

ON NUMERICAL INVARIANT OF GRAPH

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ABSTRACT. Let $G = (V(G), E(G))$ be a finite graph with n vertices, where $V(G) = \{v_1, \dots, v_n\}$ denote vertex set of G and $E(G)$ denote an edge set G . In this paper, we have introduced the graph $G_{p,q}$ for G . The motivation for introducing the graph $G_{p,q}$ are as follows:

Many mathematicians studied the properties of graph G by using boundary operator and co-boundary operator which consider only the relation between vertices and an edges of a graph G . There is no place for complete subgraph (whose vertices greater than 2) of a graph G in boundary operator and co-boundary operator of G .

Hence to study the relation between complete subgraph of G and also to study the properties of G , we have introduced an edge-boundary operator and an edge-co-boundary operator on $G_{p,q}$.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a finite graph with n vertices, where $V(G) = \{v_1, \dots, v_n\}$ denote vertex set of G and $E(G)$ denote an edge set G . The graph

G and H are said to be isomorphic if there exist a one-to-one correspondence which preserves adjacency[1]. An invariants of a graph G is a number associated with G which has same value for any graph isomorphic to G . One can define a graph upto isomorphism by using complete set of invariants. There are lot of invariants of a graph. Invariants of a graph include *Betti Number, Algebraic connectivity, Graph genus, Wiener index* for more invariants of a graph see[4]. We refer the readers to [1, 6, 5] for basic definitions concerning graphs.

Till date no complete set of invariants of a graph is known[1]. This statement motivate us to study the invariants of a graph. In this paper, we have studied the boundary operator, co-boundary operator with respect to complete subgraph of G . In[2], *P. Slepian*, studied the circuit theory by using linear algebra and network theory. In [3], *A. Nerode et.al* studied an Algebraic proof of Kirchhoffs Network theorem. To prove the Kirchoff's theorem, he used the term boundary operator and co-boundary operator. In [7], *Younng-Jin Kim and Woong Kook* studied the Harmonic cycle for a graph. In [8], *Oswald Veblen*, studied the numerical invariant of a simplicial complex. In his study, he consider linear graph. Many mathematicians studied the properties of graph G by using boundary operator and co-boundary operator which consider only the relation between vertices and an edges of a graph G . There is no place for complete subgraph (whose vertices greater than 2) of a graph G in boundary operator and co-boundary operator of G .

Hence to study the relation between complete subgraph of G and also to study the properties of G , in section (3), we have introduced the graph $G_{p,q}$ for G . We have introduced an edge-boundary operator and an edge-co-boundary operator on $G_{p,q}$.

In section2, we have studied the theory of α -chain and 1-chain associated with the graph G . In section 3, we have define a graph $G_{p,q}$ by using G , then we have studied the boundary operator and co-boundary operator for $G_{p,q}$. We called boundary operator of $G_{p,q}$ as an edge-boundary operator and the co-boundary operator of $G_{p,q}$ as an edge-co-boundary operator. In section4, we have introduced the optimal drawing of a graph G on a manifold. In section 5, we have introduced the theory of α -chain and 1-chain for an optimal drawing associated with a graph G on a manifold.

2. 0-CHAIN AND 1-CHAIN

The 0-chain of G is a formal linear combination $\sum \alpha_i v_i$ of vertices of G . The 1-chain of G is a formal linear combination $\sum \alpha_i x_i$ of edges of G .

Lemma 2.1. [1] $CV_0(G) = \{\sum \alpha_i v_i : \alpha_i \in \mathbb{F}_2 \text{ and } v_i \in V(G)\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $u = \sum \alpha_i v_i$ and $v = \sum \beta_i v_i$ then $u + v = \sum (\alpha_i + \beta_i) v_i$.
and scalar multiplication $\gamma u = \sum \gamma \alpha_i v_i$, where $\gamma \in \mathbb{F}_2$.

Lemma 2.2. [1] $CV_1(G) = \{\sum \alpha_i x_i : \alpha_i \in \mathbb{F}_2 \text{ and } x_i \in E(G)\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $x = \sum \alpha_i x_i$ and $y = \sum \beta_i x_i$ then $x + y = \sum (\alpha_i + \beta_i) x_i$.
and scalar multiplication $\gamma x = \sum \gamma \alpha_i x_i$, where $\gamma \in \mathbb{F}_2$.

The boundary operator ∂ send 1-chain to 0-chain by using following rule:

- (a) ∂ is linear ;
- (b) If $x = uv$ then $\partial x = u + v$

.

The co-boundary operator δ send 0-chain to 1-chain by using following rule:

- (a) δ is linear ;
- (b) If $\delta v = \sum \varepsilon_i x_i$, where $\varepsilon_i = 1$ whenever x_i is incident with v .

Illustration of boundary and co-boundary operator are given in following example.

Example 2.3. Let $G = (V(G), E(G))$ be a graph, where

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and $E(G) = \{x_1 = \{v_2, v_3\}, x_2 = \{v_1, v_2\}, x_3 = \{v_1, v_3\}, x_4 = \{v_2, v_4\}, x_5 = \{v_2, v_5\}, x_6 = \{v_3, v_5\}, x_7 = \{v_3, v_6\}, x_8 = \{v_5, v_6\}, x_9 = \{v_4, v_5\}, x_{10} = \{v_3, v_4\}\}$.

