

SOLUTION OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS WITH SINGULARITY USING VARIABLE MESH FINITE DIFFERENCE METHOD

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ABSTRACT. We use non-polynomial spline with variable mesh to establish a numerical scheme for the solution of boundary value problem with singularity. The discrete equation of the problem is developed based on the condition of the class C^1 of non-polynomial spline at the inner nodes and it is not valid for singularity. At singularity $t = 0$, the problem is modified in order to have a three term relationship. The method's tridiagonal scheme is analyzed using the well-known discrete imbedding invariant algorithm. We discuss the error analysis of the scheme and two examples with layer at one end of the boundary are consider to demonstrate the practical utility of the scheme. Maximum absolute errors are present in tabular form to show the efficiency of the proposed method.

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1. INTRODUCTION

Spline functions occupy an important role in the theory of numerical approximations as they have number of applications primarily in computing the smooth approximations of boundary value problems. A class of singular perturbation problem with the singularity of the form given below is considered.

$$(1) \quad \varepsilon \varphi''(t) + \frac{k}{t} \varphi'(t) + q(t) \varphi(t) = r(t), \quad 0 \leq t \leq 1,$$

with boundary conditions

$$(2) \quad \varphi(0) = \gamma_1, \quad \varphi(1) = \gamma_2$$

where γ_1, γ_2 are given constants, ε is a perturbation parameter such that $0 < \varepsilon \ll 1$ and $q(t), r(t)$ are regular functions. Let $p(t) = \frac{k}{t}$. If $p(t) \geq \tilde{M} > 0$ throughout the domain $[0, 1]$, where $\tilde{M}$ is a positive constant, then the boundary layer exist in the neighbourhood of $t = 0$.

If $p(t) \leq \tilde{N} < 0$ throughout the interval $[0, 1]$, where $\tilde{N}$ is a negative constant, then the boundary layer will be in the neighbourhood of $t = 1$.

The coefficient of diffusion term in many important physical problems is small compared to convection term. Sources of these include problems with heat transport with large Peclet numbers, fluid flow problems with large Reynolds and high Hartman problems for MHD ducts. These problems solutions depict the layer behaviour near the boundary of the domain where the solution derivatives are varying very rapidly is known as perturbation problems. Problems of the above type of Eq. (1) belongs to a similar class of singular perturbation problem in addition with having singularity at $t = 0$. A wide range of books and papers have been published, including [2-3, 10, 13, 14], detailing various approaches for solving singularly perturbed double-point boundary value issues.

Difference schemes based on cubic spline functions for the second order differential equations, which are exhibiting the boundary layer behaviour at near the end of the domain and having singularity at $t = 0$, are studied by many authors. Such as, the researchers in [4] generated uniform mesh with the help of fitted mesh technique and used B-splines to solve such problems, Rashidinia and his team [12] derived second order accuracy method using the uniform mesh cubic polynomial spline. In [9], researchers used tension spline to develop non-uniform mesh numerical methods and to solve problems in both rectangular and polar coordinates,

authors in [8] reported three convergent numerical schemes of second order using spline in tension, Mohanty and his team [7] derived difference schemes for the both singular and nonsingular problems using spline in compression. In [11], Phaneendra and his team presented fourth order Non-polynomial spline finite difference method on uniform mesh for the same type of the problems. Also, some difference schemes using spline functions are developed by Kadalbajoo and Patidar in [5], Bava in [1] to solve singularly perturbed boundary value problems and Kadalbajoo and Reddy [6] used deviating argument technique for the similar type of problems.

In this paper we used non-polynomial spline to derive a non-uniform mesh finite difference scheme for the solution of the proposed Eq. (1) and (2). The structure of this paper is as follows: In section 2, we describe non-polynomial spline and its truncation error. In section 3, the description of the numerical method for solving Eq. (1) is presented. Section 4 provides the error analysis of the derived scheme and section 5 provides numerical results with maximum absolute errors. Finally, in the last segment, discussions and conclusion are given.

2. NON-POLYNOMIAL SPLINE

Partition the interval $[0, 1]$ with variable step size $h_i = t_i - t_{i-1}$ for $i = 1$ to n and $h_{i+1} = \sigma_i h_i$ into the subdivision of the form $0 = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = 1$

The exact and approximate solution of the Eq.(1) is $\varphi(t)$, φ_i obtained by the non-polynomial cubic spline $\psi_i(t)$ passing through the points (t_i, φ_i) and (t_{i+1}, φ_{i+1}) respectively. The polynomial $\psi_i(t)$ satisfies interpolatory conditions at t_i and t_{i+1} , also the condition of the class C^1 at the common nodes (t_i, φ_i) .

