

ALMOST PSEUDO RICCI SYMMETRIC MANIFOLD ADMITTING SCHOUTEN TENSOR

MOHABBAT ALI AND MOHD VASIULLA

DEPARTMENT OF APPLIED SCIENCES & HUMANITIES, JAMIA MILLIA ISLAMIA

NEW DELHI-110025, INDIA

E-MAILS: ALI.MATH509@GMAIL.COM AND VSMLK45@GMAIL.COM

(Received: 07 May 2021 , Accepted: 22 October 2021)

ABSTRACT. The object of the present paper is to study almost pseudo Ricci symmetric manifold admitting Schouten tensor and investigated some geometric properties of this manifold.

AMS Classification: 53C25.

Keywords: Almost pseudo Ricci manifold; Schouten tensor; Codazzi type Ricci tensor; divergence-free tensor.

1. INTRODUCTION

Let (M^n, g) be a Riemannian manifold of dimension n and ∇ be the Levi-Civita connection with respect to the metric g . A Riemannian manifold is called locally symmetric [3] if $\nabla K = 0$, where K is the Riemannian curvature tensor.

During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to extend such as conformally symmetric manifolds by Chaki and Gupta [5], recurrent manifolds introduced by Walker [19], conformally recurrent manifolds by Adati and Miyazawa [2], conformally symmetric Ricci-recurrent spaces by Roter [14], pseudo Riemannian manifolds with

recurrent concircular curvature tensor by Olszak [17], semi symmetric manifolds by Tamassy and Binh [18] and many others.

In the paper [6], Chaki and Roy had shown that a general relativistic space-time with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla \text{Ric} = 0$ where Ric is the Ricci tensor of the spacetime. If however $\nabla \text{Ric} \neq 0$, then such a spacetime may be called pseudo Ricci symmetric. It can be said that the Ricci symmetric space is a special case of the pseudo Ricci symmetric space. Therefore, it is interesting to study the properties of pseudo Ricci symmetric spacetimes in general relativity.

Let R be the Ricci tensor of type (1,1) defined as

$$(1) \quad \text{Ric}(Y, Z) = g(Y, Z),$$

for all vector fields Y and Z .

A non-flat Riemannian manifold is said to be pseudo Ricci symmetric [13] if

$$(2) \quad (\nabla_X \text{Ric})(Y, Z) = 2A(X)\text{Ric}(Y, Z) + A(Y)\text{Ric}(X, Z) + A(Z)\text{Ric}(Y, X),$$

where A is non-zero 1-form such that

$$(3) \quad g(X, \mu) = A(X),$$

for all vector fields X and μ is the associated vector field corresponding to the 1-form A .

A non-flat Riemannian manifold is said to be an almost pseudo Ricci symmetric [7] if

$$(4) \quad (\nabla_X \text{Ric})(Y, Z) = [A(X) + B(X)]\text{Ric}(Y, Z) + A(Y)\text{Ric}(X, Z) + A(Z)\text{Ric}(Y, X),$$

where A and B are two non-zero 1-forms and ∇ be the operator of covariant differentiation with respect to the metric g . In this case A and B are called the associated 1-forms. Such type of manifold is denoted by $A(PRS)_n$.

If $B = A$, then the equation (4) reduces to (2), that is, $A(PRS)_n$ reduces to a pseudo Ricci symmetric manifold [4]. Thus, pseudo Ricci symmetric manifold is a particular case of $A(PRS)_n$. Tamassy and Binh [18] introduced the notion of weakly Ricci symmetric manifold which is the generalization of pseudo Ricci symmetric manifold in the sense of Chaki. It may be mentioned that an $A(PRS)_n$ is not a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh [18].

Let $g(X, \mu) = A(X)$ and $g(X, \sigma) = B(X)$ for all X . Then μ, σ are called basic vector fields of the manifold corresponding to the associated 1-forms A and B , respectively.

Almost pseudo Ricci symmetric manifolds on some structures have been studied by many authors such as De and Gazi [10], Shaikh, Hui and Bagewadi [16], De and Mallick [11], De and Pal [12] and many others.