The 1-chain $\sigma_1 = x_1 + x_2 + x_5 + x_7 + x_8$ has boundary

$$\partial \sigma_1 = (v_2 + v_3) + (v_1 + v_2) + (v_2 + v_5) + (v_3 + v_6) + (v_5 + v_6).$$

$$\partial \sigma_1 = v_1 + v_2.$$

The 0-chain $\sigma_0 = v_1 + v_6 + v_4$ has co-boundary

$$\delta \sigma_0 = (x_2 + x_3) + (x_7 + x_8) + (x_4 + x_9).$$

Definition 2.4. Let $CV_0(G)$ and $CV_1(G)$ be vector spaces over a field $\mathbb{F}_2$ associated with graph G . Let $\partial : CV_1(G) \rightarrow CV_0(G)$ be a boundary operator. The set all vectors $\alpha \in CV_1(G)$ such that $\partial(\alpha) = \theta'$, where θ' is the zero element in $CV_0(G)$, is said to be the kernel of ∂ and it is denoted by $Ker\partial$.
Therefore $Ker\partial = \{\alpha \in CV_1(G) : \partial(\alpha) = \theta'\}$.

Lemma 2.5. $Ker\partial$ is a subspace of $CV_1(G)$.

Proof. $Ker\partial = \{\alpha \in CV_1(G) : \partial(\alpha) = \theta'\}$. $\theta \in Ker\partial$ since $\partial(\theta) = \theta'$. Therefore $Ker\partial$ is non-empty.

Case-I. $Ker\partial = \{\theta\}$. Then $Ker\partial$ is a subspace of CV_1 .

Case-II. $Ker\partial \neq \{\theta\}$. Let $\alpha \in Ker\partial$. Then $\partial(\alpha) = \theta'$. Let $c \in \mathbb{F}_2$. Then $\partial(c\alpha) = c\partial(\alpha) = c\theta' = \theta'$. This implies $c\alpha \in Ker\partial$

Let $\alpha, \beta \in Ker\partial$. Then $\partial(\alpha) = \theta', \partial(\beta) = \theta'$.

$\partial(\alpha + \beta) = \partial(\alpha) + \partial(\beta) = \theta' + \theta' = \theta'$.

This implies $\alpha + \beta \in Ker\partial$.

Thus $\alpha, \beta \in Ker\partial \Rightarrow \alpha + \beta \in Ker\partial$ and $\alpha \in Ker\partial \Rightarrow c\alpha \in Ker\partial$ for all $c \in \mathbb{F}_2$. Hence $Ker\partial$ is a subspace of CV_1 .

□

The kernel of the boundary operator ∂ is also called as cycle space of the graph G .

Definition 2.6. Let $CV_0(G)$ and $CV_1(G)$ be vector spaces over a field $\mathbb{F}_2$ associated with graph G . Let $\partial : CV_1(G) \rightarrow CV_0(G)$ be a boundary operator. The set of all vectors $\partial(\alpha) \in CV_0(G)$ such that $\alpha \in CV_1(G)$ is said to be the image of ∂ and it is denoted by $Im\partial$.

Therefore $Im\partial = \{\partial(\alpha) : \alpha \in CV_1(G)\}$.

Lemma 2.7. $Im\partial$ is a subspace of $CV_0(G)$.

The image of the boundary operator ∂ is also called as range space of the graph G . The dimension of $Ker\partial$ is denoted by $dimKer\partial$. The dimension of $Im\partial$ is denoted by $dimIm\partial$. Let G_1 and G_2 be two graphs. Let $CV_0(G_1)$ and $CV_1(G_1)$ vector spaces associated with G_1 . Let $CV_0(G_2)$ and $CV_1(G_2)$ vector spaces associated with G_2 . Suppose G_1 is isomorphic to G_2 then there is a one-to-one correspondence

between their vertices and edges. Hence the vector spaces $CV_o(G_1)$ and $CV_1(G_1)$ are isomorphic to $CV_o(G_2)$ and $CV_1(G_2)$ respectively.

According to the definition of invariant of a graph, dimension of $CV_0(G)$ and dimension of $CV_1(G)$ are the numerical invariant of a graph G . Also $\dim Ker \partial$ and $\dim Im \partial$ are the numerical invariants of a graph G , where ∂ be a boundary operator associated with G .

Theorem 2.8. *Let $CV_0(G)$ and $CV_1(G)$ be vector spaces over a field $\mathbb{F}_2$ associated with a graph G . If $\partial : CV_1(G) \rightarrow CV_0(G)$ be a boundary operator associated with a graph G then*

$$(1) \quad \dim Ker \partial + \dim Im \partial = \dim CV_1(G)$$

Proof. Case-I. $Ker \partial = CV_1(G)$. Then $Im \partial$ consists of θ' only, where θ' is null element in $CV_0(G)$. Therefore $\dim Im \partial = 0$. Thus $\dim Ker \partial + \dim Im \partial = \dim CV_1(G)$.

Thus $\dim Ker \partial + \dim Im \partial = |E(G)|$.

We know that if the graph contain an edge $x = uv$ then $\partial x = u + v \neq \theta'$.

Hence case-I is not possible.

Case-II Let $Ker \partial = \{\theta\}$ and let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of $CV_1(G)$.

Then $\{\partial(\alpha_1), \dots, \partial(\alpha_n)\}$ is a basis of $Im \partial$.

Therefore $\dim Ker \partial = 0$ and $\dim CV_1(G) = |E(G)|$, $\dim Im \partial = |E(G)|$.

Hence the theorem is holds good.

From the case-II, we can observe that $Ker \partial = \{\theta\}$ is possible only if graph G doesn't contain any cycle. When G is the tree then only case-II is possible.

Case-III. Let $Ker \partial$ be a non-trivial proper subspace of $CV_1(G)$.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be a basis of $Ker \partial$.

This basis can be extended to a basis of $CV_1(G)$ and let the extended basis of $CV_1(G)$ be $\{\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_{|E(G)|}\}$.

The $Im \partial$ is generated by the vector $\partial(\alpha_1), \partial(\alpha_2), \dots, \partial(\alpha_{|E(G)|})$.