The cubic non-polynomial spline function $\psi_i(t)$ has the form for each i^{th} segment

$$(3) \quad \psi_i(t) = \tilde{a}_i + \tilde{b}_i(t - t_i) + \tilde{c}_i \sin \kappa(t - t_i) + \tilde{d}_i \cos \kappa(t - t_i), \quad i = 0, 1, 2, \dots, n-1$$

where $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ and $\tilde{d}_i$ are constants and κ is a free parameter.

A non-polynomial function $\psi(t)$ of class $C^2[0, 1]$ interpolates $\varphi(t)$ at the grid points t_i , $i = 0, 1, 2, \dots, n$ is based on a parameter κ and reduces $\psi(t)$ to ordinary cubic spline in $[0, 1]$ as $\kappa \rightarrow 0$.

In order to derive the expression for the coefficients of Eq. (3) in terms of φ_i , φ_{i+1} , M_i and M_{i+1} , we first express

$$\psi_i(t_i) = \varphi_i, \quad \psi_i(t_{i+1}) = \varphi_{i+1}, \quad \psi_i''(t_i) = M_i, \quad \psi_i''(t_{i+1}) = M_{i+1}$$

Then, we get the following expressions for the unknowns through algebraic manipulation

$$\tilde{a}_i = \varphi_i + \frac{M_i}{\kappa^2}, \quad \tilde{b}_i = \frac{\varphi_{i+1} - \varphi_i}{h} + \frac{M_{i+1} - M_i}{\kappa \tilde{\theta}}, \quad \tilde{c}_i = \frac{M_i \cos \tilde{\theta} - M_{i+1}}{\kappa^2 \sin \tilde{\theta}}, \quad \tilde{d}_i = -\frac{M_i}{\kappa^2}$$

where $\tilde{\theta} = \kappa h_{i+1}$, $i = 0, 1, 2, \dots, n-1$.

Using the condition of the class C^1 at (t_i, φ_i) , i.e. $\psi'_{i-1}(t_i) = \psi'_i(t_i)$ for $i = 1, 2, \dots, n-1$ we get the following relationship.

$$(4) \quad \sigma \varphi_{i-1} - (1 + \sigma) \varphi_i + \varphi_{i+1} = h_i h_{i+1} \left[\tilde{\alpha}_1 M_{i-1} + \tilde{\beta} (1 + \sigma) M_i + \tilde{\alpha}_2 \sigma M_{i+1} \right]$$

where

$$\tilde{\alpha}_1 = \frac{-1}{\kappa^2 h_i^2} + \frac{1}{\kappa h_i \sin(\kappa h_i)} \quad \tilde{\beta} = \frac{1}{\kappa h_i \tilde{\theta}} + \frac{\sin \kappa (h_i + h_{i+1})}{\kappa (h_i + h_{i+1}) \sin(\kappa h_i) \sin \tilde{\theta}}$$

$$\tilde{\alpha}_2 = \frac{-1}{\tilde{\theta}^2} + \frac{1}{\tilde{\theta} \sin \tilde{\theta}} M_j = \varphi'(t_j), \quad j=i, i \pm 1 \text{ and } \tilde{\theta} = \kappa h_{i+1}$$

Noticed that, if $\kappa \rightarrow 0$ the non-polynomial spline relation (4) is compatible with the normal cubic spline that is $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \frac{1}{6}$, $\tilde{\beta} = \frac{1}{3}$

The local truncation error $\tilde{T}_i(h)$ in relation to our scheme (4) is

$$T(h) =$$

$$\left[-\frac{\sigma}{2} (1 + \sigma) + \sigma^2 (\tilde{\alpha}_1 + \tilde{\beta} + \tilde{\alpha}_2) \right] \varphi_i^{(2)} h_i^2 + \left[-\frac{\sigma}{6} (-1 + \sigma^2) - \sigma^2 (\tilde{\alpha}_1 - \sigma \tilde{\alpha}_2) \right] \varphi_i^{(3)} h_i^3$$

$$+ \left[-\frac{\sigma}{24} (1 + \sigma^3) + \frac{\sigma^2}{2} (\tilde{\alpha}_1 + \sigma^2 \tilde{\alpha}_2) \right] \varphi_i^{(4)} h_i^4$$

$$+ \left[-\frac{\sigma}{120} (-1 + \sigma^4) - \frac{\sigma^2}{6} (\tilde{\alpha}_1 - \sigma^3 \tilde{\alpha}_2) \right] \varphi_i^{(5)} h_i^5 + 0 (h^6)$$

