta(\psi^k, \alpha)_x + \sum_{k-1 \geq i \geq 1} \Delta(\psi^i, \alpha)_{(\psi^{i+1} \circ \dots \circ \psi^{k-1})(x)}. \end{aligned}$$

For the second item, a direct computation with the help of Corollary 4.2 shows that

$$\begin{aligned} \Delta(\psi^1, \alpha)_{\psi^2(x)} &= \frac{1}{\|\alpha\|_{L^2}} \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \int_{\mathcal{O}_{\psi^2(x)}^\Phi} \alpha \\ &= \Delta(\psi^1, \alpha)_x + \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \left(\int_{\mathcal{O}_x^\Phi} \alpha - \int_{\mathcal{O}_{\psi^2(x)}^\Phi} \alpha \right). \end{aligned}$$

For the third item, assume l positive, and for each $i = 0, 1, \dots, l-1$; consider a smooth increasing function λ_i , defined from $[i/l, i+1/l]$ onto $[0, 1]$. Define an isotopy Φ^l as follows: $\Phi^l(t) = \Phi(\lambda_i(t)) \circ \psi^i$, whenever $t \in [i/l, i+1/l]$, and $i = 0, 1, \dots, l-1$. Since $\Phi \in Iso(\psi)_\Omega$, we have $\Phi^l \in Iso(\psi^l)_\Omega$. So, we obtain $\tilde{S}_\Omega(\Phi^l) = l\tilde{S}_\Omega(\Phi)$, and $\mathcal{O}_x^{\Phi^l} = \bigcup_{i=0}^{l-1} \mathcal{O}_{\psi^i(x)}^\Phi$: this in turn with Corollary 4.2 implies that

$$\begin{aligned} \Delta(\psi^l, \alpha)_x &= \frac{1}{\|\alpha\|_{L^2}} \langle P(\alpha), \tilde{S}_\Omega(\Phi^l) \rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \int_{\mathcal{O}_x^{\Phi^l}} \alpha \\ &= \frac{l}{\|\alpha\|_{L^2}} \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \sum_{i=0}^{l-1} \int_{\mathcal{O}_{\psi^i(x)}^\Phi} \alpha \end{aligned}$$

$$= l\Delta(\psi, \alpha)_x + \frac{\text{Vol}_\Omega(M)}{\|\alpha\|_{L^2}} \left(l \int_{\mathcal{O}_x^\Phi} \alpha - \sum_{i=0}^{l-1} \int_{\mathcal{O}_{\psi^i(x)}^\Phi} \alpha \right).$$

When $\epsilon = -1$, i.e., l is negative, the result follows from similar arguments. $\square$

Remark 4.4. *Since M is compact, the C^0 –compact-open topology on $\text{Diff}_0^\infty(M)$ is metrizable [10]: we shall denote the corresponding metric by d_{C^0} , and call it the C^0 –metric.*

Lemma 4.5. *For each $x \in M$ (fixed), and for each $\alpha \in \mathcal{Z}^1(M)$, the map $\Delta(\cdot, \alpha)_x : \text{Diff}_0^\infty(M) \rightarrow \mathbb{R}, \psi \mapsto \Delta(\psi, \alpha)_x$, is continuous with respect to the C^0 –metric on $\text{Diff}_0^\infty(M)$.*

Proof. Let $(\psi_i) \subseteq \text{Diff}_0^\infty(M)$ be a sequence which converges to $\psi \in \text{Diff}_0^\infty(M)$ with respect to the C^0 –metric. Suppose that for i sufficiently large

$$(25) \quad d_{C^0}(\psi_i, \psi) \leq r(g)/i,$$

where $r(g)$ is the injectivity radius of the Riemannian metric g on M (fixed). Since for all i sufficiently large, the metric condition in (25) holds, then for all $y \in M$, the points $\psi_i(y)$ and $\psi(y)$ can be connected through a unique minimal geodesic χ_y^i . Let z be any point of M which realizes the supremum of the function $y \mapsto |\nu_x^{\psi_i, \alpha}(y) - \nu_x^{\psi, \alpha}(y)|$, and for all i sufficiently large, denote by $\spadesuit(z, x, \psi_i, \psi)$ the smooth 2–chain delimited by the curves $\psi_i(\gamma_z)$, $\psi(\gamma_z)$, χ_z^i , and χ_x^i , where γ_z is any smooth curve x to z . Compute,

$$\nu_x^{\psi_i, \alpha}(z) - \nu_x^{\psi, \alpha}(z) = \int_{\gamma_z} \psi_i^*(\alpha) - \int_{\gamma_z} \psi^*(\alpha) = \int_{\psi_i(\gamma_z)} \alpha - \int_{\psi(\gamma_z)} \alpha,$$

and since α is closed, derive from Stokes' theorem that

$$0 = \int_{\spadesuit(z, x, \psi_i, \psi)} d\alpha = \int_{\partial \spadesuit(z, x, \psi_i, \psi)} \alpha,$$

i.e.,

$$\begin{aligned}
 \sup_y |\nu_x^{\psi_i, \alpha}(y) - \nu_x^{\psi, \alpha}(y)| &= \left| \int_{\gamma_z} \psi_i^*(\alpha) - \int_{\gamma_z} \psi^*(\alpha) \right| \\
 &= \left| \int_{\psi_i(\gamma_z)} \alpha - \int_{\psi(\gamma_z)} \alpha \right| \\
 &= \left| \int_{\chi_z^i} \alpha - \int_{\chi_x^i} \alpha \right| \\
 &\leq \left| \int_{\chi_z^i} \alpha \right| + \left| \int_{\chi_x^i} \alpha \right| \\
 (26) \qquad \qquad \qquad &\leq 2\|\alpha\|_{L^2} d_{C^0}(\psi_i, \psi),
 \end{aligned}$$

because the length of a minimal geodesic is bounded from above by the distance between its endpoints. Hence,

$$\begin{aligned}
 |\Delta(\psi_i, \alpha)_x - \Delta(\psi, \alpha)_x| &\leq \frac{\text{Vol}_\Omega(M)}{\|\alpha\|_{L^2}} \sup_y |\nu_x^{\psi_i, \alpha}(y) - \nu_x^{\psi, \alpha}(y)| \\
 &\leq 2\text{Vol}_\Omega(M) d_{C^0}(\psi_i, \psi) \rightarrow 0, i \rightarrow \infty.
 \end{aligned}$$

□

4.1. Flux geometry through orbits.

4.1.1. *The de Rham groups, projection, and linear sections.* Fix a linear section $\mathcal{S}$ of the natural projection $P : \mathcal{Z}^*(M) \rightarrow H^*(M, \mathbb{R})$. That is, $\mathcal{S} : H^*(M, \mathbb{R}) \rightarrow \mathcal{Z}^*(M)$ is a linear map such that $P \circ \mathcal{S} = \text{id}_{H^*(M, \mathbb{R})}$. Therefore, each $\alpha \in \mathcal{Z}^*(M)$ decomposes as

$$(27) \qquad \qquad \qquad \alpha = \mathcal{S}(P(\alpha)) + (\alpha - \mathcal{S}(P(\alpha))),$$

with $P(\mathcal{S}(P(\alpha))) = P(\alpha)$ and $P((\alpha - \mathcal{S}(P(\alpha)))) = 0$. Note that a form α such that $P(\alpha) = 0$ is called an exact form. In the decomposition above, for each $\alpha \in \mathcal{Z}^*(M)$, the form $\mathcal{S}(P(\alpha))$ is called the non-exact part of α while $(\alpha - \mathcal{S}(P(\alpha)))$ is called the exact part of α . Let $\mathcal{H}^*(M, \mathcal{S})$ denote the image $\mathcal{S}(P(\mathcal{Z}^*(M))) \subset \mathcal{Z}^*(M)$: $\mathcal{H}^*(M, \mathcal{S})$ is isomorphic to $H^*(M, \mathbb{R})$, and it follows from Hodge's theory that the latter set is a finite dimensional vector space over $\mathbb{R}$ whose dimension is denoted $b_*(M)$, and is called the $*$ -th Betti number of the manifold M [17]. For $*$ = 1, we fix a basis $(H_i)_{1 \leq i \leq b_1(M)}$ on $\mathcal{H}^1(M, \mathcal{S})$ and equip it with a norm $|\cdot|$.

We shall also write $\overline{\mathcal{B}_\mathcal{S}^*(0, 1)}$ to mean the closure of the open unit ball $\mathcal{B}_\mathcal{S}^*(0, 1)$ of the finite dimensional vector space $\mathcal{H}^*(M, \mathcal{S})$: The set $\overline{\mathcal{B}_\mathcal{S}^*(0, 1)}$ is compact.