As $\partial(\alpha_1) = \partial(\alpha_2) = \dots = \partial(\alpha_k) = \theta'$. Therefore $Im \partial$ is generated by the vectors $\partial(\alpha_{k+1}), \partial(\alpha_{k+2}), \dots, \partial(\alpha_{|E(G)|})$.

Let us consider the relation

$$c_{k+1} \partial(\alpha_{k+1}) + c_{k+2} \partial(\alpha_{k+2}) + \dots, c_{|E(G)|} \partial(\alpha_{|E(G)|}) = \theta' \text{ for some scalar } c_i \in$$

$\mathbb{F}_2$. Since ∂ is a linear, this implies that

$$\partial(c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \cdots, c_{|E(G)|}\alpha_{|E(G)|}) = \theta'.$$

This shows that $c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \cdots, c_{|E(G)|}\alpha_{|E(G)|}$ is in $\text{Ker}\partial$. Since $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is an assume basis for

$\text{Ker}\partial$, $c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \cdots, c_{|E(G)|}\alpha_{|E(G)|} = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_k\alpha_k$ for some $b_i \in \mathbb{F}_2$. Therefore $b_1\alpha_1 + b_2\alpha_2 + \cdots + b_k\alpha_k - (c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \cdots, c_{|E(G)|}\alpha_{|E(G)|}) = \theta$.

Therefore linear independence of $\{\alpha_1, \alpha_2, \cdots, \alpha_{|E(G)|}\}$ requires $b_1 = b_2 = \cdots = b_k = c_{k+1} = \cdots = c_{|E(G)|} = 0$. The fact that $c_{k+1} = \cdots = c_{|E(G)|} = 0$ prove that the set $\{\partial(\alpha_{k+1}), \partial(\alpha_{k+2}), \cdots, \partial(\alpha_{|E(G)|})\}$ is linearly independent and so this is a basis for $\text{Im}\partial$.

$\dim \text{Im}\partial = |E(G)| - k$ and $\dim \text{Ker}\partial + \dim \text{Im}\partial = k + (|E(G)| - k) = |E(G)|$. Thus

$$(2) \quad \dim \text{Ker}\partial + \dim \text{Im}\partial = \dim CV_1(G)$$

□

Corollary 2.9. *Let G be a graph and $\dim \text{Im}\partial = |E(G)| - 1$. Then G contain exactly one cycle.*

Corollary 2.10. *Let G be a graph and $\dim \text{Im}\partial = |E(G)| - k$. Then G may contain cycles greater than or equal to k .*

Example 2.11. *Let $G = (V(G), E(G))$ be a graph, where $G = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{x_1 = \{v_1, v_2\}, x_2 = \{v_2, v_3\}, x_3 = \{v_3, v_4\}, x_4 = \{v_1, v_4\}, x_5 = \{v_4, v_2\}\}$. Let $CV_0(G)$ and $CV_1(G)$ be vector spaces over a field $\mathbb{F}_2$ associated with a graph G . Let $\partial : CV_1(G) \rightarrow CV_0(G)$ be a boundary operator associated with a graph G . Then one can observe that G has three cycle i.e $x_1 + x_4 + x_5$ and $x_2 + x_3 + x_5$ and $x_1 + x_2 + x_3 + x_4$.*

We can observe that one cycle is express as formal linear combination of other two cycle i.e $x_1 + x_2 + x_3 + x_4 = x_1 + x_4 + x_5 + x_2 + x_3 + x_5$. Hence basis for $\text{Ker}\partial$ is the set $\{x_2 + x_3 + x_5, x_1 + x_4 + x_5\}$. Hence $\dim \text{Ker}\partial = 2$.

From above, we conclude that number of cycles in the graph are greater than or equal to $\dim \text{Ker}\partial$.

If $\alpha = \sum_i \alpha_i x_i$ be a cycle then number of edges in the cycle α is the length of cycle. In example(2.11), the length of cycle $x_1 + x_2 + x_3 + x_4$ is 4.

Definition 2.12. Let $CV_0(G)$ and $CV_1(G)$ be vector spaces over a field $\mathbb{F}_2$ associated with graph G . Let $\partial : CV_1(G) \rightarrow CV_0(G)$ be a boundary operator. Let α, β be two cycle of G . Then $\gamma = \alpha + \beta$ is also a cycle of G . If length of cycle γ is strictly less than the sum of length of cycle α and β then we call the cycle γ as RP-cycle of G .

Theorem (5.7), gives the information about minimum number of RP-cycle of G but it will not give exact number of RP-cycle of G .

From above discussion in section 2, one can observe that boundary operator and co-boundary operator studied the relation between vertices and edges of a graph G . There is no place for complete subgraph (whose vertices greater than 2) in boundary operator and co-boundary operator of G . Hence to study the relation between complete subgraph of G and also to study the properties of G , in section (3), we have introduced the graph $G_{p,q}$ for G .

3. EDGE- (p, q) GRAPH OF A GRAPH G

Define for each p , $G_{p,q} = (V(G_{p,q}), E(G_{p,q}))$ as an edge- (p, q) graph of a graph G , where $V(G_{p,q}) = \{K_p : K_p \text{ denote a complete subgraph of } G \text{ with } p \text{ vertices}\}$ and $E(G_{p,q}) = \{x = \{K_r, K_s\} : K_r, K_s \in V(G_{p,q}) \text{ and intersection of } K_r \text{ and } K_s \text{ contain complete subgraph with vertices } q\}$.

The 0-chain of $G_{p,q}$ is a formal linear combination $\sum \alpha_i v_i$ of vertices of $G_{p,q}$. The 1-chain of $G_{p,q}$ is a formal linear combination $\sum \alpha_i x_i$ of edges of $G_{p,q}$.