For the choice of

$$\tilde{\alpha}_1 = \frac{1 + \sigma - \sigma^2}{12\sigma} \quad \tilde{\alpha}_2 = \frac{\sigma^2 + \sigma - 1}{12\sigma^2} \quad \tilde{\beta} = \frac{\sigma^3 + 4\sigma^2 + 4\sigma + 1}{12\sigma^2}$$

the above scheme Eq. (4) indicates third order convergent and the truncation error will be

$$(5) \quad \tilde{T}_i(h) = \frac{1}{360} \left[\sigma (\sigma^2 - 1) (\sigma + 2) (2\sigma + 1) \right] \varphi_i^{(5)} h_i^5 + 0 (h^6).$$

3. DESCRIPTION OF THE METHOD

At the grid points t_i , Eq. (1) may be discretized by $\varepsilon\varphi_i'' + p(t_i)\varphi_i' + q(t_i)\varphi_i = r_i$

Using spline's second derivatives, we have

$$(6) \quad \varepsilon M_j = r(t_j) - p(t_j)\varphi_j' - q(t_j)\varphi_j \quad \text{for } j=i, i\pm 1$$

Using Eq. (6) in Eq. (4) and with the help of the following approximations to the first derivative of φ

$$\begin{aligned} \varphi_{i+1}' &= \frac{1}{h_{i+1}} \left[\frac{2\sigma+1}{\sigma+1} \varphi_{i+1} - (1+\sigma)\varphi_i + \frac{\sigma^2}{1+\sigma} \varphi_{i-1} \right] \\ \varphi_i' &= \frac{1}{h_i} \left[\frac{1}{\sigma(\sigma+1)} \varphi_{i+1} + \frac{\sigma-1}{\sigma} \varphi_i - \frac{\sigma}{\sigma+1} \varphi_{i-1} \right] \\ (7) \quad \varphi_{i-1}' &= \frac{1}{h_i} \left[\frac{-1}{\sigma(\sigma+1)} \varphi_{i+1} + \frac{1+\sigma}{\sigma} \varphi_i - \frac{2+\sigma}{1+\sigma} \varphi_{i-1} \right] \end{aligned}$$

We get the tridiagonal system

$$\begin{aligned} & \left(-\varepsilon\sigma - \tilde{\alpha}_1 p_{i-1} h_i \left(\frac{2\sigma^2+\sigma^3}{1+\sigma} \right) - \tilde{\beta} p_i h_i \left(\frac{\sigma^3}{\sigma+1} \right) + \tilde{\alpha}_2 p_{i+1} h_i \left(\frac{\sigma^3}{\sigma+1} \right) + \right. \\ & \tilde{\alpha}_1 q_{i-1} h_i^2 (\sigma^2) \varphi_{i-1} \\ & + \left(\varepsilon(1+\sigma) + \tilde{\alpha}_1 p_{i-1} h_i (\sigma + \sigma^2) + \tilde{\beta} p_i h_i (\sigma^2 - \right. \\ (8) \quad & \left. \sigma - \tilde{\alpha}_2 p_{i+1} h_i (\sigma + \sigma^2) + \tilde{\beta} q_i h_i^2 (\sigma^2) \varphi_i \right. \\ & + \left. \left(-\varepsilon - \tilde{\alpha}_1 p_{i-1} h_i \left(\frac{\sigma}{1+\sigma} \right) + \tilde{\beta} p_i h_i \left(\frac{\sigma}{\sigma+1} \right) + \tilde{\alpha}_2 p_{i+1} h_i \left(\frac{2\sigma^2+\sigma}{\sigma+1} \right) + \right. \right. \\ & \left. \tilde{\alpha}_2 q_{i+1} h_i^2 (\sigma^2) \varphi_{i+1} \right. \\ & \left. = -h_{i+1}^2 \left(\tilde{\alpha}_1 r_{i-1} + \tilde{\beta} r_i + \tilde{\alpha}_2 r_{i+1} \right), \right. \end{aligned}$$

where $h_{i+1} = \sigma h_i$.