Proposition 4.6. *Let (M, Ω) be a closed oriented manifold. For each $\varphi \in (G_\Omega(M) \setminus \{id_M\})$, let Φ and Ψ be two volume-preserving isotopies in $Iso(\varphi)_\Omega$. If there exists a point $z_0 \in M$ for which the orbits $\mathcal{O}_{z_0}^\Psi$ and $\mathcal{O}_{z_0}^\Phi$ are homotopic relatively to fix endpoints, then*

$$\tilde{S}_\Omega(\Phi) = \tilde{S}_\Omega(\Psi).$$

Proof. Suppose that the 1-cycle $(-\mathcal{O}_{z_0}^\Psi) \sqcup \mathcal{O}_{z_0}^\Phi$ is null homologous. Consider α to be any closed 1-form such that $P(\alpha) \neq 0$. Corollary 4.2 implies that

$$(28) \quad \Delta(\varphi, \alpha)_{z_0} = \frac{1}{\|\alpha\|_{L^2}} \langle P(\alpha), \tilde{S}_\Omega(\Theta) \rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}} \int_{\mathcal{O}_{z_0}^\Theta} \alpha,$$

for all $\Theta \in Iso(\varphi)_\Omega$. In particular, since each of the isotopies Φ and Ψ belongs to $Iso(\varphi)_\Omega$, we then obtain from (28) that

$$(29) \quad \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle - Vol_\Omega(M) \int_{\mathcal{O}_{z_0}^\Phi} \alpha = \langle P(\alpha), \tilde{S}_\Omega(\Psi) \rangle - Vol_\Omega(M) \int_{\mathcal{O}_{z_0}^\Psi} \alpha,$$

i.e.,

$$(30) \quad \langle P(\alpha), \tilde{S}_\Omega(\Psi) - \tilde{S}_\Omega(\Phi) \rangle = Vol_\Omega(M) \int_{(-\mathcal{O}_{z_0}^\Psi) \sqcup \mathcal{O}_{z_0}^\Phi} \alpha,$$

for all closed 1-form α . But, since $(-\mathcal{O}_{z_0}^\Psi) \sqcup \mathcal{O}_{z_0}^\Phi$ is null homologous, and α is closed, one derives from Stokes' theorem that: $\int_{(-\mathcal{O}_{z_0}^\Psi) \sqcup \mathcal{O}_{z_0}^\Phi} \alpha = 0$, i.e., $\langle P(\alpha), \tilde{S}_\Omega(\Psi) - \tilde{S}_\Omega(\Phi) \rangle = 0$, for all closed 1-form α . This implies that $\tilde{S}_\Omega(\Psi) = \tilde{S}_\Omega(\Phi)$. $\square$

Proof of Theorem 1.2 : The Poincaré pairing

$$\langle, \rangle : H^1(M, \mathbb{R}) \times H^{(n-1)}(M, \mathbb{R}) \rightarrow \mathbb{R},$$

being a continuous bilinear mapping, let μ_0 denote any positive constant such that

$$(31) \quad |\langle A, B \rangle| \leq \mu_0 \|A\|_{L^2} \|B\|_{L^2},$$

for all $(A, B) \in H^1(M, \mathbb{R}) \times H^{(n-1)}(M, \mathbb{R})$. Let α (fixed) be any closed 1-form such that $P(\alpha) \in \overline{\mathcal{B}_S^1(0, 1)}$, and consider the continuous linear mapping

$$L_\alpha : H^{(n-1)}(M, \mathbb{R}) \rightarrow \mathbb{R}, B \mapsto \langle P(\alpha), B \rangle.$$

It follows from (31) that

$$(32) \quad |L_\alpha(B)| = |\langle P(\alpha), B \rangle| \leq \mu_0 \|B\|_{L^2},$$

for all $B \in H^{(n-1)}(M, \mathbb{R})$ because $P(\alpha) \in \overline{\mathcal{B}_S^1(0, 1)}$.

Since by construction the set Γ_Ω is countable, we can assume that Γ_Ω contains more than one element. Otherwise, $\Gamma_\Omega = \{0\}$, and hence discrete. Let $\beta, \tilde{\beta} \in \Gamma_\Omega$: From the characterization of elements in Γ_Ω , there exist $\Phi, \Psi \in \pi_1(G_\Omega(M))$ such that $\beta = \tilde{S}_\Omega(\Phi)$, and $\tilde{\beta} = \tilde{S}_\Omega(\Psi)$. We define another loop at the identity by setting $\Theta := \Phi *_r \Psi^{-1}$, and we have the following two possibilities:

Either: There exists a $z_0 \in M$ such that the 1-cycle $\mathcal{O}_{z_0}^\Theta = \mathcal{O}_{z_0}^\Phi \sqcup (-\mathcal{O}_{z_0}^\Psi)$ is null homologous; in this situation, Lemme 3.5 implies that $\frac{1}{\text{Vol}_\Omega(M)} \langle P(\alpha), \beta - \tilde{\beta} \rangle = \frac{1}{\text{Vol}_\Omega(M)} \langle P(\alpha), \tilde{S}_\Omega(\Theta) \rangle = \int_{\mathcal{O}_{z_0}^\Theta} \alpha = 0$, for all closed 1-form α , i.e., $\beta = \tilde{\beta}$.

Or: For each $y \in M$, the 1-cycle $\mathcal{O}_y^\Theta$ is not null homologous; in this case, since $\beta \neq \tilde{\beta}$, assume in addition that they are arbitrarily closed in the natural topology of Γ_Ω , i.e., in the topology induced by the vector space structure of $H^{(n-1)}(M, \mathbb{R})$. Since $H^{(n-1)}(M, \mathbb{R})$ is a finite dimensional real vector space, all the norms on $H^{(n-1)}(M, \mathbb{R})$ are equivalent, in particular, we shall equip $H^{(n-1)}(M, \mathbb{R})$ with the L^2 -Hodge norm $\|, \|_{L^2}$: Let l be any arbitrary positive integer, and assume that

$$(33) \quad \|\beta - \tilde{\beta}\|_{L^2} < \frac{\text{Vol}_\Omega(M)}{2\mu_0 l},$$

where μ_0 is the constant defined in (31). On the other hand, let $z \in M$ (fixed) and choose any closed 1-form α_0 on M with $P(\alpha_0) \in \overline{\mathcal{B}_S^1(0, 1)}$ such that $\int_{\mathcal{O}_z^\Theta} \alpha_0 \neq 0$: This is also possible because $\mathcal{O}_z^\Theta$ is a non-null homologous loop by assumption. With the help of Lemma 3.5 together with (33), we derive that

$$(34) \quad \begin{aligned} 0 &< \left| \int_{\mathcal{O}_z^\Theta} \alpha_0 \right| \\ &= \left| \frac{\langle P(\alpha_0), \tilde{S}_\Omega(\Theta) \rangle}{\text{Vol}_\Omega(M)} \right| \\ &\leq \frac{\mu_0}{\text{Vol}_\Omega(M)} \|\beta - \tilde{\beta}\|_{L^2} \\ &\leq 1/(2l), \end{aligned}$$

for all positive integer l . This is a contradiction because the top right-hand side of the above inequalities tends to zero as l tends to infinity. Therefore, we have proved that in Γ_Ω two elements are arbitrarily closed, if and only if, they are equal. $\square$

Proof of Theorem 1.5 : Let $\Psi = \{\psi^t\}$ be any smooth isotopy in $\ker S_\Omega$, and for each fixed t , consider the reparametrized path $\Psi_t : s \mapsto \psi^{(st)}$. Since M is compact, the map $L_\Psi : t \mapsto \Psi_t$ is Lipschitz continuous with respect to the C^0 -metric, and by Proposition 3.3, the map $\tilde{S}_\Omega$ is continuous with respect to the C^0 -metric. On the other hand, since Ψ is a path in $\ker S_\Omega$, then for each t , we have $\Psi_t(1) \in \ker S_\Omega$, i.e., $\tilde{S}_\Omega(\Psi_t) \in \Gamma_\Omega$ for each t . Thus, the composition $\tilde{S}_\Omega \circ L_\Psi$ is a continuous map from the connected topological space $[0, 1]$ into the discrete topological space Γ_Ω : hence, this map must be a constant map. That is,

$$\begin{aligned} (\tilde{S}_\Omega \circ L_\Psi)(t) &= (\tilde{S}_\Omega \circ L_\Psi)(0), \\ &= 0, \end{aligned} \tag{35}$$

for all t . In particular, for $t = 1$, we have

$$\begin{aligned} 0 &= (\tilde{S}_\Omega \circ L_\Psi)(1), \\ &= \tilde{S}_\Omega(\Psi). \end{aligned} \tag{36}$$

□

Proof of Lemma 1.3 : Let $\{\phi_i\} \subset \ker S_\Omega$ be a sequence which converges in the C^1 -metric to some $\phi \in G_\Omega(M)$. Let $\{\Phi_i = (\phi_i^t)\}$ be a sequence of vanishing-flux isotopies with $\Phi_i \in Iso(\phi_i)_\Omega$ for each i , and let $\Phi := (\phi_t) \in Iso(\phi)_\Omega$ be an arbitrary element. Since the group $G_\Omega(M)$ is locally connected by smooth arcs (see [3]), fix a small path connected open neighborhood of the identity map, say $\mathcal{V}(id_M)$, and since $\psi^i := \phi_i \circ \phi^{-1} \xrightarrow{C^1} id_M$, we may assume that for all i sufficiently large, one has $\psi^i \in \mathcal{V}(id_M)$. For each i sufficiently large, one can choose a smooth curve σ_i from ψ^i to the identity, lying inside $\mathcal{V}(id_M)$ such that σ_i tends to the constant path identity in the C^1 -metric as i tends to infinity (if necessary, when σ_i is only C^1 , then approximate it in the C^∞ -topology by another curve ϱ_i having the same endpoints than σ_i , [10]). Assume this done.