Lemma 3.1. $CV_0(G_{p,q}) = \{\sum \alpha_i v_i : \alpha_i \in \mathbb{F}_2 \text{ and } v_i \in V(G_{p,q})\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $u = \sum \alpha_i v_i$ and $v = \sum \beta_i v_i$ then $u + v = \sum (\alpha_i + \beta_i) v_i$.
and scalar multiplication $\gamma u = \sum \gamma \alpha_i v_i$, where $\gamma \in \mathbb{F}_2$.

Lemma 3.2. $CV_1(G_{p,q}) = \{\sum \alpha_i x_i : \alpha_i \in \mathbb{F}_2 \text{ and } x_i \in E(G_{p,q})\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $x = \sum \alpha_i x_i$ and $y = \sum \beta_i x_i$ then $x + y = \sum (\alpha_i + \beta_i) x_i$.
and scalar multiplication $\gamma x = \sum \gamma \alpha_i x_i$, where $\gamma \in \mathbb{F}_2$.

An edge-boundary operator ∂ send 1-chain to 0-chain by using following rule:

- (a) ∂ is linear ;
- (b) If $x = uv$ then $\partial x = u + v$

An edge-coboundary operator δ send 0-chain to 1-chain by using following rule:

- (a) δ is linear ;
- (b) If $\delta v = \sum \varepsilon_i x_i$, where $\varepsilon_i = 1$ whenever x_i is incident with v .

Definition 3.3. Let $CV_0(G_{p,q})$ and $CV_1(G_{p,q})$ be vector spaces over a field $\mathbb{F}_2$ associated with graph $G_{p,q}$. Let $\partial : CV_1(G_{p,q}) \rightarrow CV_0(G_{p,q})$ be a boundary operator. The set all vectors $\alpha \in CV_1(G_{p,q})$ such that $\partial(\alpha) = \theta'$, where θ' is the zero element in $CV_0(G_{p,q})$, is said to be the kernel of ∂ and it is denoted by $Ker\partial$. Therefore $Ker\partial = \{\alpha \in CV_1(G_{p,q}) : \partial(\alpha) = \theta'\}$.

Lemma 3.4. $Ker\partial$ is a subspace of $CV_1(G_{p,q})$.

Proof. $Ker\partial = \{\alpha \in CV_1(G_{p,q}) : \partial(\alpha) = \theta'\}$. $\theta \in Ker\partial$ since $\partial(\theta) = \theta'$. Therefore $Ker\partial$ is non-empty.

Case-I. $Ker\partial = \{\theta\}$. Then $Ker\partial$ is a subspace of CV_1 .

Case-II. $Ker\partial \neq \{\theta\}$. Let $\alpha \in Ker\partial$. Then $\partial(\alpha) = \theta'$. Let $c \in \mathbb{F}_2$. Then $\partial(c\alpha) = c\partial(\alpha) = c\theta' = \theta'$. This implies $c\alpha \in Ker\partial$

Let $\alpha, \beta \in Ker\partial$. Then $\partial(\alpha) = \theta', \partial(\beta) = \theta'$.

$\partial(\alpha + \beta) = \partial(\alpha) + \partial(\beta) = \theta' + \theta' = \theta'$.

This implies $\alpha + \beta \in Ker\partial$.

Thus $\alpha, \beta \in Ker\partial \Rightarrow \alpha + \beta \in Ker\partial$ and $\alpha \in Ker\partial \Rightarrow c\alpha \in Ker\partial$ for all $c \in \mathbb{F}_2$. Hence $Ker\partial$ is a subspace of CV_1 .

□

The kernel of the boundary operator ∂ is also called as cycle space of the graph of $G_{p,q}$.

Definition 3.5. Let $CV_0(G_{p,q})$ and $CV_1(G_{p,q})$ be vector spaces over a field $\mathbb{F}_2$ associated with graph $G_{p,q}$. Let $\partial : CV_1(G_{p,q}) \rightarrow CV_0(G_{p,q})$ be a boundary operator. The set of all vectors $\partial(\alpha) \in CV_0(G_{p,q})$ such that $\alpha \in CV_1(G_{p,q})$ is said to be the image of ∂ and it is denoted by $Im\partial$.

Therefore $Im\partial = \{\partial(\alpha) : \alpha \in CV_1(G_{p,q})\}$.

Lemma 3.6. $Im\partial$ is a subspace of $CV_0(G_{p,q})$.

The image of the boundary operator ∂ is also called as range space of the graph of $G_{p,q}$. The dimension of $Ker\partial$ is denoted by $dimKer\partial$. The dimension of $Im\partial$ is denoted by $dimIm\partial$. Let G_1 and G_2 be two graphs. Let $G_{p,q}^1$ and $G_{p,q}^2$ are the graph associated with G_1 and G_2 respectively. Let $CV_o(G_{p,q}^1)$ and $CV_1(G_{p,q}^1)$ vector spaces associated with $G_{p,q}^1$. Let $CV_o(G_{p,q}^2)$ and $CV_1(G_{p,q}^2)$ vector spaces associated with $G_{p,q}^2$. Suppose G_1 is isomorphic to G_2 then there is a one-to-one correspondence between their vertices and edges of $G_{p,q}^1$ and $G_{p,q}^2$. Hence the vector spaces $CV_o(G_{p,q}^1)$ and $CV_1(G_{p,q}^1)$ are isomorphic to $CV_o(G_{p,q}^2)$ and $CV_1(G_{p,q}^2)$ respectively.