For $i = 1$, the coefficients φ_{i-1} , φ_i and φ_{i+1} becomes infinity in Eq. (8), thus we need to develop a boundary formula for this case. By using L-Hospital rule, from

Eq. (6), we get

$$(9) \quad \varphi_i'' = \frac{q_i \varphi_i + r_i}{\varepsilon - k} \text{ i.e., } M_i = \frac{q_i \varphi_i + r_i}{\varepsilon - k}$$

Again using (4), we get the following boundary formula for $i = 1$

$$(10) \quad \begin{aligned} & \left[-\sigma + \frac{\tilde{\alpha}_1 h_2^2 q_0}{\varepsilon - k} \right] \varphi_0 + \left[(1 + \sigma) + \frac{\tilde{\beta} h_2^2 q_1}{\varepsilon - k} \right] \varphi_1 + \left[-\sigma + \frac{\tilde{\alpha}_2 h_2^2 q_2}{\varepsilon - k} \right] \varphi_2 \\ &= \frac{-h_2^2}{\varepsilon - k} \left[\tilde{\alpha}_1 r_0 + \tilde{\beta} r_1 + \tilde{\alpha}_2 r_2 \right] \end{aligned}$$

Along with the Eq.(10), We solve the tridiagonal scheme (8) for $i = 1, 2, \dots, n-1$ to find the estimates $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_{n-1}$ of the solution $\varphi(t)$ at $t_1, t_2, \dots, t_{n-1}$.

Remark: For $\varepsilon = 1$ (regular problem), $\sigma = 1$ (uniform mesh), $p(t) = 0$, we have

$$\begin{aligned} & \left(-1 + \frac{h^2}{12} p_{i-1} \right) \varphi_{i-1} + \left(2 + \frac{10h^2}{12} p_i \right) \varphi_i + \left(-1 + \frac{h^2}{12} p_{i+1} \right) \varphi_{i+1} \\ &= \frac{h^2}{12} (q_{i-1} + 10 q_i + q_{i+1}) \end{aligned}$$

which is Numerov method of order four for the regular problem

$$\varphi''(t) = p(t)\varphi(t) + q(t), \text{ with } \varphi(0) = \gamma_0, \varphi(1) = \gamma_1$$

4. ERROR ANALYSIS

With the help of approximations of φ given in (7), we get

$$(11) \quad \begin{aligned} \hat{e}'_{i-1} &= \varphi'(t_{i-1}) - \varphi'_{i-1} = \left(\frac{\sigma^2 + 2\sigma + 1}{6(1+\sigma)} \right) h_i^2 \varphi^{(2)}(t_i) + \left(\frac{\sigma^3 - 3\sigma - 2}{24(1+\sigma)} \right) h_i^3 \varphi^{(3)}(t_i) \\ &+ \left(\frac{\sigma^4 + 4\sigma + 3}{120(1+\sigma)} \right) h_i^4 \varphi^{(4)}(\xi_1^{(i)}) \end{aligned}$$

$$(12) \quad \begin{aligned} \hat{e}'_{i+1} &= \varphi'(t_{i+1}) - \varphi'_{i+1} = \left(\frac{\sigma^3 + 2\sigma^2 + \sigma}{6(1+\sigma)} \right) h_i^2 \varphi^2(t_i) + \left(\frac{2\sigma^4 + 3\sigma^3 - \sigma}{24(1+\sigma)} \right) h_i^3 \varphi^3(t_i) \\ &+ \left(\frac{3\sigma^5 + 4\sigma^4 + \sigma}{120(1+\sigma)} \right) h_i^4 \varphi^4(\xi_2^{(i)}) \end{aligned}$$

$$(13) \quad \hat{e}'_i = \varphi'(t_i) - \varphi'_i = -\frac{\sigma}{6} h_i^2 \varphi^{(2)}(t_i) + \left(\frac{\sigma(1-\sigma)}{24} \right) h_i^3 \varphi^{(3)}(t_i) - \left(\frac{\sigma^3 - \sigma^2 + \sigma}{120} \right) h_i^4 \varphi^{(4)}(\xi_3^{(i)})$$

where $t_{i-1} < \xi_1^{(i)}, \xi_2^{(i)}, \xi_3^{(i)} < t_i$.