Fix any sufficiently large integer i , define $\Psi^i : t \mapsto (\phi_i^t \circ \phi_t^{-1})$, and consider the following juxtaposition of paths,

$$\mathfrak{J}_i(t) = \begin{cases} \Psi^i(\lambda(t)), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \sigma_i(\tau(t)), & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $\lambda : [0, 1/2] \rightarrow [0, 1]$, and $\tau : [1/2, 1] \rightarrow [0, 1]$ are smooth functions such that $\lambda(1/2) = 1 = \tau(1)$, and $\lambda(0) = 0 = \tau(1/2)$.

By construction, we have $\mathfrak{J}_i(0) = id_M = \mathfrak{J}_i(1)$, i.e., $\tilde{S}_\Omega(\mathfrak{J}_i) \in \Gamma_\Omega$. That is, for all i sufficiently large, we have

$$(\star) : 0 = \text{dist}(\tilde{S}_\Omega(\mathfrak{J}_i), \Gamma_\Omega) = \text{dist}(\tilde{S}_\Omega(\sigma_i) - \tilde{S}_\Omega(\Phi), \Gamma_\Omega),$$

because $\tilde{S}_\Omega(\mathfrak{J}_i) = \tilde{S}_\Omega(\sigma_i) + \tilde{S}_\Omega(\Phi_i) - \tilde{S}_\Omega(\Phi)$, and $\tilde{S}_\Omega(\Phi_j) = 0$ for all j , where $\text{dist}(a, b)$ stands for the distance between a and b . Taking the limit in $(\star)$ as i tends to infinity yields,

$$\begin{aligned} 0 &= \lim_i \text{dist}(\tilde{S}_\Omega(\sigma_i) - \tilde{S}_\Omega(\Phi), \Gamma_\Omega) \\ &= \text{dist}(\lim_i \tilde{S}_\Omega(\sigma_i) - \tilde{S}_\Omega(\Phi), \Gamma_\Omega) \\ &= \text{dist}(-\tilde{S}_\Omega(\Phi), \Gamma_\Omega), \end{aligned} \tag{37}$$

because by construction, σ_i tends to the constant path identity in the C^1 -metric as $i \rightarrow \infty$. That is, $\tilde{S}_\Omega(\Phi) \in \overline{\Gamma_\Omega}$, and this implies that $\tilde{S}_\Omega(\Phi) \in \Gamma_\Omega$ because by Theorem 1.2, the group Γ_Ω is closed. Finally, we have proved that any volume-preserving isotopy from the identity to ϕ must have its flux in Γ_Ω : This means that $\phi \in \ker S_\Omega$. $\square$

Proof of Lemma 1.4 : Let $\{\Phi_i\}$ be a sequence of vanishing-flux isotopies which converges uniformly to a volume-preserving isotopy Φ . We have to show that $\tilde{S}_\Omega(\Phi) = 0$. Since Φ is smooth, then for each closed 1-form α , we have

$$\begin{aligned} (38) \quad \left| \int_M \mathcal{F}_\alpha(\Phi)(1) \Omega \right| &\leq \left| \int_M \mathcal{F}_\alpha(\Phi_i)(1) \Omega \right| + \left| \int_M (\mathcal{F}_\alpha(\Phi_i)(1) - \mathcal{F}_\alpha(\Phi)(1)) \Omega \right| \\ &\leq 0 + 4 \text{Vol}_\Omega(M) \|\alpha\|_{L^2 d_{C^0}(\Phi_i, \Phi)} \rightarrow 0, i \rightarrow \infty, \end{aligned}$$

because by Proposition 3.3 we have, $0 = \langle P(\alpha), \tilde{S}_\Omega(\Phi_i) \rangle = \int_M \mathcal{F}_\alpha(\Phi_i)(1) \Omega$, for each i , and since $\lim_{C^0}(\Phi_i) = \Phi$, it follows from Lemma 3.10–[15] that

$$\begin{aligned} \left| \int_M \mathcal{F}_\alpha((\Phi_i)(1) - \mathcal{F}_\alpha(\Phi)(1)) \Omega \right| &\leq \text{Vol}_\Omega(M) \sup_{x \in M} |\mathcal{F}_\alpha(\Phi_i)(1)(x) - \mathcal{F}_\alpha(\Phi)(1)(x)|, \\ &\leq 4 \text{Vol}_\Omega(M) \|\alpha\|_{L^2 d_{C^0}(\Phi_i, \Phi)}, \end{aligned}$$

for i large enough. Thus, the right-hand side in (38) tends to zero as i goes at infinity, and hence the left-hand side in (38) vanishes. Since Φ is a volume-preserving isotopy, it follows from Proposition 3.3 that, $0 = \int_M \mathcal{F}_\alpha(\Phi)(1)\Omega = \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle$, for each closed 1-form α . That is, $\langle P(\beta), \tilde{S}_\Omega(\Phi) \rangle = 0$, for all closed 1-form β , which implies that $\tilde{S}_\Omega(\Phi) = 0$. $\square$

4.1.2. Application to quasi-morphisms for symplectic closed 2-surfaces. We assume that (M, ω) is a 2-dimensional closed symplectic manifold. Firstly, observe that Proposition 4.3 implies that for each non-trivial closed 1-form α , and each $x \in M$, we have

$$(39) \quad \Delta(\psi \circ \phi, \alpha)_x = \Delta(\psi, \alpha)_x + \Delta(\phi, \alpha)_x + \frac{\text{Vol}_\Omega(M)}{\|\alpha\|_{L^2}} \left(\int_{\mathcal{O}_x^\Psi} \alpha - \int_{\mathcal{O}_{\phi(x)}^\Psi} \alpha \right),$$

for all $\psi, \phi \in G_\omega(M)$, and for all $\Psi \in \text{Iso}(\psi)_\omega$, where $G_\omega(M)$ is the identity component in the group of all symplectic diffeomorphisms of (M, ω) . That is, the quantity $\left(\int_{\mathcal{O}_x^\Psi} \alpha - \int_{\mathcal{O}_{\phi(x)}^\Psi} \alpha \right)$ seems to impeach $\Delta(\cdot, \alpha)_x$ to have the properties of a homomorphism in the usual sense. However, the goal of this subsection is to check whether $\Delta(\cdot, \alpha)_x$ can give rise to a quasi-morphism or not.

Let G be a group. A mapping $\Delta : G \rightarrow \mathbb{R}$ is a quasi-morphism if there exists a real number $R > 0$ such that $|\Delta(fg) - \Delta(f) - \Delta(g)| \leq R$, for all $f, g \in G$.

In addition, if $\Delta(f^m) = m\Delta(f)$, for each $f \in G$ and for each $m \in \mathbb{Z}$, then Δ is called a homogeneous quasi-morphism (see [8]). The minimal value of the real number R in the above definition is sometimes called the defect of the quasi-morphism Δ . This is due to the fact that if $R = 0$, then Δ will be a homomorphism in the usual sense. Furthermore, for each quasi-morphism Δ , the map $\psi \mapsto \lim_{m \rightarrow \infty} \{\Delta(\psi^m)/m\}$, induces a homogeneous quasi-morphism.

Proposition 4.7. *Assume $(\widetilde{M}, \omega)$ to be a 2-dimensional closed symplectic manifold with $H^1(\widetilde{M}, \mathbb{R}) \neq 0$. Let α be a non-trivial closed 1-form and $p \in \widetilde{M}$ be any point. Therefore, the mapping $\Delta(\cdot, \alpha)_p : G_\omega(\widetilde{M}) \rightarrow \mathbb{R}, \psi \mapsto \Delta(\psi, \alpha)_p$, is a continuous quasi-morphism whose defect is less than or equal to twice the square of the symplectic area of $\widetilde{M}$.*

Proof.

Let α be a non-trivial closed 1-form and $p \in \widetilde{M}$. Proposition 4.3 implies that for all $\psi, \phi \in G_\omega(\widetilde{M})$, we have

$$\frac{1}{Vol_\Omega(\widetilde{M})} |\Delta(\psi \circ \phi, \alpha)_p - \Delta(\psi, \alpha)_p - \Delta(\phi, \alpha)_p| \leq \left| \int_{\mathcal{O}_{\phi(p)}^\Psi} \frac{1}{\|\alpha\|_{L^2}} \alpha \right| + \left| \int_{\mathcal{O}_p^\Psi} \frac{1}{\|\alpha\|_{L^2}} \alpha \right|,$$

for each $\Phi \in Iso(\phi)_\omega$. Since $\int_{\mathcal{O}_p^\Psi} \frac{1}{\|\alpha\|_{L^2}} \alpha$ (resp. $\int_{\mathcal{O}_{\phi(p)}^\Psi} \frac{1}{\|\alpha\|_{L^2}} \alpha$) is the algebraic value of the symplectic area swept by the orbit $\mathcal{O}_p^\Psi$ (resp. $\mathcal{O}_{\phi(p)}^\Psi$) under the symplectic flow generated by the closed 1-form $\frac{1}{\|\alpha\|_{L^2}} \alpha$, then $\left| \int_{\mathcal{O}_p^\Psi} \frac{1}{\|\alpha\|_{L^2}} \alpha \right|$ (resp. $\left| \int_{\mathcal{O}_{\phi(p)}^\Psi} \frac{1}{\|\alpha\|_{L^2}} \alpha \right|$) is less than or equal to the symplectic area $A(\widetilde{M})$ of $\widetilde{M}$. Therefore,

$$(40) \quad |\Delta(\psi \circ \phi, \alpha)_p - \Delta(\psi, \alpha)_p - \Delta(\phi, \alpha)_p| \leq 2A(\widetilde{M})Vol_\Omega(\widetilde{M}) = 2[A(\widetilde{M})]^2,$$

for all $\psi, \phi \in G_\omega(\widetilde{M})$ because here the area form is the volume form. $\square$