According to the definition of invariant of a graph, dimension of $CV_0(G_{p,q})$ and dimension of $CV_1(G_{p,q})$ are the numerical invariant of a graph $G_{p,q}$. Hence it is also an invariant of a graph G . Also $dimKer\partial$ and $dimIm\partial$ are the numerical invariants of a graph $G_{p,q}$, where ∂ be a boundary operator associated with $G_{p,q}$. Hence $dimKer\partial$ and $dimIm\partial$ are the numerical invariants of a graph G ,

Theorem 3.7. *Let $G_{p,q}$ be a graph associated with G . Let $CV_0(G_{p,q})$ and $CV_1(G_{p,q})$ be vector spaces over a field $\mathbb{F}_2$ associated with a graph $G_{p,q}$. If $\partial : CV_1(G_{p,q}) \rightarrow CV_0(G_{p,q})$ be a boundary operator associated with a graph $G_{p,q}$ then*

$$(3) \quad dimKer\partial + dimIm\partial = dimCV_1(G_{p,q})$$

Proof. Case-I. $Ker\partial = CV_1(G_{p,q})$. Then $Im\partial$ consists of θ' only, where θ' is null element in $CV_0(G_{p,q})$. Therefore $dimIm\partial = 0$. Thus $dimKer\partial + dimIm\partial = dimCV_1(G_{p,q})$.

Thus $dimKer\partial + dimIm\partial = |E(G_{p,q})|$.

We know that if the graph contain an edge $x = uv$ then $\partial x = u + v \neq \theta'$.

Hence case-I is not possible.

Case-II Let $Ker\partial = \{\theta\}$ and let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of $CV_1(G_{p,q})$.

Then $\{\partial(\alpha_1), \dots, \partial(\alpha_n)\}$ is a basis of $Im\partial$.

Therefore $dimKer\partial = 0$ and

$$dimCV_1(G_{p,q}) = |E(G_{p,q})|, \quad dimIm\partial = |E(G_{p,q})|.$$

Hence the theorem is holds good.

From the case-II, we can observe that $Ker\partial = \{\theta\}$ is possible only if graph $G_{p,q}$ doesn't contain any cycle. When $G_{p,q}$ is the tree then only case-II is possible.

Case-III. Let $Ker\partial$ be a non-trivial proper subspace of $CV_1(G_{p,q})$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be a basis of $Ker\partial$. This basis can be extended to a basis of $CV_1(G_{p,q})$ and let the extended basis of $CV_1(G_{p,q})$ be $\{\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_{|E(G_{p,q})|}\}$.

The $Im\partial$ is generated by the vector $\partial(\alpha_1), \partial(\alpha_2), \dots, \partial(\alpha_{|E(G_{p,q})|})$. As $\partial(\alpha_1) = \partial(\alpha_2) = \dots = \partial(\alpha_k) = \theta'$. Therefore $Im\partial$ is generated by the vectors $\partial(\alpha_{k+1}), \partial(\alpha_{k+2}), \dots, \partial(\alpha_{|E(G_{p,q})|})$.

Let us consider the relation

$$c_{k+1}\partial(\alpha_{k+1}) + c_{k+2}\partial(\alpha_2) + \dots, c_{|E(G_{p,q})|}\partial(\alpha_{|E(G_{p,q})|}) = \theta'$$

for some scalar $c_i \in \mathbb{F}_2$. Since ∂ is a linear, this implies that $\partial(c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \dots, c_{|E(G_{p,q})|}\alpha_{|E(G_{p,q})|}) = \theta'$. This shows that $c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \dots, c_{|E(G_{p,q})|}\alpha_{|E(G_{p,q})|}$ is in $Ker\partial$. Since $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is an assume basis for $Ker\partial$, $c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \dots, c_{|E(G_{p,q})|}\alpha_{|E(G_{p,q})|} = b_1\alpha_1 + b_2\alpha_2 + \dots + b_k\alpha_k$ for some $b_i \in \mathbb{F}_2$. Therefore $b_1\alpha_1 + b_2\alpha_2 + \dots + b_k\alpha_k - (c_{k+1}\alpha_{k+1} + c_{k+2}\alpha_2 + \dots, c_{|E(G_{p,q})|}\alpha_{|E(G_{p,q})|}) = \theta$.

Therefore linear independence of $\{\alpha_1, \alpha_2, \dots, \alpha_{|E(G_{p,q})|}\}$ requires $b_1 = b_2 = \dots = b_k = c_{k+1} = \dots = c_{|E(G_{p,q})|} = 0$. The fact that $c_{k+1} = \dots = c_{|E(G_{p,q})|} = 0$ prove that the set $\{\partial(\alpha_{k+1}), \partial(\alpha_{k+2}), \dots, \partial(\alpha_{|E(G_{p,q})|})\}$ is linearly independent and so this is a basis for $Im\partial$.

$dim Im\partial = |E(G_{p,q})| - k$ and $dim Ker\partial + dim Im\partial = k + (|E(G_{p,q})| - k) = |E(G_{p,q})|$. Thus

$$(4) \quad dim Ker\partial + dim Im\partial = dim CV_1(G_{p,q})$$

□

Corollary 3.8. *Let $G_{p,q}$ be a graph associated with G and $dim Im\partial = |E(G_{p,q})| - 1$. Then $G_{p,q}$ contain exactly one cycle.*

Corollary 3.9. *Let $G_{p,q}$ be a graph associated with G and $dim Im\partial = |E(G_{p,q})| - k$. Then $G_{p,q}$ may contain cycles greater than or equal to k .*

4. OPTIMAL DRAWING ON A MANIFOLD

In this section, we have given the theory of an optimal drawing on a manifold. Let M be a Riemannian manifold of dimension $2m$. Let the elements of M is denoted by

$(x^1, \dots, x^{2m})$, where $x^i \in \mathbb{R}$. Let $v_i \in M$ and $v_j \in M$. The function $e : [0, 1] \rightarrow M$ is called as an edge joining v_i and v_j if it satisfy following properties:

- $e(0) = v_i$ and $e(1) = v_j$;
- e must be a continuous function;
- non-self-intersecting.

and it is denoted by e or $\{v_i, v_j\}$. In [9],[10], *R.K.Guy*, studied the crossing number of a graph G on the plane. He also told in his paper that crossing number of a complete graph K_5 on the surface of sphere is one and crossing number of K_5 on torus is zero. This motivates the following question: Can we classify the surface by using crossing Number of a graph G ? This also motivates the following question, can we classify the manifold by using crossing number of graph? Many mathematician studied the crossing number of a graph G , see [11, 12, 13, 15, 19].