Substituting $\varepsilon M_j = p(t_j)\varphi'(t_j) + q(t_j)\varphi(t_j) + r(t_j)$, $j=i, i\pm 1$ in Eq. (3), we get

$$(14) \quad \begin{aligned} \varepsilon \sigma \varphi_{i-1} - \varepsilon(1 + \sigma) \varphi_i + \varepsilon \varphi_{i+1} = & h_{i+1}^2 \left[\tilde{\alpha}_1 \left(p_{i-1} \varphi'_{i-1} + q_{i-1} \varphi_{i-1} + r_{i-1} \right) \right] \\ & + h_{i+1}^2 \left[\tilde{\beta} \left(p_i \varphi'_i + q_i \varphi_i + r_i \right) \right] \\ & + h_{i+1}^2 \left[\tilde{\alpha}_2 \left(p_{i+1} \varphi'_{i+1} + q_{i+1} \varphi_{i+1} + r_{i+1} \right) \right] \end{aligned}$$

Putting exact solution in Eq. (14), we get

$$(15) \quad \begin{aligned} & \varepsilon \sigma \varphi(t_{i-1}) - \varepsilon(1 + \sigma) \varphi(t_i) + \varepsilon \varphi(t_{i+1}) \\ & = h_{i+1}^2 [\tilde{\alpha}_1 (p_{i-1} \varphi'(t_{i-1}) + q_{i-1} \varphi(t_{i-1}) + r_{i-1})] \\ & \quad + h_{i+1}^2 [\tilde{\beta} (p_i \varphi'(t_i) + q_i \varphi(t_i) + r_i)] \\ (16) \quad & + h_{i+1}^2 [\tilde{\alpha}_2 (p_{i+1} \varphi'(t_{i+1}) + q_{i+1} \varphi(t_{i+1}) + r_{i+1})] \\ (17) \quad & + \tilde{T}_i(h) \end{aligned}$$

where

$$(18) \quad \tilde{T}_i(h) = \frac{\varepsilon}{360} [\sigma (\sigma^2 - 1) (\sigma + 2) (2\sigma + 1)] \varphi_i^{(5)} h_i^5 + 0 (h^6)$$

for the choice of $\tilde{\alpha}_1 = \frac{1+\sigma-\sigma^2}{12\sigma}$, $\tilde{\alpha}_2 = \frac{\sigma^2+\sigma-1}{12\sigma^2}$, $\tilde{\beta} = \frac{\sigma^3+4\sigma^2+4\sigma+1}{12\sigma^2}$

Subtracting Eq. (14) from Eq. (15) and putting $\hat{e}_j = \varphi(t_j) - \varphi_j j = ii \pm 1$, we get

$$(19) \quad \begin{aligned} & (\varepsilon \sigma - \sigma^2 h_i^2 \tilde{\alpha}_1 q_{i-1}) \hat{e}_{i-1} + \left(-(1 + \sigma) \varepsilon - \sigma^2 h^2 \tilde{\beta} q_i \right) \hat{e}_i + (\varepsilon - \sigma^2 h_i^2 \tilde{\alpha}_2 q_{i+1}) \hat{e}_{i+1} = \\ & h_i^4 \left(\tilde{\alpha}_1 \left(\frac{\sigma^4 + 2\sigma^3 + \sigma^2}{6(1+\sigma)} \right) p_{i-1} - \tilde{\beta} \left(\frac{\sigma^3}{6} \right) p_i + \tilde{\alpha}_2 \left(\frac{\sigma^5 + 2\sigma^4 + \sigma^3}{6(1+\sigma)} \right) p_{i+1} \right) \varphi^{(2)}(t_i) \\ & + h_i^5 \left(\tilde{\alpha}_1 \left(\frac{\sigma^5 - 3\sigma^3 - 2\sigma^2}{24(1+\sigma)} \right) p_{i-1} - \tilde{\beta} \left(\frac{\sigma^3 - \sigma^4}{24} \right) p_i + \tilde{\alpha}_2 \left(\frac{2\sigma^6 + 3\sigma^5 - \sigma^3}{24(1+\sigma)} \right) p_{i+1} \right) \varphi^{(3)}(t_i) \\ & + O(h_i^6) + \tilde{T}_i(h) \end{aligned}$$