Remark 4.8. Let h be any smooth function of a compact manifold M . If ϕ is any diffeomorphism of M isotopic to the identity map with respect to an isotopy $\Phi := \{\phi_t\}$, then the smooth function $x \mapsto \mathcal{F}_{dh}(\Phi)(1)(x)$ is supported in $\text{supp}(\phi)$. Assume on the contrary that there exists $z \in \text{supp}(\mathcal{F}_{dh}(\Phi)(1))$ such that $z \notin \text{supp}(\phi)$. Then,

$$0 \neq \mathcal{F}_{dh}(\Phi)(1)(z) = \int_{\mathcal{O}_z^\Phi} dh = h(\phi(z)) - h(z),$$

which is a contradiction since $z \notin \text{supp}(\phi)$, implies that $h(\phi(z)) - h(z) = h(z) - h(z) = 0$. In particular, given any oriented closed manifold (M, Ω) , and any smooth function h of M , then for any vanishing-flux volume-preserving diffeomorphism ψ , we have that

$$\text{supp}(\Delta(\psi, dh)) \subseteq \text{supp}(\psi).$$

Proof of Theorem 1.6 : Let $z \in M$ be a point whose orbit under γ is smooth, i.e., $\mathcal{O}_z^\gamma$ is smooth. By definition of $\gamma := (\gamma_t)_{t \in [0,1]}$, there exists $\Phi_i = (\phi_i^t)$ a sequence of vanishing flux isotopies which converges uniformly to the loop γ . Since $\gamma_1 = id_M$, then the sequence of time-one maps $\{\phi_i^1\}$ converges uniformly to the identity map id_M , and then it follows from Lemma 4.5 that $\lim_{C^0} \Delta(\phi_i^1, \alpha)_z = \Delta(id_M, \alpha)_z = 0$. On the other hand, by Corollary 4.2, we also have

$$\|\alpha\|_{L^2} \Delta(\phi_i^1, \alpha)_z = -Vol_\Omega(M) \int_{\mathcal{O}_z^{\Phi_i}} \alpha,$$

for each i , and for all non-trivial closed 1-form α . Hence, combining the above arguments give $\lim_{C^0} \left(\int_{\mathcal{O}_z^{\Phi_i}} \alpha \right) = 0$, for all non-trivial closed 1-form α . Observe that, $|\int_{\mathcal{O}_z^\gamma} \alpha| \leq |\int_{\mathcal{O}_z^{\Phi_i}} \alpha - \int_{\mathcal{O}_z^\gamma} \alpha| + |\int_{\mathcal{O}_z^{\Phi_i}} \alpha|$, for all i , and derive that $\int_{\mathcal{O}_z^\gamma} \alpha = 0$, because $\lim_{C^0} \left(\int_{\mathcal{O}_z^{\Phi_i}} \alpha \right) = 0$, and as in the proof of Lemma 4.5, for i sufficiently large, one can construct a smooth 2-chain whose boundary consists of $\mathcal{O}_z^\gamma$, $\mathcal{O}_z^{\Phi_i}$ and a minimizing geodesic χ_z^i that connects $\gamma_1(z)$ to $\phi_i^1(z)$; and derive from Stokes' theorem that $|\int_{\mathcal{O}_z^{\Phi_i}} \alpha - \int_{\mathcal{O}_z^\gamma} \alpha| = |\int_{\chi_z^i} \alpha| \leq \|\alpha\|_{L^2} d_{C^0}(\gamma, \Phi_i) \rightarrow 0, i \rightarrow \infty$. Therefore, we have proved that $\int_{\mathcal{O}_z^\gamma} \alpha = 0$, for all closed 1-form α , i.e., the 1-cycle $\mathcal{O}_z^\gamma$ is a boundary. $\square$

Proof of Lemma 1.7 : Let Φ be a volume-preserving isotopy out of $\ker \tilde{S}_\Omega$, and let ϕ denote its time-one map. Since $\tilde{S}_\Omega(\Phi) \neq 0$, and $\langle, \rangle$ is non-degenerate, we can choose a harmonic 1-form $\mathcal{H}_0$ such that $\langle P(\mathcal{H}_0), \tilde{S}_\Omega(\Phi) \rangle \neq 0$. Assume this done. Let $r(g)$ be the injectivity radius of the Riemannian metric g on M , and put

$$(41) \quad \delta_0 := \frac{1}{\beta_0} \min\{r(g), \frac{|\langle P(\mathcal{H}_0), \tilde{S}_\Omega(\Phi) \rangle|}{\|\mathcal{H}_0\|_{L^2} \text{Vol}_\Omega(M)}\},$$

for some $\beta_0 \in [4, +\infty[$. If $d_{C^0}(\Phi, Id) \leq \delta_0$, then one deduces as in the proof of Lemma 4.5 that for all $x \in M$, we have

$$(42) \quad |\Delta(\phi, \mathcal{H}_0)_x - 0| = |\Delta(\phi, \mathcal{H}_0)_x - \Delta(id_M, \mathcal{H}_0)_x| \leq 2\text{Vol}_\Omega(M) d_{C^0}(\Phi, Id),$$

because $d_{C^0}(\phi, id_M) \leq d_{C^0}(\Phi, Id)$. On the other hand, derive from (42) and Corollary 4.2 that

$$(43) \quad \frac{|\langle P(\mathcal{H}_0), \tilde{S}_\Omega(\Phi) \rangle|}{\text{Vol}_\Omega(M) \|\mathcal{H}_0\|_{L^2}} \leq \frac{1}{\text{Vol}_\Omega(M)} |\Delta(\phi, \mathcal{H}_0)_x| + \frac{1}{\|\mathcal{H}_0\|_{L^2}} \left| \int_{\mathcal{O}_x^\Phi} \mathcal{H}_0 \right|,$$

$$\leq 2d_{C^0}(\Phi, Id) + \text{length}(\mathcal{O}_x^\Phi),$$

for each $x \in M$. Now, when $z \in M$ is such that the length of the orbit $\mathcal{O}_z^\Phi$ with respect to the Riemannian metric g minimizes the distance between its endpoints, then $\text{length}(\mathcal{O}_z^\Phi) \leq d_{C^0}(\Phi, Id)$, and this together with (43) implies that,

$$(44) \quad \beta_0 < 3.$$

This is a contradiction. $\square$

5. METRIC GEOMETRY FOR VOLUME-PRESERVING DIFFEOMORPHISMS

The goal of this section is to introduce and study a right-invariant metric on $G_\Omega(M)$. In particular, we shall establish several comparison results in the context of Hofer and Hofer-like geometries [4], [15].

5.1. Norm on the group $G_\Omega(M)$. For each $\psi \in G_\Omega(M)$, set

$$(45) \quad \|\psi\|^\infty := \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} |\tilde{\Delta}(\psi, \alpha)_z| \right),$$

where $\tilde{\Delta}(\psi, \alpha)_z := \|\alpha\|_{L^2} \Delta(\psi, \alpha)_z$ and

$$\mathcal{B}(1) := \{\alpha \in \mathcal{Z}^1(M) : \|\alpha\|_{L^2} = 1\}.$$

We have the following facts.

Proposition 5.1. *The rule $\|\cdot\|^\infty$ has the following properties:*

- (1) *Positivity:* $\|\psi\|^\infty \geq 0$, for all $\psi \in G_\Omega(M)$.
- (2) *Triangle inequality:* $\|\psi \circ \phi\|^\infty \leq \|\psi\|^\infty + \|\phi\|^\infty$, for all $\psi, \phi \in G_\Omega(M)$.
- (3) *Duality:* $\|\psi^{-1}\|^\infty = \|\psi\|^\infty$, for all $\psi \in G_\Omega(M)$.
- (4) *If $\|\psi\|^\infty = 0$, then $\psi = id_M$.*
- (5) *Right-invariance:* The following rule

$$(46) \quad d^\Omega(\phi, \psi) := \|\phi \circ \psi^{-1}\|^\infty,$$

for all $\psi, \phi \in G_\Omega(M)$.

Hence, the norm $\|\cdot\|^\infty$ induces a right-invariant metric on $G_\Omega(M)$.

Proof. (1) Positivity: This follows from the definition of $\|\cdot\|^\infty$.

- (2) Triangle inequality: Let $\psi, \phi \in G_\Omega(M)$, $\alpha \in \mathcal{B}(1)$, and derive with the help of Proposition 4.3 that:

$$|\Delta(\psi \circ \phi, \alpha)_z| = |\Delta(\phi, \alpha)_z + \Delta(\psi, \alpha)_{\phi(z)}| \leq |\Delta(\phi, \alpha)_z| + |\Delta(\psi, \alpha)_{\phi(z)}|,$$

for all $z \in M$, which implies the desired inequality.