In this section, we have introduced an optimal drawing of a graph G on a manifold. We have studied boundary operator corresponding to an optimal drawing of a graph G on a manifold.

4.1. Definition of a optimal drawing on a manifold. Let $G = (V(G), E(G))$ be a finite undirected graph with no loops or multiple edges with a vertex set $V(G) = \{v_1, \dots, v_n\}$ and an edge set $E(G)$, where $E(G) \subseteq \{\{v_i, v_j\} : v_i, v_j \in V(G) \text{ and } v_i \neq v_j\}$.

A good drawing D_M of G on M is a mapping of G into M such that the set $V(G)$ is mapped to $|V(G)|$ of distinct points, called vertices and if $\{v_i, v_j\} \in E(G)$ then there is an edge on M whose end points are the vertices v_i and v_j such that

- (i) the points on the edge $\{v_i, v_j\}$ except v_i and v_j are called interior points;
- (ii) any pair of edge have at most one interior point in common; such a point is called a crossing;
- (iii) two edge sharing common vertex have no other point in common.

A good drawing D_M of G on M is called an optimal drawing of G on M if D_M has the minimum number of crossings among all possible good drawings of G on M . The number of crossings in an optimal drawing of G on M is called the crossing number of G on M .

Definition 4.1. *Two optimal drawings are said to be isomorphic on a Manifold M when there is a one-to-one correspondence between the vertices and if any pair of edges in one drawing crosses, the corresponding pair of edges in the second drawing*

Definition 4.2. *An invariant of an optimal drawing D_M of a graph G on a manifold M is a number associated with D_M of a graph G on M which has the same value for any optimal drawing of G on M isomorphic to D_M .*

A complete set of invariants of an optimal drawing of a graph G on M determines an optimal drawing upto an isomorphism. No decent complete set of invariants of an optimal drawing of a graph on a manifold is known.

Definition 4.3. *Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G) = \{1, \dots, n\}$. Let M be a Riemannian manifold. Let D_M be an optimal drawing of G on M with crossing number k . Give labels $n+1, \dots, n+k$ to the crossings of D_M . Hence the number of vertices of D_M are $1, \dots, n+k$ out of which the vertices $1, \dots, n$ correspond to vertices of the graph G and the vertices $n+1, \dots, n+k$ correspond to the crossings of D_M .*

5. 0-CHAIN AND 1-CHAIN OF AN OPTIMAL DRAWING ON A MANIFOLD

In this section, we have given the two vector spaces associated with an optimal drawing D_M of G on a Manifold M .

We defined the vector spaces over the field $\mathbb{F}_2 = \{0, 1\}$. Let $G = (V(G), E(G))$ be a graph. Let $V(D) = \{v_1, \dots, v_n\}$. Let D_M be an optimal drawing of G on a manifold M with a crossing number k . Give labels $v_{n+1}, \dots, v_{n+k}$ to the crossing of D_M . Therefore $V(D_M) = \{v_1, \dots, v_n, \dots, v_{n+k}\}$. Let $E(D_M) = \{e_{ij} = \{v_i, v_j\} : v_i, v_j \in V(D_M) \text{ and there is an edge between } v_i \text{ and } v_j\}$.

The 0-chain of D_M is a formal linear combination $\sum \alpha_i v_i$ of vertices of $V(D_M)$. The 1-chain of D_M is a formal linear combination $\sum \alpha_i x_i$ of edges of D_M .

Lemma 5.1. *$CV_0(D_M) = \{\sum \alpha_i v_i : \alpha_i \in \mathbb{F}_2 \text{ and } v_i \in V(D_M)\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $u = \sum \alpha_i v_i$ and $v = \sum \beta_i v_i$ then $u + v = \sum (\alpha_i + \beta_i) v_i$. and scalar multiplication $\gamma u = \sum \gamma \alpha_i v_i$, where $\gamma \in \mathbb{F}_2$.*

Lemma 5.2. *$CV_1(D_M) = \{\sum \alpha_i x_i : \alpha_i \in \mathbb{F}_2 \text{ and } x_i \in E(D_M)\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $x = \sum \alpha_i x_i$ and $y = \sum \beta_i x_i$ then*

$$x + y = \sum (\alpha_i + \beta_i)x_i.$$

and scalar multiplication $\gamma x = \sum \gamma \alpha_i x_i$, where $\gamma \in \mathbb{F}_2$.

The boundary operator ∂ on an optimal drawing D_M send 1-chain to 0-chain by using following rule:

- (a) ∂ is linear ;
- (b) If $x = uv$ then $\partial x = u + v$

.

The co-boundary operator δ on an optimal drawing D_M send 0-chain to 1-chain by using following rule:

- (a) δ is linear ;
- (b) If $\delta v = \sum \varepsilon_i x_i$, where $\varepsilon_i = 1$ whenever x_i is incident with v .

Definition 5.3. Let $CV_0(D_M)$ and $CV_1(D_M)$ be vector spaces over a field $\mathbb{F}_2$ associated with an optimal drawing D_M on a manifold M . Let $\partial : CV_1(D_M) \rightarrow CV_0(D_M)$ be a boundary operator. The set all vectors $\alpha \in CV_1(D_M)$ such that $\partial(\alpha) = \theta'$, where θ' is the zero element in $CV_0(D_M)$, is said to be the kernel of ∂ and it is denoted by $Ker\partial$.