Let $p_{i+1} = p_i + h_{i+1} p'_i + \frac{h_{i+1}^2}{2} p_i^{(2)}(\eta_1^{(i)})$, $p_{i-1} = p_i - h_i p'_i + \frac{h_i^2}{2} p_i^{(2)}(\eta_2^{(i)})$ where $t_{i-1} < \eta_1^{(i)}, \eta_2^{(i)} < t_i$. Using these expressions and Eq. (19), we have

$$(20) \quad (\varepsilon \sigma - \sigma^2 h_i^2 \tilde{\alpha}_1 q_{i-1}) \hat{e}_{i-1} + \left(-(1 + \sigma) \varepsilon - \sigma^2 h^2 \tilde{\beta} q_i \right) \hat{e}_i + (\varepsilon - \sigma^2 h_i^2 \tilde{\alpha}_2 q_{i+1}) \hat{e}_{i+1} = \tilde{T}_i(h)$$

where

$$\begin{aligned}\tilde{T}_i(h) = h_i^4 & \left(\tilde{\alpha}_1 \left(\frac{\sigma^4 + 2\sigma^3 + \sigma^2}{6(1+\sigma)} \right) - \tilde{\beta} \left(\frac{\sigma^3}{6} \right) + \right. \\ & \left. \tilde{\alpha}_2 \left(\frac{\sigma^5 + 2\sigma^4 + \sigma^3}{6(1+\sigma)} \right) p_i \varphi^{(2)}(t_i) + O(h_i^5) \right)\end{aligned}$$

It can be seen easily that $\tilde{T}_i(h) = O(h^4)$ for the choice of

$$\tilde{\alpha}_1 = \frac{1 + \sigma - \sigma^2}{12\sigma} \quad \tilde{\alpha}_2 = \frac{\sigma^2 + \sigma - 1}{12\sigma^2} \quad \tilde{\beta} = \frac{\sigma^3 + 4\sigma^2 + 4\sigma + 1}{12\sigma^2}$$

Let $\tilde{J} = \text{trid}[\sigma\varepsilon \quad -(1+\sigma)\varepsilon \quad \varepsilon]$, $\tilde{D} = \text{trid}[\sigma^2\tilde{\alpha}_1 \quad \sigma^2\tilde{\beta} \quad \sigma^2\tilde{\alpha}_2]$ are $n-1 \times n-1$ tridiagonal matrices and $\hat{E} = (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{n-1})^t$, $\hat{E} = (\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{n-1})^t$ are vectors of order $n-1$.

The matrix vector form of Eq. (10) can be written as

$$(21) \quad (\tilde{J} + h^2 \tilde{D} \tilde{G}) \hat{E} = \tilde{T}_i(h)$$

Following [3], for small enough h , we have

$$\left\| (\tilde{J} - h^2 \tilde{D} \tilde{G})^{-1} \right\| \leq \left\| \tilde{J}^{-1} \right\| \leq \frac{1}{6} \left(\frac{1}{8h^2} + \frac{1}{2} \right).$$

Hence,

$$\left\| \hat{E} \right\| \leq \left\| (\tilde{J} - h^2 \tilde{D} \tilde{G})^{-1} \right\| \left\| \tilde{T}_i(h) \right\|$$

Therefore $\left\| \hat{E} \right\| = O(h^2)$ for the choice of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\beta}$ mentioned above and it is concluded that our approach gives a second order convergent method.

5. NUMERICAL EXAMPLES

In order to test the viability of the proposed method, we consider two problems of the type Eq.(1). The mesh ratio σ is chosen based on the location of the boundary layer. We choose the starting value of the step length given by $h_1 = \frac{\sigma-1}{\sigma^{N-1}}$ for $\sigma > 1$ as it gives more mesh points near the left end $t = 0$ and $h_1 = \frac{1-\sigma}{1-\sigma^n}$ for $\sigma < 1$ as it gives more mesh points near the right end $t = 1$.

Example 1. $-\varepsilon\varphi'' + \frac{1}{t}\varphi' + (1+t^2)\varphi = f(x)$, $0 < t < 1$.

The exact solution is $y(x) = \exp(x^2)$. The errors are tabulated in Table 1 for various values of ε and h .

Example 2. $-\varepsilon\varphi'' + \frac{1}{t}\varphi' = f(x)$, $0 < x < 1$.