- (3) Duality: From Proposition 4.3, we derive that

$$0 = \Delta(id_M, \alpha) = \Delta(\psi \circ \psi^{-1}, \alpha) = \Delta(\psi^{-1}, \alpha) + \Delta(\psi, \alpha) \circ \psi^{-1},$$

i.e., $\Delta(\psi^{-1}, \alpha) = -\Delta(\psi, \alpha) \circ \psi^{-1}$. This implies that the desired equality.

- (4) Non-degeneracy: If $\|\psi\|^\infty = 0$, then $\Delta(\psi, \alpha) = 0$, for all $\alpha \in \mathcal{Z}^1(M)$. With the above vanishing condition, it follows from Corollary 4.2 that $\langle P(\alpha), \tilde{S}_\Omega(\Psi) \rangle = Vol_\Omega(M) \int_{\mathcal{O}_\Psi} \alpha$, for all $x \in M$, for all $\alpha \in \mathcal{Z}^1(M)$, and for each $\Psi \in Iso(\psi)_\Omega$. Now, assume that there exists $z_0 \in M$ such that $\psi(z_0) \neq z_0$. Then, pick an open neighborhood U_{z_0} of z_0 that does not contain $\psi(z_0)$, and consider a bump function ρ supported in U_{z_0} such that $\rho(z_0) \neq 0$. In particular, for $\alpha = d\rho$, we derive from the above arguments that

$$0 = \langle P(\alpha), \tilde{S}_\Omega(\Psi) \rangle = Vol_\Omega(M) \int_{\mathcal{O}_\Psi} \alpha = \rho(\psi(z_0)) - \rho(z_0),$$

for each $\Psi \in Iso(\psi)_\Omega$. That is, $\rho(\psi(z_0)) = \rho(z_0) \neq 0$: This is a contradiction because the open neighborhood U_{z_0} of z_0 does not contain $\psi(z_0)$, and the bump function ρ is supported in U_{z_0} . If necessary, in case $\|d\rho\|_{L^2} \neq 1$, then replace the function ρ by the function $\varrho := \frac{\rho}{\|d\rho\|_{L^2}}$.

- (5) Right-invariance: We have,

$$\begin{aligned} d^\Omega(\phi \circ \rho, \psi \circ \rho) &= \|(\phi \circ \rho) \circ (\psi \circ \rho)^{-1}\|^\infty \\ (47) \quad &= \|\phi \circ \rho \circ \rho^{-1} \circ \psi^{-1}\|^\infty, \\ &= d^\Omega(\phi, \psi), \end{aligned}$$

for all $\rho, \psi, \phi \in G_\Omega(M)$.

□

Here is a consequence of the proof of Lemma 4.5.

Corollary 5.2. *If $\psi \in \ker S_\Omega$ is sufficiently C^0 -closed to the identity map, then*

$$(48) \quad \|\psi\|^\infty \leq 2Vol_\Omega(M)d_{C^0}(\psi, id_M).$$

6. COMPARISON WITH OTHER METRICS

6.1. Comparison with the usual Hofer norm. In this subsection, we assume that $\Omega_0 := \frac{1}{n!}\omega^n$, where ω is a symplectic form, and $\|\cdot\|_{Hofer}$ denote the usual Hofer norm defined on the group $Ham(M, \omega)$, of Hamiltonian diffeomorphisms: for any Hamiltonian isotopy $\Phi = \{\phi_t\}$ such that $\iota_{\phi_t}\omega = dH_t$, for each t , its L^∞ -Hofer (resp. $L^{(1,\infty)}$ -Hofer) length is defined by

$$(49) \quad l_H^\infty(\Phi) = \max_t osc(H_t),$$

and

$$(50) \quad l_H^{(1,\infty)}(\Phi) = \int_0^1 \text{osc}(H_t) dt.$$

For any diffeomorphism $\phi \in \text{Ham}(M, \omega)$, its Hofer norm is defined by

$$(51) \quad \|\phi\|_{\text{Hofer}} := \inf_{\Phi} (l_H^{\infty}(\Phi)) = \inf_{\Phi} \left(l_H^{(1,\infty)}(\Phi) \right),$$

where the infimum is taken over all the Hamiltonian isotopies Φ with time one map ϕ (see [14]).

We have the following fact.

Proposition 6.1. *Let $\Omega := \frac{1}{n!}\omega^n$, where ω is a symplectic form. Then, for any $\phi \in \text{Ham}(M, \omega)$, we have*

$$(52) \quad \|\phi\|^{\infty} \leq 2\|\phi\|_{\text{Hofer}}.$$

In particular, our construction implies the non-degeneracy of the Hofer-norm.

Proof. Assume that $\text{Vol}_{\Omega}(M) = 1$. Let $\phi \in \text{Ham}(M, \omega)$. For each $\alpha \in \mathcal{B}(1)$, we have

$$(53) \quad \begin{aligned} \Delta(\phi, \alpha)_x &= - \int_{\mathcal{O}_x^{\Phi}} \alpha \\ &= - \int_{\mathcal{O}_x^{\Phi}} \iota_{X_{\alpha}} \omega \\ &= \int_0^1 \omega(\dot{\phi}_t, X_{\alpha}) \circ \phi_t(x) dt, \end{aligned}$$

for any Hamiltonian isotopy $\Phi = \{\phi_t\}$ with time-one map ϕ such that $\iota_{\dot{\phi}_t} \omega = dH_t$, for each t , where $\iota_{X_{\alpha}} \omega = \alpha$. Let $\Psi_{\alpha} = \{\psi_t^{\alpha}\}$ be the symplectic flow generated by

X_α . Then, (53) implies that

$$\begin{aligned}
 \sup_{x \in M} |\Delta(\phi, \alpha)(x)| &= \sup_{x \in M} \left| \int_0^1 \omega(\dot{\phi}_t, X_\alpha) \circ \phi_t(x) dt \right| \\
 &= \sup_{x \in M} \left| \int_0^1 dH_t(X_\alpha) \circ \phi_t(x) dt \right| \\
 &\leq \int_0^1 \sup_{z \in M} |dH_t(X_\alpha)(z)| dt \\
 &= \int_0^1 \left(\sup_{y \in M} |dH_t(X_\alpha)(\psi_s^\alpha(y))| \right) dt, \text{ for each } s, \\
 (54) \qquad &= \int_0^1 \left(\sup_{y \in M} \left| \frac{d}{ds} (H_t(\psi_s^\alpha(y))) \right| \right) dt, \text{ for each } s,
 \end{aligned}$$

which implies that

$$\sup_{x \in M} |\Delta(\phi, \alpha)(x)| \leq \int_0^1 \left(\int_0^1 \sup_{y \in M} \left| \frac{d}{ds} (H_t(\psi_s^\alpha(y))) \right| ds \right) dt,$$

for any Hamiltonian isotopy $\Phi = \{\phi_t\}$ with time-one map ϕ such that $\iota_{\dot{\phi}_t} \omega = dH_t$, for each t . On the other hand, for each s fixed, let x_{min}^s (resp. x_{max}^s) be the point of M at which the function $z \mapsto \frac{d}{ds} (H_t(\psi_s^\alpha(z)))$ realizes its minimum (resp. maximum). We know that: either $\sup_{z \in M} \left| \frac{d}{ds} (H_t(\psi_s^\alpha(z))) \right| = \frac{d}{ds} (H_t(\psi_s^\alpha(x_{max}^s)))$, or $\sup_{z \in M} \left| \frac{d}{ds} (H_t(\psi_s^\alpha(z))) \right| = -\frac{d}{ds} (H_t(\psi_s^\alpha(x_{min}^s)))$. That is, either

$$\int_0^1 \sup_{z \in M} \left| \frac{d}{ds} (H_t(\psi_s^\alpha(z))) \right| ds = \int_0^1 \frac{d}{ds} (H_t(\psi_s^\alpha(x_{max}^s))) ds = H_t(\psi_1^\alpha(x_{max}^1)) - H_t(x_{max}^0),$$

or

$$\int_0^1 \sup_{z \in M} \left| \frac{d}{ds} (H_t(\psi_s^\alpha(z))) \right| ds = - \int_0^1 \frac{d}{ds} (H_t(\psi_s^\alpha(x_{min}^s))) ds = -H_t(\psi_1^\alpha(x_{min}^1)) + H_t(x_{min}^0).$$

Therefore, we derive from the above estimates that,

$$\begin{aligned}
 \sup_{x \in M} |\Delta(\phi, \alpha)(x)| &\leq \max \left(\int_0^1 (-H_t(\psi_1^\alpha(x_{min}^1)) + H_t(x_{min}^0)) dt, \int_0^1 (H_t(\psi_1^\alpha(x_{max}^1)) - H_t(x_{max}^0)) dt \right) \\
 &\leq 2 \int_0^1 \text{osc}(H_t) dt,
 \end{aligned}$$

for any Hamiltonian isotopy $\Phi = \{\phi_t\}$ with time-one map ϕ such that $\iota_{\dot{\phi}_t} \omega = dH_t$, for each t . Thus, we derive that

$$\begin{aligned}
 \|\phi\|^\infty &\leq 2 \inf_{\Phi \in Iso(\phi)_\omega} \left(l_H^{(1, \infty)}(\Phi) \right) \\
 (55) \qquad &= 2 \|\phi\|_{Hofers}.
 \end{aligned}$$

□

6.2. Comparison with the Hofer-like metric. In this subsection, we assume again that $\Omega_0 := \frac{1}{n!}\omega^n$. Let e_{HL} denote the usual Hofer-like energy defined on the group $G_\omega(M)$ [4], [15]. Also, let us recall the definition of the supremum norm (i.e., the uniform sup norm) of a 1-form α : for each $x \in M$, we know that α induces a linear map $\alpha_x : T_x M \rightarrow T_x M$, whose norm is given by

$$(56) \quad \|\alpha_x\|^g := \sup \left\{ \|\alpha_x(X)\|^g ; X \in T_x M, \|X\|_g = 1 \right\},$$

where $\|\cdot\|_g$ is the norm induced on each tangent space $T_x M$ (at the point x) by the Riemannian metric g : The uniform sup norm of α , say $|\alpha|_0$, is defined as

$$(57) \quad |\alpha|_0 := \sup_{x \in M} \|\alpha_x\|^g.$$

We shall need the following results.