A non-flat Riemannian manifold is called generalized recurrent [15] if

$$(5) \quad (\nabla_X \text{Ric})(Y, Z) = A(X)\text{Ric}(Y, Z) + B(X)g(Y, Z).$$

where A and B are non-zero 1-forms. If the associated 1-form B becomes zero, then the manifold reduces to a Ricci recurrent manifold, that is

$$(6) \quad (\nabla_X \text{Ric})(Y, Z) = A(X)\text{Ric}(Y, Z).$$

In a Riemannian manifold, the Ricci tensor is called Codazzi type [1] if

$$(7) \quad (\nabla_X \text{Ric})(Y, Z) = (\nabla_Y \text{Ric})(X, Z).$$

The Schouten tensor in a Riemannian manifold is given by [9]

$$(8) \quad P(Y, Z) = \frac{1}{n-2} \left[\text{Ric}(Y, Z) - \frac{r}{2(n-1)} g(Y, Z) \right].$$

There is a decomposition formula in which the Riemannian curvature tensor decomposes into non-conformally invariant part, the Schouten tensor and a conformally invariant part, the conformal curvature tensor

$$(9) \quad K = P \odot g + C$$

where C is the conformal curvature tensor of g , and $\odot$ denotes the Kulkarni-Nomizu product, K is the Riemannian curvature tensor and P is the Schouten tensor, see [9]. Since the Weyl tensor is conformally invariant, to study the deformation of the conformal metric, we only need to understand the Schouten tensor (see [9], [8]).

The above results will be used in next sections.

2. SCHOUTEN FLAT $A(PRS)_n$

Let us assume that our manifold is Schouten-flat, then from (8) we have

$$(10) \quad \text{Ric}(Y, Z) = \frac{r}{2(n-1)} g(Y, Z).$$

Assuming that our manifold is $A(PRS)$ admitting Schouten tensor.

Taking covariant derivaative of (10), we get

$$(11) \quad (\nabla_X \text{Ric})(Y, Z) = \frac{1}{2(n-1)} (\nabla_X r) g(Y, Z).$$

which in view of (4) and (10) the relation (11) becomes

$$(12) \quad (\nabla_X \text{Ric})(Y, Z) = r\{[A(X) + B(X)]g(Y, Z) + A(Y)g(X, Z) + A(Z)g(Y, X)\}.$$

Contracting of (12) over Y and Z , we get

$$(13) \quad n(\nabla_X r) = r[(n+2)A(X) + nB(X)].$$

Again, contracting (12) over X and Y , then we have

$$(14) \quad (\nabla_Z r) = r[(n+2)A(Z) + B(Z)].$$

Replace Z by X in (14) and then comparing from (13), we get

$$(n-1)rA(X) = 0.$$

Since the sscalar curvature r is non-zero, then from above we have $A(X) = 0$.

Hence we obtain the following theorem:

Theorem 2.1. *A Schouten-flat $A(PRS)_n$ of non-vanishing scalar curvature reduces to a recurrent manifold with the recurrence vector field corresponding to the 1-form B .*

3. $A(PRS)_n$ ADMITTING NON-ZERO SCHOUTEN TENSOR

In this section, we assume that our manifold $A(PRS)_n$ is of non-zero Schouten tensor.

Taking covariant differentiation of (8) with respect to X , we get

$$(15) \quad (\nabla_X P)(Y, Z) = \frac{1}{n-2} \left[(\nabla_X \text{Ric})(Y, Z) - \frac{1}{2(n-1)} (\nabla_X r) g(Y, Z) \right].$$

