Journal of Dynamical Systems & Geometric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 19(1) June 2021, pp. 51–56
DOI : 10.1080/1726037X.2021.1909218

ROLE OF DIGITAL ROOT IN NUMBER THEORY

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(Received: 9 May 2020, Accepted: 10 December 2020)

Abstract. In Vedic mathematics, digital root of a number is known as Beejank. Before the development of computer devices, the idea was used to check the results of mathematical operations.

In this paper, we highlight the properties of digital root of a number and some results of digital roots. There are many interesting properties of digital root of a number. Perfect square number and a perfect cube number have specific digital roots.

In modern mathematics, digital roots make partition of the set of natural numbers. Nowadays digital root of a number has many applications in generation of random sequence of numbers.

AMS Classification: 11A67.

Keywords: Digital root; Square; Cube; Partition.

1. Introduction

The Beejank or digital root of a natural number n is obtained by computing the sum of its digits, then computing the sum of the digits of the resulting number, and so on, till a single digit number is obtained denoted by B(n).

Digital root, also known as seed number, is a value obtained from a positive integer by adding the digits of that number repeatedly until a single digit is obtained. Digital root of a natural root is the digital root of the sum of the digits of the number.

For example:

1) B(83) = B(8 + 3) = B(11) = B(1+1) = B(2) = 2. So, B(83) =2

2) B(5138) = B(5+1+3+8) = B(17) = B(1+7) = B(8) = 8

3) B(183573) = B(1+8+3+5+7+3) = B(27) = B(2+7)= B(9) = 9

In Vedic mathematics, digital root of a natural number is known as Beejank, Beejank means seed number. Mathematical operations such as addition, subtraction, multiplication and division are verified using Beejank. Before the development of computer devices, the idea was used to check the results of mathematical operations. In this paper, we will discuss the properties of Beejank of number.

2. Some properties of digital root

2.1. **Property.** If $n \in N = \{1, 2, 3, \ldots\}$, $1 \leq n \leq 9$, then $B(n) = n$ and for any $n \in N$, then $B(n) \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
This is true because digital root is a single digit number.

2.2. **Property.** If a natural number is not divisible by 9 then the digital root is the remainder when it is divided by 9 and if the natural number is a multiple of 9, then its digital root is 9.
In other way, "The difference between any number n and $B(n)$ is a multiple of 9" i.e., $n - B(n) = 9k$, where k is non-negative number.

Proof: For any number $n = a_0 + 10a_1 + 100a_2 + 1000a_3 + \ldots$, digital root is obtained by sum of digits $B(n) = B(a_0 + a_1 + a_2 + a_3 + \ldots)$.
Then, $n - (a_0 + a_1 + a_2 + a_3 + \ldots) = (10 - 1)a_1 + (100 - 1)a_2 + (1000 - 1)a_3 + \ldots$, each term is divisible by 9.
Hence it follows that $n - B(n)$ is divisible by 9.

Note: 1. Above result is true for $(a_0 + a_1 + a_2 + a_3 + \ldots)$ in place of n.
2. If $n \in N$ is divided by 9 and r is the (least non-negative) remainder, i.e. by Division Algorithm, there exist unique non-negative integers q, r such that $n = 9q+r$ where $0 \leq r < 9$, then

$$B(n) = \begin{cases} r, \ if \ r \neq 0 \\ 9, \ if \ r = 0 \end{cases}.$$

Now, for any $n \in N, k \in N \cup \{0\}$; both n and $n + 9k$ have same remainder, so $B(n) = B(n + 9k) = B(B(n))$.

The above findings can be used to get simple proof of the following property.

2.3. **Property.** For any two natural numbers m and n, following relations are true $B(m + n) = B(B(m) + B(n))$ and $B(m \times n) = B(B(m) \times B(n))$.

From the Property 2.2, if number n is not divisible by 9, then B(n) is equal to remainder. When number n is divisible by 9 or if number n is multiple of 9, then remainder is 9, which clearly states that $B(m + n) = B(B(m) + B(n))$.

Similarly, we can extend this idea for multiplication $B(m \times n) = B(B(m) \times B(n))$.

Proof: Let $m, n \in N$ any.

Case (i): If $9|m$ then $B(m) = 9$.

Then $B(m + n) = B(n) = B(B(n)) = B(9 + B(n)) = B(B(m) + B(n))$.

And $B(mn) = 9 = B(9 \cdot B(n)) = B(B(m) \cdot B(n))$.

Similarly, the result follows for $9|n$.

Case (ii): Consider $9 \nmid m, 9 \nmid n$.

Then by Division Algorithm, we have non-negative integers p, r, q, s such that $m = 9p + r, n = 9q + s$ where $r, s \in \{1, 2, 3, 4, 5, 6, 7, 8\}$,

i.e. $B(m) = r, \ B(n) = s$ and

$m + n = 9(p + q) + r + s, \ mn = 9(9pq + r + s) + rs$, which give

$B(m + n) = B(r + s) = B(B(m) + B(n))$ and $B(mn) = B(rs) = B(B(m) \cdot B(n))$.

2.4. **Property.** Any number multiple of 3 has digital root 3, 6 or 9.