Proposition 6.2. *Let (M, ω) be a closed symplectic manifold of dimension $2n$. Then, the mapping*

$$\omega^{n-1} : H^1(M, \mathbb{R}) \longrightarrow H^{2n-1}(M, \mathbb{R}), [\alpha] \mapsto [\omega^{(n-1)} \wedge \alpha],$$

is continuous as a linear map defined on a finite dimensional real-vector space: there exists a positive constant $C(\omega)$ that depends on ω such that

$$\|[\omega^{(n-1)} \wedge \alpha]\|_{L^2} \leq C(\omega) \|\alpha\|_{L^2} = C(\omega) \|\alpha\|_{L^2}.$$

Proposition 6.3. *Let (M, ω) be a closed symplectic manifold of dimension $2n$. Let X be a harmonic vector field (i.e., a vector field such that $\iota_X \omega$ is a harmonic 1-form). Then, for all closed 1-form α , there exists a positive constant which depends on α such that*

$$\|\alpha(X)\|_{C^0} := \sup_{z \in M} |\alpha(X)(z)| \leq C(\alpha) \|\iota_X \omega\|_{L^2}.$$

Proof. Let $b_1(M)$ the first Betti number of the manifold M , and let $h_1, \dots, h_{b_1(M)}$ be a basis of the space of harmonic 1-forms such that $\|h_i\|_{L^2} = 1$, for all i . Let α be a closed 1-form such that $\iota_{X_\alpha} \omega = \alpha$ for some symplectic vector field X_α [4], [15]. Assume that

$$\iota_{X_\alpha} \omega = \sum_{i=1}^{b_1(M)} \lambda_i h_i,$$

where λ'_i 's are real numbers, and derive from the properties of the symplectic form that

$$\begin{aligned}
 \sup_{z \in M} |\alpha(X)(z)| &= \sup_{z \in M} |\iota_{X_\alpha} \omega(X)(z)| \\
 &= \sup_{z \in M} |-\iota_X \omega(X_\alpha)(z)| \\
 (58) \qquad \qquad &\leq \left(\sum_{i=1}^{b_1(M)} |\lambda_i| \right) \max_{1 \leq i \leq b_1(M)} \|h_i(X_\alpha)\|_{C^0}.
 \end{aligned}$$

(59)

Since the space of harmonic 1-forms is a real-vector space of finite dimension, then any given two norms on the latter are equivalent that is, there exist a positive constant C_1 such that $\sum_{i=1}^{b_1(M)} |\lambda_i| \leq C_1 \|\iota_X \omega\|_{L^2}$. This implies that

$$\|\alpha(X)\|_{C^0} \leq \left(C_1 \max_{1 \leq i \leq b_1(M)} \|h_i(X_\alpha)\|_{C^0} \right) \|\iota_X \omega\|_{L^2}.$$

Take $C(\alpha) := C_1 \max_{1 \leq i \leq b_1(M)} \|h_i(X_\alpha)\|_{C^0}$.

□

We shall say that a Riemannian metric g is compatible with the symplectic form ω if there exists a complex structure J such that $g(X, Y) = \omega(X, JY)$, for all vector fields X, Y .

Lemma 6.4. (Continuous dependence of $C(\alpha)$ on α , and local boundedness) *Let (M, ω) be a closed symplectic manifold of dimension $2n$. Assume that the Riemannian metric g is compatible with the symplectic form ω . Then,*

- (1) *the map $\alpha \mapsto C(\alpha)$ is Lipschitz continuous with respect to the L^2 –Hodge norm, with Lipschitz constant*

$$\kappa := \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{\text{Vol}_g(M)}},$$

which depends on a certain fixed basis $h_1, \dots, h_{b_1(M)}$ of the space of all harmonic 1–forms,

- (2) *there exists a positive constant η which depends on a certain fixed basis $h_1, \dots, h_{b_1(M)}$ of the space of all harmonic 1–forms such that for all $\alpha \in \mathcal{B}(1)$, we have $C(\alpha) \leq \eta$, with*

$$\eta := 2 \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{\text{Vol}_g(M)}} + C(h_1).$$

Proof. Let α, β be two closed 1–forms. Compute,

$$\begin{aligned} |C(\alpha) - C(\beta)| &= C_1 \left| \max_{1 \leq i \leq b_1(M)} \|h_i(X_\alpha)\|_{C^0} - \max_{1 \leq i \leq b_1(M)} \|h_i(X_\beta)\|_{C^0} \right| \\ &\leq C_1 \max_{1 \leq i \leq b_1(M)} \|h_i(X_\alpha - X_\beta)\|_{C^0} \\ (60) \quad &\leq C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \int_M \|X_\alpha - X_\beta\|_g. \end{aligned}$$

Thus,

$$\begin{aligned} |C(\alpha) - C(\beta)|^2 \text{Vol}_g(M) &= \int_M |C(\alpha) - C(\beta)|^2 d\text{Vol}_g \\ &\leq \left(C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \right)^2 \int_M \|X_\alpha - X_\beta\|_g^2 d\text{Vol}_g, \\ (61) \quad &= \left(C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \right)^2 \int_M g(X_\alpha - X_\beta, X_\alpha - X_\beta) d\text{Vol}_g. \end{aligned}$$

From the fact that $(\iota_{X_\alpha} \omega)(Z) = \alpha(Z) = g(X^\alpha, Z)$, for all vector field Z , where X^α is the vector field such that $\flat(X^\alpha) = \alpha$, and with the compatibility of the metric g , we derive that

$$g(JX_\alpha, Z) = \omega(X_\alpha, Z) = g(X^\alpha, Z),$$

for all vector field Z : Hence, $X_\alpha = -J(X^\alpha)$. It follows from the above estimates that,

$$\begin{aligned}
 |C(\alpha) - C(\beta)|^2 Vol_g(M) &\leq \left(C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \right)^2 \int_M g(X_\alpha - X_\beta, X_\alpha - X_\beta) dVol_g, \\
 &= \left(C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \right)^2 \int_M g(J(X^\alpha - X^\beta), J(X^\alpha - X^\beta)) dVol_g, \\
 &= \left(C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \right)^2 \int_M g(X^\alpha - X^\beta, X^\alpha - X^\beta) dVol_g, \\
 &= \left(C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \right)^2 \int_M g(\sharp(\alpha - \beta), \sharp(\alpha - \beta)) dVol_g, \\
 (62) \quad &= \left(C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0 \right)^2 \|\alpha - \beta\|_{L^2}^2.
 \end{aligned}$$

That is,

$$|C(\alpha) - C(\beta)| \leq \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{Vol_g(M)}} \|\alpha - \beta\|_{L^2},$$

for all closed 1-forms α, β . For the second item, we derive from the assumption $\|\alpha\|_{L^2} = 1 = \|\beta\|_{L^2}$, that we have

$$\begin{aligned}
 |C(\alpha) - C(\beta)| &\leq \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{Vol_g(M)}} \|\alpha - \beta\|_{L^2} \\
 &\leq \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{Vol_g(M)}} (\|\alpha\|_{L^2} + \|\beta\|_{L^2}), \\
 &= 2 \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{Vol_g(M)}}.
 \end{aligned}$$

In particular, for $\beta = h_1$, it follows that $C(\alpha) \leq 2 \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{Vol_g(M)}} + C(h_1)$. $\square$

Theorem 6.5. *Let (M, ω) be a closed symplectic manifold of dimension $2n$. Assume that the Riemannian metric g is compatible with the symplectic form ω . For all $\varphi \in G_\omega(M)$, we have*

$$\|\varphi\|^\infty \leq 3 \max\{2, C(\omega), 2 \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{Vol_g(M)}} + C(h_1)\} e_{HL}(\varphi),$$

where $C(\omega)$ and $C(h_1)$ are constants in Proposition 6.2 and Proposition 6.3; and e_{HL} is the Hofer-like energy.