Contracting of (4) over Y and Z , then we obtain

$$(16) \quad (\nabla_X r) = [A(X) + B(X)]r + 2A(R(X)).$$

By virtue of (4) and (16) in (15), we get

$$(17) \quad \begin{aligned} (\nabla_X P)(Y, Z) = & \frac{1}{n-2} \left\{ [A(X) + B(X)] \text{Ric}(Y, Z) + A(Y) \text{Ric}(X, Z) \right. \\ & + A(Z) \text{Ric}(Y, X) - \frac{r}{2(n-1)} [A(X) + B(X)] g(Y, Z) \\ & \left. - \frac{1}{(n-1)} A(R(X)) g(Y, Z) \right\}. \end{aligned}$$

Contracting of (17) over X and Y , we have

$$(18) \quad \begin{aligned} (\text{div} P)(Z) = & \frac{1}{n-2} \left[\frac{(4n-5)}{2(n-1)} A(R(Z)) + r \frac{(2n-3)}{2(n-1)} A(Z) \right. \\ & \left. + B(R(Z)) - \frac{r}{2(n-1)} B(Z) \right]. \end{aligned}$$

If the Schouten tensor P is divergence free, then from (18)

$$(19) \quad \frac{1}{n-2} \left[\frac{(4n-5)}{2(n-1)} A(R(Z)) + r \frac{(2n-3)}{2(n-1)} A(Z) + B(R(Z)) - \frac{r}{2(n-1)} B(Z) \right] = 0.$$

Thus, we are in the position to state the following:

Theorem 3.1. *In an $A(PRS)_n$, if the Schouten tensor is divergence free then $\frac{r}{2(n-1)}$ is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector σ where $g(X, \sigma) = B(X)$ if and only if $\frac{r(2n-3)}{4n-5}$ is an eigenvalue of Ricci tensor Ric corresponding to the eigenvector μ where $g(X, \mu) = A(X)$.*

Let the Schouten tensor P is recurrent then from (6), we have

$$(20) \quad (\nabla_X P)(Y, Z) = \alpha(X) P(Y, Z),$$

where α is 1-form. Using (8) and (17) in (20), we obtain

$$(21) \quad \begin{aligned} \alpha(X) \left[\text{Ric}(Y, Z) - \frac{r}{2(n-1)} \right] = & [A(X) + B(X)] \text{Ric}(Y, Z) + A(Y) \text{Ric}(X, Z) \\ & + A(Z) \text{Ric}(Y, X) - \frac{r}{2(n-1)} [A(X) + B(X)] g(Y, Z) \\ & - \frac{1}{(n-1)} A(R(X)) g(Y, Z). \end{aligned}$$

Contracting of (21) over X and Y , we get

$$(22) \quad \begin{aligned} \alpha(R(Z)) - \frac{r}{2(n-1)} \alpha(Z) = & \frac{1}{2(n-1)} \left[(4n-5) A(R(Z)) + r(2n-3) A(Z) \right] \\ & + B(R(Z)) - \frac{r}{2(n-1)} B(Z). \end{aligned}$$

This leads us to the following:

Theorem 3.2. *In an $A(PRS)_n$, let us assume that the Schouten tensor is recurrent with the recurrence vector corresponding to 1-form α . If $\frac{r}{2(n-1)}$ is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvectors both σ and ρ where $g(X, \sigma) = B(X)$, $g(X, \rho) = \alpha(X)$ then $-r\frac{(2n-3)}{(4n-5)}$ is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector μ where $g(X, \mu) = A(X)$.*

Let us suppose that the scalar curvature r is non-zero constant, then from (15) reduces to

$$(23) \quad (\nabla_X P)(Y, Z) = \frac{1}{n-2}(\nabla_X Ric)(Y, Z).$$

From (4) and (23), we have

$$(24) \quad (\nabla_X P)(Y, Z) = \frac{1}{n-2} \left\{ [A(X) + B(X)] Ric(Y, Z) + A(Y) Ric(X, Z) + A(Z) Ric(Y, X) \right\}.$$

Interchanging X and Y in (24) and then subtracting them, we get

$$(25) \quad (\nabla_X P)(Y, Z) - (\nabla_Y P)(X, Z) = \frac{1}{n-2} [B(X) Ric(Y, Z) - B(Y) Ric(X, Z)].$$