Therefore $Ker\partial = \{\alpha \in CV_1(D_M) : \partial(\alpha) = \theta'\}$.

Lemma 5.4. $Ker\partial$ is a subspace of $CV_1(D_M)$.

Proof. $Ker\partial = \{\alpha \in CV_1(D_M) : \partial(\alpha) = \theta'\}$. $\theta \in Ker\partial$ since $\partial(\theta) = \theta'$. Therefore $Ker\partial$ is non-empty.

Case-I. $Ker\partial = \{\theta\}$. Then $Ker\partial$ is a subspace of CV_1 .

Case-II. $Ker\partial \neq \{\theta\}$. Let $\alpha \in Ker\partial$. Then $\partial(\alpha) = \theta'$. Let $c \in \mathbb{F}_2$. Then $\partial(c\alpha) =$

$$c\partial(\alpha) = c\theta' = \theta'. \text{ This implies } c\alpha \in Ker\partial$$

$$\text{Let } \alpha, \beta \in Ker\partial. \text{ Then } \partial(\alpha) = \theta', \partial(\beta) = \theta'.$$

$$\partial(\alpha + \beta) = \partial(\alpha) + \partial(\beta) = \theta' + \theta' = \theta'.$$

This implies $\alpha + \beta \in Ker\partial$.

Thus $\alpha, \beta \in Ker\partial \Rightarrow \alpha + \beta \in Ker\partial$ and $\alpha \in Ker\partial \Rightarrow c\alpha \in Ker\partial$ for all $c \in \mathbb{F}_2$. Hence $Ker\partial$ is a subspace of CV_1 .

□

The kernel of the boundary operator ∂ is also called as cycle space of an optimal drawing of D_M .

Definition 5.5. Let $CV_0(D_M)$ and $CV_1(D_M)$ be vector spaces over a field $\mathbb{F}_2$ associated with graph G . Let $\partial : CV_1(D_M) \rightarrow CV_0(D_M)$ be a boundary operator. The set of all vectors $\partial(\alpha) \in CV_0(D_M)$ such that $\alpha \in CV_1(D_M)$ is said to be the image of ∂ and it is denoted by $Im\partial$.

Therefore $Im\partial = \{\partial(\alpha) : \alpha \in CV_1(D_M)\}$.

Lemma 5.6. $Im\partial$ is a subspace of $CV_0(D_M)$.

The image of the boundary operator ∂ is also called as range space of the graph of G . The dimension of $Ker\partial$ is denoted by $dimKer\partial$. The dimension of $Im\partial$ is denoted by $dimIm\partial$. Let D_M^1 and D_M^2 be two optimal drawing of a graph G on a manifold M . Let $CV_o(D_M^1)$ and $CV_1(D_M^1)$ vector spaces associated with D_M^1 . Let $CV_o(D_M^2)$ and $CV_1(D_M^2)$ vector spaces associated with D_M^2 . Suppose D_M^1 is isomorphic to D_M^2 then there is a one-to-one correspondence between their vertices and edges. Hence the vector spaces $CV_o(D_M^1)$ and $CV_1(D_M^1)$ are isomorphic to $CV_o(D_M^2)$ and $CV_1(D_M^2)$ respectively.

According to the definition of invariant of an optimal drawing, dimension of $CV_0(D_M)$ and dimension of $CV_1(D_M)$ are the numerical invariant of D_M on M . Also $dimKer\partial$ and $dimIm\partial$ are the numerical invariants of D_M , where ∂ be a boundary operator associated with D_M .

Theorem 5.7. Let $CV_0(D_M)$ and $CV_1(D_M)$ be vector spaces over a field $\mathbb{F}_2$ associated with an optimal drawing D_M of a graph G on a manifold M . If $\partial : CV_1(D_M) \rightarrow CV_0(D_M)$ be a boundary operator associated with D_M then

$$(5) \quad dimKer\partial + dimIm\partial = dimCV_1(D_M)$$

Proof. Case-I. $Ker\partial = CV_1(D_M)$. Then $Im\partial$ consists of θ' only, where θ' is null element in $CV_0(D_M)$. Therefore $dimIm\partial = 0$. Thus $dimKer\partial + dimIm\partial = dimCV_1(D_M)$.

Thus $dimKer\partial + dimIm\partial = |E(D_M)|$.

We know that if an optimal drawing D_M contain an edge $x = uv$ then $\partial x = u + v \neq \theta'$.

Hence case-I is not possible.

Case-II Let $Ker\partial = \{\theta\}$ and let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of $CV_1(D_M)$.

Then $\{\partial(\alpha_1), \dots, \partial(\alpha_n)\}$ is a basis of $Im\partial$.

Therefore $\dim Ker\partial = 0$ and $\dim CV_1(D_M) = |E(D_M)|$, $\dim Im\partial = |E(D_M)|$. Hence the theorem is holds good.

From the case-II, we can observe that $Ker\partial = \{\theta\}$ is possible only if optimal drawing D_M doesn't contain any cycle. When D_M is the tree then only case-II is possible.

Case-III. Let $Ker\partial$ be a non-trivial proper subspace of $CV_1(D_M)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be a basis of $Ker\partial$.

This basis can be extended to a basis of $CV_1(D_M)$ and let the extended basis of $CV_1(D_M)$ be $\{\alpha_1, \alpha_2, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_{|E(D_M)|}\}$.