Proof. In this proof we assume that $Vol_\Omega(M) = 1$ for simplicity. Let $\varphi \in G_\omega(M)$, and $\Phi = (\phi_t)$ be any symplectic isotopy from the identity to φ . Assume that $\phi_t = \rho_t \circ \psi_t$, be the Hodge-decomposition of Φ where $\rho := (\rho_t)$ is a harmonic isotopy (i.e., $\iota_{\dot{\rho}_t} \omega =: \mathcal{H}_t$, is a harmonic 1-form for all t) and $\Psi = (\psi_t)$ is a Hamiltonian isotopy (i.e., $\iota_{\dot{\psi}_t} \omega = dU_t$, for all t) [4], [15]. We have

$$\begin{aligned}
 \|\varphi\|^\infty &= \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} |\tilde{\Delta}(\psi, \alpha)_z| \right), \\
 &= \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} |\langle [\alpha \wedge \omega^{(n-1)}], [\int_0^1 \mathcal{H}_t dt] \rangle - \int_{\mathcal{O}_z^\Phi} \alpha| \right) \\
 &= \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} |\langle [\alpha \wedge \omega^{(n-1)}], [\int_0^1 \mathcal{H}_t dt] \rangle - \int_{\mathcal{O}_z^\Psi} \alpha - \int_{\mathcal{O}_{\rho_1(z)}^\rho} \alpha| \right), \\
 &\leq \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} |\langle [\alpha \wedge \omega^{(n-1)}], [\int_0^1 \mathcal{H}_t dt] \rangle| \right) + \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} \left| \int_{\mathcal{O}_z^\Psi} \alpha \right| \right), \\
 (63) \quad &+ \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} \left| \int_{\mathcal{O}_{\rho_1(z)}^\rho} \alpha \right| \right).
 \end{aligned}$$

Therefore, we use Proposition 6.2 and Proposition 6.3 to derive that

$$\begin{aligned}
 \|\varphi\|^\infty &\leq \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} |\langle [\alpha \wedge \omega^{(n-1)}], [\int_0^1 \mathcal{H}_t dt] \rangle| \right) + \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} \left| \int_{\mathcal{O}_z^\Psi} \alpha \right| \right), \\
 &+ \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} \left| \int_{\mathcal{O}_{\rho_1(z)}^\rho} \alpha \right| \right), \\
 &\leq C(\omega) \max_t \|\mathcal{H}_t\|_{L^2} + 2 \max_t \text{osc}(U_t) + \eta \max_t \|\mathcal{H}_t\|_{L^2}, \\
 &\leq 3 \max\{C(\omega), \eta, 2\} \left(\max_t \|\mathcal{H}_t\|_{L^2} + \max_t \text{osc}(U_t) \right), \\
 (64) \quad &\leq 3 \max\{C(\omega), \eta, 2\} l_{HL}^\infty(\Phi),
 \end{aligned}$$

for all symplectic isotopy from the identity to φ , where $C(\omega)$ and η are constants in Proposition 6.2 and Lemma 6.4, l_{HL}^∞ is the L^∞ -version of the Hofer-like length [4], [15]. Finally, taking the infimum over the set of all symplectic isotopies from the identity to φ in the above estimate yields,

$$\|\varphi\|^\infty \leq 3 \max\{2, C(\omega), 2 \frac{C_1 \max_{1 \leq i \leq b_1(M)} |h_i|_0}{\sqrt{Vol_g(M)}} + C(h_1)\} e_{HL}(\varphi).$$

□

6.3. Displacement energy.

Definition 6.6. *For each non-empty open subset $\mathcal{U} \subset M$, we define the displacement energies of $\mathcal{U}$ as:*

$$(65) \quad E^\Omega(\mathcal{U}) := \inf\{\|\psi\|^\infty : \psi \in G_\Omega(M), \psi(\mathcal{U}) \cap \mathcal{U} = \emptyset\},$$

and

$$(66) \quad E_\Omega(\mathcal{U}) := \inf\{\|\psi\|^\infty : \psi \in \ker S_\Omega, \psi(\mathcal{U}) \cap \mathcal{U} = \emptyset\}.$$

Theorem 6.7.

For each non-empty open subset $\mathcal{U} \subset M$, we have $E^\Omega(\mathcal{U}) > 0$.

The proof of the above theorem will follow from an adaptation of similar result from symplectic geometry. First, let us generalize the a result due to Polterovich [14] (see Proposition 1.5.B. [5]).

Proposition 6.8. *For any non-empty open subset $B \subset M$, there exists a pair of vanishing-flux volume-preserving diffeomorphisms ϕ , and ψ each of which supported in B such that the commutator $[\phi, \psi] := \psi^{-1} \circ \phi^{-1} \circ \psi \circ \phi$ is different from the identity map.*

We shall say that a map ψ displaces a non-empty subset $\mathcal{U} \subset M$ if $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$. Let us denote by $D_\Omega(\mathcal{U})$ the set of all $\phi \in G_\Omega(M)$ that displace $\mathcal{U}$. We have following fact.

Proposition 6.9. *Consider a non-empty open subset B in M . Let ϕ and ψ be defined as in Proposition 6.8 and let $f \in D_\Omega(B)$. We have $[[f, \phi^{-1}], \psi] = [\phi, \psi]$.*

Proof.

This is a verbatim repetition of the proof of similar fact found in [5]. □

We shall need the following arguments which can be found in [15].

Let $z \in M$ (fixed), for all $\epsilon > 0$, for each $x \in M$, choose a curve γ_x from z to x such that its length $l(\gamma_x) := \int_0^1 \|\dot{\gamma}_x(s)\|_g ds$ satisfies

$$(67) \quad l(\gamma_x) \leq \text{diam}(M)(1 + \epsilon),$$

where $\text{diam}(M)$ stands for the diameter of M with respect to the Riemannian metric g . For any smooth isotopy $\Phi = \{\phi_t\}$, and any family of closed 1-forms $\alpha = \{\alpha_t\}$, we have

$$(68) \quad \max_{t \in [0,1]} \text{osc} \left(\int_0^t \alpha_t(\dot{\phi}_u) \circ \phi_u du \right) \leq 2(1 + \epsilon) \text{diam}(M) (1 + \sup_{s,v} |T_{\gamma_y(s)} \phi_v|) \max_{t \in [0,1]} \|\alpha_t\|_{L^2},$$

where $\text{osc}(f) = \max f - \min f$, y is a certain point of M , and $T_{\gamma_y(s)} \phi_v$ is the tangent map of ϕ_v at $\gamma_y(s)$ (see [15]).

Corollary 6.10.

Let (M, Ω) be a closed oriented manifold. For any vanishing flux isotopy $\Phi = \{\phi_t\}$, and for any closed 1-form α , we have

$$(1) \quad \min_x \left(\left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x) \right) \leq 0, \text{ and}$$

$$(2) \quad 0 \leq \max_x \left(\left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x) \right) \leq \text{osc} \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right).$$

Proof.

Since $\Phi = \{\phi_t\}$ has a trivial flux and for any closed 1-form α , we have that $\int_M \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right) \Omega = \langle P(\alpha), \tilde{S}_\Omega(\Phi) \rangle = 0$. Thus, the min-max principle for integral implies that

$$(69) \quad \min_x \left(\left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x) \right) \int_M \Omega \leq \int_M \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right) \Omega$$

$$\leq \max_x \left(\left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x) \right) \int_M \Omega,$$

i.e., $\min_x \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x) \leq 0 \leq \max_x \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x)$. For (2), we have

$$(70) \quad \text{osc} \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right) = \max_x \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x) - \min_x \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x)$$

$$\geq \max_x \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x),$$

since $-\min_x \left(\int_0^1 \alpha(\dot{\phi}_u) \circ \phi_u du \right)(x) \geq 0$, from item (1). □

We are able to give the proof of Theorem 6.7

Proof of Theorem 6.7 : For each non-empty open subset $\mathcal{U} \subset M$, either $D_\Omega(\mathcal{U})$ is empty, or non-empty. In the case, $D_\Omega(\mathcal{U}) = \emptyset$, we have $E^\Omega(\mathcal{U}) = +\infty$: We are done. Assume that $D_\Omega(\mathcal{U})$ is non-empty. By the characterization of the infimum, for each positive integer k , there exists a volume-preserving diffeomorphism f_k such that

$$(71) \quad \|f_k\|^\infty < E^\Omega(\mathcal{U}) + \frac{1}{k},$$

and $f_k \in D_\Omega(\mathcal{U})$. Fix the integer k large enough, and apply Proposition 6.8 with $B := \mathcal{U}$, to derive that there exists a pair of volume-preserving diffeomorphisms ϕ , and ψ each of which supported in $\mathcal{U}$ such that the commutator

$[\phi, \psi] := \psi^{-1} \circ \phi^{-1} \circ \psi \circ \phi$ is different from the identity map, and since $f_k \in D_\Omega(\mathcal{U})$, we derive from Proposition 6.9 that for k large enough, we have $[[f_k, \phi^{-1}], \psi] = [\phi, \psi]$.