The exact solution of this problem is $y(x) = x \sinh x$. The errors are tabulated in Table 1 for various values of ε and h .

Example 3. $\varepsilon\varphi'' + \frac{1}{t}\varphi' + \varphi = 0$, $0 \leq t \leq 1$ with boundary condition $y(0) =$

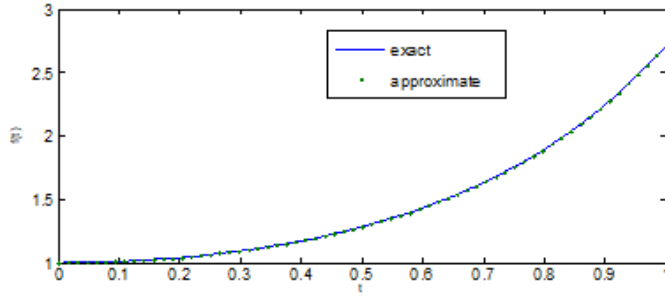
TABLE 1. Maximum absolute errors for different values of ε and h

$\varepsilon \downarrow n \rightarrow$	2^3	2^4	2^5	2^6	2^7	2^8
Example-1						
2^{-3}	7.00e-003	1.70e003	4.39e00-4	1.04e004	4.66e-005	5.18e005
2^{-4}	1.00e-002	2.22e-003	5.51e-004	1.34e-004	4.04e-005	4.97e-005
2^{-5}	1.29e-002	3.00e-003	6.57e-004	1.60e-004	3.82e-005	4.58e-005
2^{-7}	1.48e-002	4.70e-003	1.10e-003	2.08e-004	4.81e-005	2.69e-005
2^{-10}	1.39e-002	5.10e-003	1.50e-003	4.07e-004	1.16e-004	3.56e-005
Example-2						
2^{-3}	3.40e-003	9.05e004	2.28e00-4	5.60e005	2.40e-005	2.61e005
2^{-4}	4.30e-003	9.94e-004	2.50e-004	6.18e-005	2.14e-005	2.51e-005
2^{-5}	5.40e-003	1.20e-003	2.72e-004	6.71e-005	1.65e-005	2.33e-005
2^{-7}	5.30e-003	1.80e-003	4.10e-004	7.68e-005	1.83e-005	1.43e-005
2^{-10}	4.30e-003	1.70e-003	5.60e-004	1.56e-004	4.69e-005	1.54e-005

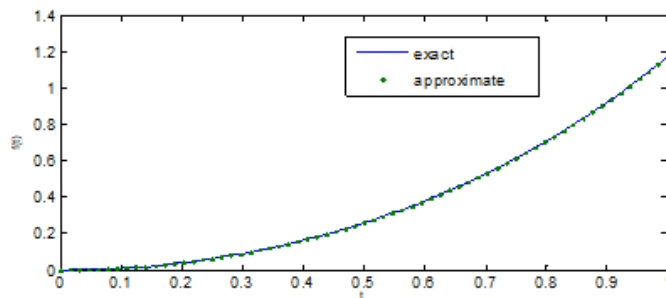
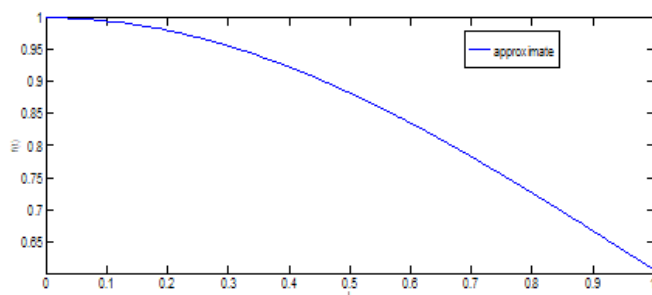
0, $y(1) = e^{-1/2}$, whose exact solution is not known. The numerical results are shown in Table 2 different values of $\varepsilon = 10^{-3}$, 10^{-4} with $h = 0.01$.

6. DISCUSSIONS AND CONCLUSION

In this paper, We used non-polynomial spline with variable mesh to establish a numerical scheme for the solution of boundary value problem with singularity. The discrete equation of the problem is developed based on the condition of the class C^1 of non-polynomial spline at the inner nodes and it is not valid for singularity. At singularity $t = 0$, the problem is modified in order to have a three term relationship. The method's tridiagonal scheme is analyzed using the well-known discrete imbedding invariant algorithm. The maximum absolute errors are tabulated for the existing regular examples selected from the literature to demonstrate the method. From the results it is found that for a smaller value of ε , the present approach estimates the exact solution very well.