If P is Codazzi type then from (7) and (25), we obtain

$$(26) \quad B(X) Ric(Y, Z) - B(Y) Ric(X, Z) = 0.$$

Contracting (26) over Y and Z , we get

$$B(X)r = B(R(X))$$

which in view of (1) the above relation yields

$$Ric(X, \sigma) = rg(X, \sigma).$$

This leads to the following theorem:

Theorem 3.3. *In an $A(PRS)_n$ of non-zero constant scalar curvature, if Schouten tensor is of Codazzi type then r is an eigenvalue of the Ricci tensor Ric corresponding to the eigenvector σ where $g(X, \sigma) = B(X)$.*

Let r is non-zero constant and Schouten tensor P is covariantly constant, then from (15) we find that

$$(27) \quad (\nabla_X Ric)(Y, Z) = 0.$$

Hence from (4) and (27), we have

$$(28) \quad [A(X) + B(X)]\text{Ric}(Y, Z) + A(Y)\text{Ric}(X, Z) + A(Z)\text{Ric}(Y, X) = 0.$$

Contracting of (28) over Y and Z , we get

$$[A(X) + B(X)]r + 2A(R(X)) = 0$$

which implies

$$A(R(X)) = -\frac{r}{2}[A(X) + B(X)].$$

Thus we obtain the following theorem:

Theorem 3.4. *If an $A(PRS)_n$ of non-vanishing constant scalar curvature and the Schouten tensor is covariantly constant then the Ricci tensor is covariantly constant and the 1-forms A and B are related by*

$$A(R(X)) = -\frac{r}{2}[A(X) + B(X)].$$

4. EXISTENCE OF $A(PRS)_n$ ADMITTING SCHOUTEN TENSOR

We define a Riemannian metric on the 4-dimensional real number space $\mathbb{R}^4$ by the formula

$$(29) \quad ds^2 = e^{x^1}(dx^1)^2 + e^{x^2}(dx^2)^2 + (dx^3)^2 + (\sin(x^3))^2(dx^4)^2.$$

Then the only non-vanishing components of the Christoffel symbols and the Ricci tensor are as follows:

$$(30) \quad \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} = \frac{1}{2}, \quad \left\{ \begin{matrix} 4 \\ 34 \end{matrix} \right\} = \cot(x^2), \quad \left\{ \begin{matrix} 3 \\ 44 \end{matrix} \right\} = -\sin(x^3)\cos(x^3),$$

and

$$(31) \quad \text{Ric}_{33} = -1, \quad \text{Ric}_{44} = -\sin(x^3)^2,$$

It can be easily shown that the scalar curvature of the manifold is -2 . Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold whose scalar curvature r is constant. Then the only non-zero components of the Schouten tensor are as follows:

$$(32) \quad P_{11} = \frac{e^{x^1}}{6}, \quad P_{22} = \frac{e^{x^2}}{6}, \quad P_{33} = -\frac{1}{3}, \quad P_{44} = -\frac{(\sin(x^3))^2}{3}.$$

Taking covariant derivatives of (32) we get $P_{ij,k} = 0$ for all i, j, k . This means that P_{ij} are covariantly constant. Also, by taking covariant derivatives of (31), we get $\text{Ric}_{ij,k} = 0$ for all i, j, k .

Now consider the associated 1-forms A and B are as follows:

$$(33) \quad A(x) = e^{x^1+x^2} + e^{x^3+x^4} \quad \text{and} \quad B(x) = e^{x^1+x^2} + 2e^{x^3+x^4}.$$

Taking covariant derivative of (33), we get

$$(34) \quad A_i(x) = \begin{cases} e^{x^1+x^2}, & \text{if } i=1,2 \\ e^{x^3+x^4}, & \text{if } i=3,4 \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} e^{x^1+x^2}, & \text{if } i=1,2 \\ e^{x^3+x^4}, & \text{if } i=3,4. \end{cases}$$

From (31) and (34) the following equations

$$A^1 R_{11} = -\frac{r}{2}[A_1 + B_1], \quad A^2 R_{22} = -\frac{r}{2}[A_2 + B_2]$$

and

$$A^3 R_{33} = -\frac{r}{2}[A_3 + B_3], \quad A^4 R_{44} = -\frac{r}{2}[A_4 + B_4]$$

are satisfied. This justify the Theorem 3.4.