On the other hand, compute

$$\begin{aligned} \|\phi, \psi\|^\infty &= \|[\phi, \psi]^{-1}\|^\infty \\ &= \|[[f_k, \phi^{-1}], \psi]^{-1}\|^\infty \\ &= \|[f_k, \phi^{-1}] \circ \psi^{-1} \circ [f_k, \phi^{-1}]^{-1} \circ \psi\|^\infty \\ &\leq \|[f_k, \phi^{-1}] \circ \psi^{-1}\|^\infty + \|[f_k, \phi^{-1}]^{-1} \circ \psi\|^\infty \\ &= 2\|[f_k, \phi^{-1}]\|^\infty \\ (72) \quad &\leq 4\|f_k\|^\infty. \end{aligned}$$

Combining (71) together with (72), implies that

$$(73) \quad 0 < \frac{\|[\phi, \psi]\|^\infty}{4} \leq \|f_k\|^\infty < E^\Omega(\mathcal{U}) + \frac{1}{k},$$

for k large enough. Q.E.D. $\square$

6.4. Some geometric aspects of $G_\omega(M)$. The following theorems show that using the metric d^Ω induced by the norm $\|\cdot\|^\infty$, one can determine the volume preservation nature of the uniform limit of a sequence of vanishing-flux volume-preserving diffeomorphisms.

Theorem 6.11.

Let $\{\phi_i\} \subset \ker S_\Omega$, let $\phi \in \ker S_\Omega$, and let $\psi : M \rightarrow M$ be any map such that

- $\{\phi_i\} \xrightarrow{C^0} \psi$ and
- $d^\Omega(\phi_i, \psi) := \|\phi_i \circ \psi^{-1}\|^\infty \rightarrow 0, i \rightarrow \infty$.

Then, we must have $\psi = \phi$.

Proof: Assume that $\psi \circ \phi^{-1} \neq id_M$. That is, there exists a non-empty open subset $B \subset M$ such that $\psi \circ \phi^{-1} \in D_\Omega(B)$. Since $\phi_i \xrightarrow{C^0} \psi$, then $\phi_i \circ \phi^{-1} \xrightarrow{C^0} \psi \circ \phi^{-1}$. Choosing an integer k large enough, we may assume $\phi_k \circ \phi^{-1} \in D_\Omega(B)$. Thus, the positivity result of the displacement energy E^Ω , implies that $0 < E^\Omega(B) \leq \|\phi_k \circ \phi^{-1}\|^\infty$, for all k large enough. The latter inequalities contradicts each other as k tends to infinity. Q.E.D. $\square$

Definition 6.12. Let $\Phi = \{\phi_t\}$, and $\Psi = \{\psi_t\}$ be two vanishing-flux isotopies. Then, we define the distance between them as :

$$d_{vol}(\Phi, \Psi) := \sup_{t \in [0,1]} d^\Omega(\phi_t, \psi_t).$$

We have following fact.

Theorem 6.13.

Let $\Phi_i = \{\phi_i^t\}$ be a sequence of vanishing-flux paths, let $\Psi = \{\psi_t\}$ be another vanishing-flux path, and let $\rho = \{\rho_t\}$ be any family of maps from M to M such that

- $\Phi_i \xrightarrow{C^0} \rho$, and
- $d_{vol}(\Phi_i, \Psi) \longrightarrow 0, i \longrightarrow \infty$.

Then, we must have $\Psi = \rho$.

Proof.

Assume that $\rho \circ \Psi^{-1} \neq Id$, i.e., there exist $t_0 \in]0, 1]$ such that $\rho_{t_0} \circ \psi_{t_0}^{-1} \neq id_M$. Since $\phi_i^{t_0} \xrightarrow{C^0} \rho_{t_0}$, then $\phi_i^{t_0} \circ \psi_{t_0}^{-1} \xrightarrow{C^0} \rho_{t_0} \circ \psi_{t_0}^{-1}$. On the other hand, the fact that $d_{vol}(\Phi_i, \Psi) \longrightarrow 0$ when $i \longrightarrow \infty$, implies that by the right-invariance of d^Ω that $d^\Omega(\phi_i^{t_0} \circ \psi_{t_0}^{-1}, id_M) \longrightarrow 0$ when $i \longrightarrow \infty$. Therefore, we have

- $\phi_i^{t_0} \circ \psi_{t_0}^{-1} \xrightarrow{C^0} \rho_{t_0} \circ \psi_{t_0}^{-1}$, and
- $d^\Omega(\phi_i^{t_0} \circ \psi_{t_0}^{-1}, id_M) \longrightarrow 0$ when $i \longrightarrow \infty$,

which by the mean of Theorem 6.11 implies that $\rho_{t_0} \circ \psi_{t_0}^{-1} = id_M$. $\square$

In the following result we estimate the metric d^Ω .

Lemma 6.14.

Let (M, Ω) be a closed oriented manifold equipped with a Riemannian metric g . Let $\text{diam}(M)$ be the Riemannian diameter of M with respect to g , and let $\phi, \psi \in \ker S_\Omega$. There exists a positive constant $K(\phi, g, \psi)$ which depends on g , ϕ , and ψ such that $d^\Omega(\phi, \psi) \leq 2(1 + \epsilon)\text{diam}(M)k(\phi, g, \psi)$, for all positive ϵ .

Proof.

Let $\phi, \psi \in \ker S_\Omega$, and let $\rho = \{\rho_t\}$ be any vanishing-flux isotopy from the identity to $\phi \circ \psi^{-1}$. By definition,

$$d^\Omega(\phi, \psi) := \|\phi \circ \psi^{-1}\|^\infty = \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} \left| \int_{\mathcal{O}_z^\rho} \alpha \right| \right).$$

Apply formula (68) and Corollary 6.10 together with the above equalities to derive that

$$d^\Omega(\phi, \psi) = \sup_{\alpha \in \mathcal{B}(1)} \left(\sup_{z \in M} \left| \int_{\mathcal{O}_z^\rho} \alpha \right| \right) \leq 2(1 + \epsilon)\text{diam}(M) \left(1 + \sup_{s,v} |T_{\gamma_y(s)} \rho_v| \right),$$

for all positive ϵ , for some point $y \in M$. Take

$$k(\phi, g, \psi) := \inf_{\rho \in \text{Iso}(\phi \circ \psi^{-1})} \left(1 + \sup_{s,v} |T_{\gamma_y(s)} \rho_v| \right),$$

where $\text{Iso}(\phi \circ \psi^{-1})$ is the space of all smooth vanishing-flux isotopies from the identity to $\phi \circ \psi^{-1}$. $\square$

Lemma 6.15.

Let (M_i, Ω_i) , $i = 1, 2$ be two closed oriented manifolds. If $\Theta : M_1 \rightarrow M_2$ is a smooth diffeomorphism such that $\Theta^*(\Omega_2) = \Omega_1$, and Θ^* maps closed forms onto closed forms, then

$$(1) \text{ for each } \phi \in \ker S_{\Omega_1}, \text{ we have } \|\Theta \circ \phi \circ \Theta^{-1}\|_2^\infty \leq 2\|\phi\|_1^\infty,$$

$$(2) \text{ for each } \psi \in \ker S_{\Omega_2}, \text{ we have } \|\Theta^{-1} \circ \psi \circ \Theta\|_1^\infty \leq 2\|\psi\|_2^\infty \text{ and}$$

$$(3) \text{ for each } \phi \in \ker S_{\Omega_1}, \text{ we have}$$

$$\frac{1}{2} \|\Theta \circ \phi \circ \Theta^{-1}\|_2^\infty \leq \|\phi\|_1^\infty \leq 2 \|\Theta \circ \phi \circ \Theta^{-1}\|_2^\infty.$$

Proof.

Let $\mathcal{Z}^1(M_i)$ be the space of closed 1-form of M_i , and let $\beta \in \mathcal{Z}^1(M_2)$. Then, $\Theta^*(\beta) \in \mathcal{Z}^1(M_1)$, and for each $\phi \in \ker S_{\Omega_1}$, we have $\phi^*(\Theta^*(\beta)) - \Theta^*(\beta) = d(\Delta_1(\phi, \Theta^*(\beta)))$, i.e.,

$$d(\Delta_2(\Theta \circ \phi \circ \Theta^{-1}, \beta)) = (\Theta \circ \phi \circ \Theta^{-1})^*(\beta) - \beta = d(\Delta_1(\phi, \Theta^*(\beta)) \circ \Theta^{-1}).$$

Thus, $\text{osc}(\Delta_2(\Theta \circ \phi \circ \Theta^{-1}, \beta)) = \text{osc}(\Delta_1(\phi, \Theta^*(\beta)) \circ \Theta^{-1})$, implies that

$$\begin{aligned} \|\Theta \circ \phi \circ \Theta^{-1}\|_2^\infty &\leq \sup_{\beta \in \mathcal{B}(1)_2} (\text{osc}(\Delta_2(\Theta \circ \phi \circ \Theta^{-1}, \beta))) \\ &= \sup_{\beta \in \mathcal{B}(1)_2} (\text{osc}(\Delta_1(\phi, \Theta^*(\beta)))) \\ (74) \qquad \qquad \qquad &\leq \sup_{\alpha \in \mathcal{B}(1)_1} (\text{osc}(\Delta_1(\phi, \alpha))). \end{aligned}$$

That is, $\|\Theta \circ \phi \circ \Theta^{-1}\|_2^\infty \leq 2\|\phi\|_1^\infty$, because $\text{osc}(f) \leq 2 \sup_x |f(x)|$. Similarly, one proves the item 2 and completes the proof of Lemma 6.15 by proving item 3 which follows from the combination of items 1 and 2. $\square$

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