FIGURE 1. Example 1 with $n = 64$ and $\epsilon = 2^{-10}$.TABLE 2: Numerical results of Example-3 for $h = 0.01$

t	Numerical solution for $\varepsilon = 10^{-3}$	Numerical solution for $\varepsilon = 10^{-4}$
0	1.000000000000000	1.000000000000000
0.01	0.999772228361881	0.999946592135979
0.02	0.999444580200142	0.999793200434538
0.03	0.999465553276671	0.999548547716945
0.04	0.998774128867018	0.999192091597038
0.05	0.998726799464504	0.998750572964155
0.10	0.994482617560363	0.995001488280677
0.20	0.979721373054119	0.980185336926093
0.30	0.955680312797009	0.955983757211175
0.40	0.922921941301340	0.923103753381187
0.50	0.882371275944413	0.882486523926594
0.90	0.666971776842780	0.666976221569099
0.95	0.636830569944083	0.636845242296312
0.96	0.630778222548422	0.630778723888910
0.97	0.624720810558951	0.624734130487353
0.98	0.618659586650185	0.618659622904657
0.99	0.612595792839100	0.612608201255755
1.00	0.606530659712633	0.606530659712633

FIGURE 2. Example 2 with $n = 64$ and $\epsilon = 2^{-10}$.FIGURE 3. Example 3 with $n = 100$ and $\epsilon = 10^{-3}$.

REFERENCES

- [1] Bawa, R.K., Spline based computational technique for linear singularly perturbed boundary value problems, Applied Mathematics and Computation, Vol. 167, Number 1, 225-236 (2005).
- [2] Bender, C.M., Orszag, S.A., Advanced mathematical methods for scientists and engineers, Mc. Graw-Hill, New York, (1978).
- [3] Henrici, P., Discrete Variable Methods in Ordinary differential equations, Wiley, New York, (1962).
- [4] Kadalbajoo, M.K., Aggarwal, V.K., Fitted mesh B-spline method for solving a class of singular singularly perturbed boundary value problems, International Journal of Computer Mathematic, Vol.82, Number 1, 67-76 (2005).
- [5] Kadalbajoo, M. K., Patidar, K.C., Numerical solution of singularly perturbed two-point boundary value problems by spline in compression, International Journal of Computer Mathematics, Vol. 77, Number 1, 263-283 (2001).
- [6] Kadalbajoo, M.K., Reddy, Y.N., Numerical solution of singularly perturbation problems via deviating arguments, Applied Mathematics and Computation, Vol. 21, Number 3, 221-232 (1987).

- [7] Mohanty, R.K., Navnit Jha and Evans, D.J., Spline in compression method for the numerical solution of singularly perturbed two-point singular boundary value problems, *International Journal of Computational Mathematics*, Vol.81, Number 5, 615-627 (2004).
- [8] Mohanty, R.K., Evans, D.J. and Aurora, U., Convergent spline in tension methods for singularly perturbed two-point singular boundary value problems, *International Journal of Computer Mathematics*, Vol. 82, Number 1, 55-66 (2005).
- [9] Mohanty, R.K., Urvashi Aurora, A family of non-uniform mesh tension spline methods for singularly perturbed two-point singular boundary value problems with significant first derivatives, *Applied Mathematics and Computation*, Vol 172, Number 531-544 (2006).
- [10] O'Malley, R.E., *Introduction to Singular Perturbations*, Academic Press, New York, (1974).
- [11] Phaneendra, K., Siva Prasad, E. and Kumara Swamy, D., Fourth order method for singularly perturbed singular boundary problems using non-polynomial spline, *Maejo International Journal of Science and technology*, Vol. 12, Number 1, 1-10 (2008).
- [12] Rashidinia, J., Ghasemi, M., Cubic spline solution of singularly perturbed boundary value problems with significant first derivatives, *Applied Mathematics and Computation*, Vol. 190, Number 2, 1762-1766 (2007).
- [13] Varga, R.S., *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, (1962).
- [14] Young, D.M., *Iterative Solution of Large Linear Systems*, Academic Press, New York (1971).