If the number is multiple of 3, then the sum of digits of the number is always divisible by 3. Continuing the same process for bigger number, single digit number which is divisible by 3 are 3, 6 or 9. So the digital roots of any number which is multiple of 3 are 3, 6 or 9.

2.5. **Property.** The digital root of a triangular number is one of the 1, 3, 6 or 9.

Proof: Triangular numbers are the number of objects arranged in an equilateral triangle. Triangular numbers or Triangular number sequence is given by 1, 3, 6, 10, 15, 21, 28, ...

Any triangular number in this series is obtained by the formula $n = \frac{m(m+1)}{2}, m \in N$.

If we divide m by 6, we have six different forms of m, which are $6k, 6k+1, 6k+2, 6k+3, 6k+4, 6k+5$.

If $m = 6k, 6k+2, 6k+3, 6k+5$, then one of the factor in $m(m+1)$, which is m or $m+1$ is always a multiple of 3. In this case, the triangular numbers are multiple of 3 and hence, triangular number has digital root 3, 6 or 9.

If $m = 6k+1$, then $n = \frac{(6k+1)(6k+2)}{2} = (6k+1)(3k+1) = 9(2k^2+k)+1$, which when divided by 9, the remainder is always 1. Hence, in this case, the digital root of a triangular number is 1.

If $m = 6k+4$, then
$n = \frac{(6k+4)(6k+5)}{2} = (3k+2)(6k+5) = 18k^2+27k+10 = 9(2k^2+3k+1)+1.$
Hence, in this case also, digital root of a triangular number is 1.

Thus, the digital roots of a triangular number are one of the 1, 3, 6 or 9.

Note: For the triangular numbers which are multiple of 3 have digital root 3, 6 or 9 and the triangular numbers which are not multiple of 3 have digit root 1.

2.6. **Property.** If number n is a perfect square, then $B(n)$ is one of the 1, 4, 7 or 9.

Given number n is a perfect square, so that $n = m^2$. If we divide m by 9, we obtain 9 forms of m depending on the remainder, which are $9k, 9k \pm 1, 9k \pm 2, 9k \pm 3, 9k \pm 4$. Squaring these expressions and obtaining the remainder after division by 9, we find that the remainder is one of the 1, 4, 7 or 9.
Hence the digital root of a perfect square is one of the 1, 4, 7, 9.

Note: The unit place of a perfect square number is always 1, 4, 9, 6, 5 or 0. If the unit place of any number is 2, 3, 7 and 8, it is never a perfect square.

Similarly, if the digital root of a natural number is one of 2, 3, 5, 6 and 8, then is also never a perfect square.

2.7. **Property.** If number n is perfect cube, then $B(n)$ is one of the 1, 8 or 9.

It can be proved by above method of property 6.

2.8. **Property.** If number n is perfect sixth power, then $B(n)$ is one of the 1 and 9.

If number n is perfect sixth power, then it is a perfect square as well as a perfect cube. Hence $B(n)$ belongs to $\{1,4,7,9\}$ and $\{1,8,9\}$, and hence in the intersection of these two sets, i.e., $B(n)$ belongs to $\{1,9\}$.

2.9. **Property.** If N is the set of natural numbers and $D = \{1,2,3,4,5,6,7,8,9\}$, then the function $B(n) : N \to D$ is well defined.

Let $a, b \in N$ and $a = b$, then digits of a and b are same, and the sum of digits of a is equal to the sum of digits of b. Hence $B(a) = B(b)$. Therefore, $B(n) : N \to D$ is well defined.

3. Equivalent numbers

Two numbers a and b are said to be equivalent numbers if and only if both numbers have same digital root.

i.e., $a \sim b \Leftrightarrow B(a) = B(b)$.

3.1. **Property.** Digital roots partition the set of positive integers.

OR, The relation digital root is an equivalence relation.

Proof: We define this relation as $aRb \Leftrightarrow a \sim b \Leftrightarrow B(a) = B(b)$.

R is reflexive: $B(a) = B(a) \Leftrightarrow a \sim a \Leftrightarrow aRa$.

R is symmetric: $aRb \Leftrightarrow a \sim b \Leftrightarrow B(a) = B(b) \Leftrightarrow B(b) = B(a) \Leftrightarrow b \sim a \Leftrightarrow bRa$.

R is transitive: aRb and $bRc \implies aRc$.

$aRb \implies B(a) = B(b)$ and $bRc \implies B(b) = B(c)$. Hence, $B(a) = B(c) \implies aRc$.

Therefore, the relation digital root is an equivalence relation and it partition the set of non-negative integers.

4. Inverse of digital root

Inverse image of a digital root $r \in \{1,2,3,4,5,6,7,8,9\}$ under B is

$B^{-1}(r) = r + 9k;\ k = 0,1,2,\ldots.$

For any $n \in N; B^{-1}(B(n)) = \{B(n) + 9k : k = 0,1,2,3,\ldots\}$.

Example: $B^{-1}(4) = 4 + 9k = \{4,13,22,31,\ldots\}$.

5. Pseudo-random generators

There are several random computational generators, which generate numbers in a pseudo-random chain. Among them, the congruent generator and the Fibonacci sequence, which are two bad generators and are used in composition to the digital root for the creation of a generator better than the originals.

6. Acknowledgment

Authors are grateful to Late Dr. Anant Vyawahare for constant encouragement and motivation throughout his life.

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