In consequence of the above, one can say that

Theorem 4.1. *There exist a manifold $(\mathbb{R}^4, g)$ which is an $A(PRS)n$ with the constant scalar curvature and covariantly constant Schouten tensor with the above mentioned choice of the 1-forms then Theorem 3.4 holds.*

REFERENCES

- [1] M. Ali, Q. Khan and M. Vasiulla, On special weakly symmetric Riemannian manifolds, BSG Proceedings, Vol. 28, 1–11 (2021).
- [2] T. Adati and T. Miyazawa, On a Riemannian space with recurrent conformal curvature, Tensor (N.S.), Vol. 18, 348–354 (1967).
- [3] E. Cartan, Bull. Soc. Math. France, Sur une classe remarquable d'espaces de Riemannian, Vol. 54, 214–264 (1926).
- [4] M. C. Chaki, On pseudo Ricci symmetric manifolds, Bulg. J. Phys, Vol. 15, 525–531 (1988).
- [5] M. C. Chaki and B. Gupta, On conformally symmetric spaces, Indian J. Math, Vol. 5, 113–295 (1963).
- [6] M. C. Chaki and S. Roy, Spacetimes with covariant-constant energy-momentum tensor, In. J. Theory Phys., Vol. 35, 1027–1032 (1996).
- [7] M. C. Chaki and T. Kawaguchi, On almost pseudo Ricci symmetric manifolds, Tensor (N.S.), Vol. 68, Number 1, 10–14 (2007).
- [8] M. C. Chaki, A. N. Roychowdhury, On Ricci-recurrent spaces of 2nd order, Indian Jour. Math., Vol. 9, Number 2, 279–287 (1967).
- [9] B. Y. Chen, K. Yano, Hypersurfaces of a conformally flat space, Tensor (N. S.), Vol. 26, 318–322 (1972).

- [10] U. C. De and A. K. Gazi, On conformally flat almost pseudo Ricci symmetric manifolds, Kyungpook Math. J., Vol. 49, 507–520 (2009).
- [11] U. C. De and S. Mallick, On almost pseudo concircularly symmetric manifold, The Journal of Math. and Comp. Sci., Vol. 20, Number 3, 317–320 (2012).
- [12] U. C. De and P. Pal, On some classes of almost pseudo Ricci symmetric manifold, Publ. Math. Debrecen, Vol. 83, Number (1-2), 207–225 (2013).
- [13] R. Deszcz, On pseudo-symmetric spaces, Bull. Soc. Math. Belg. Serie A, Vol. 44, 1–34 (1992).
- [14] K. Olszak and Z. Olszak, On pseudo-Riemannian manifolds with recurrent concircular curvature tensor, Acta Math. Hungar, Dol: 10.1007/510474-012-0216-5.
- [15] W. Roter, On conformally symmetric Ricci-recurrent spaces, Colloq. Math, Vol. 31, 87–96 (1974).
- [16] A. A. Shaikh, S. K. Hui and C. S. Bagewadi, On quasi-conformally almost pseudo Ricci symmetric manifolds, Tamsui Oxford Journal of Math. Sci., Vol. 26, Number 2, 203–219 (2010).
- [17] Z. I. Szabo, Structure theorems on Riemannian spaces satisfying $R(X Y) \cdot R = 0$, J. Diff. Geo, Vol. 17, 531–582 (1982).
- [18] L. Tamassy and T. Q. Binh, On weakly symmetries of Einstein and Sasakian manifolds, Tensor (N.S.), Vol. 53, 140-148 (1993).
- [19] A. G. Walker, On Ruse's space of recurrent curvature, Proc. London Math. Soc., Vol. 52, 36–64 (1951).