The $Im\partial$ is generated by the vector $\partial(\alpha_1), \partial(\alpha_2), \dots, \partial(\alpha_{|E(D_M)|})$. As $\partial(\alpha_1) = \partial(\alpha_2) = \dots = \partial(\alpha_r) = \theta'$. Therefore $Im\partial$ is generated by the vectors $\partial(\alpha_{r+1}), \partial(\alpha_{r+2}), \dots, \partial(\alpha_{|E(D_M)|})$.

Let us consider the relation

$$c_{r+1}\partial(\alpha_{r+1}) + c_{r+2}\partial(\alpha_{r+2}) + \dots, c_{|E(D_M)|}\partial(\alpha_{|E(D_M)|}) = \theta'$$

for some scalar $c_i \in \mathbb{F}_2$. Since ∂ is a linear, this implies that $\partial(c_{r+1}\alpha_{r+1} + c_{r+2}\alpha_{r+2} + \dots, c_{|E(D_M)|}\alpha_{|E(D_M)|}) = \theta'$. This shows that $c_{r+1}\alpha_{r+1} + c_{r+2}\alpha_{r+2} + \dots, c_{|E(D_M)|}\alpha_{|E(D_M)|}$ is in $Ker\partial$. Since $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is an assume basis for $Ker\partial$, $c_{r+1}\alpha_{r+1} + c_{r+2}\alpha_{r+2} + \dots, c_{|E(D_M)|}\alpha_{|E(D_M)|} = b_1\alpha_1 + b_2\alpha_2 + \dots + b_r\alpha_r$ for some $b_i \in \mathbb{F}_2$. Therefore $b_1\alpha_1 + b_2\alpha_2 + \dots + b_r\alpha_r - (c_{r+1}\alpha_{r+1} + c_{r+2}\alpha_{r+2} + \dots, c_{|E(D_M)|}\alpha_{|E(D_M)|}) = \theta$.

Therefore linear independence of $\{\alpha_1, \alpha_2, \dots, \alpha_{|E(D_M)|}\}$ requires $b_1 = b_2 = \dots = b_r = c_{r+1} = \dots = c_{|E(D_M)|} = 0$. The fact that $c_{r+1} = \dots = c_{|E(D_M)|} = 0$ prove that the set $\{\partial(\alpha_{r+1}), \partial(\alpha_{r+2}), \dots, \partial(\alpha_{|E(D_M)|})\}$ is linearly independent and so this is a basis for $Im\partial$.

$\dim Im\partial = |E(D_M)| - r$ and $\dim Ker\partial + \dim Im\partial = r + (|E(D_M)| - r) = |E(D_M)|$. Thus

$$(6) \quad \dim Ker\partial + \dim Im\partial = \dim CV_1(D_M)$$

□

Corollary 5.8. *Let D_M be an optimal drawing of a graph G on a manifold M and $\dim Im\partial = |E(D_M)| - 1$. Then D_M contain exactly one cycle.*

Corollary 5.9. *Let D_M be an optimal drawing of a graph G on a manifold M and $\dim \text{Im} \partial = |E(D_M)| - r$. Then D_M may contain cycles greater than or equal to r .*

6. 1-CHAIN AND 0-CHAIN OF AN OPTIMAL DRAWING ON A MANIFOLD WHICH NOT INCLUDE CROSSING

In section 5, we have seen the 0-chain and 1-chain of an optimal drawing of a graph on a manifold. In this section, we have consider the 0-chain and 1-chain of an optimal drawing on a manifold which not include vertices corresponding to crossings. Let $G = (V(G), E(G))$ be a graph. Let $V(D) = \{v_1, \dots, v_n\}$. Let D_M be an optimal drawing of G on a manifold M with a crossing number k . Give labels $v_{n+1}, \dots, v_{n+k}$ to the crossing of D_M . Therefore $V(D_M) = \{v_1, \dots, v_n, \dots, v_{n+k}\}$. Let $E(D_M) = \{e_{ij} = \{v_i, v_j\} : v_i, v_j \in V(D_M) \text{ and there is an edge between } v_i \text{ and } v_j\}$.

The RP-0-chain of D_M is a formal linear combination $\sum \alpha_i v_i$ of vertices of $V(D_M)$ such that $v_i \notin \{v_{n+1}, \dots, v_{n+k}\}$.

The RP-1-chain of D_M is a formal linear combination $\sum \alpha_i x_i$ of edges of D_M such that $v_i \notin \{v_{n+1}, \dots, v_{n+k}\}$.

Lemma 6.1. *$\text{RPCV}_0(D_M) = \{\sum \alpha_i v_i : \alpha_i \in \mathbb{F}_2 \text{ and } v_i \in V(D_M) \text{ such that } v_i \notin \{v_{n+1}, \dots, v_{n+k}\}\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $u = \sum \alpha_i v_i$ and $v = \sum \beta_i v_i$ then $u + v = \sum (\alpha_i + \beta_i) v_i$. and scalar multiplication $\gamma u = \sum \gamma \alpha_i v_i$, where $\gamma \in \mathbb{F}_2$.*

Lemma 6.2. *$\text{RPCV}_1(D_M) = \{\sum \alpha_i x_i : \alpha_i \in \mathbb{F}_2 \text{ and } x_i \in E(D_M) \text{ such that } v_i \notin \{v_{n+1}, \dots, v_{n+k}\}\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $x = \sum \alpha_i x_i$ and $y = \sum \beta_i x_i$ then $x + y = \sum (\alpha_i + \beta_i) x_i$. and scalar multiplication $\gamma x = \sum \gamma \alpha_i x_i$, where $\gamma \in \mathbb{F}_2$.*

RP-boundary operator ∂ on an optimal drawing D_M send RP-1-chain to RP-0-chain by using following rule:

- (a) ∂ is linear ;
- (b) If $x = uv$ then $\partial x = u + v$

.

RP-co-boundary operator δ on an optimal drawing D_M send RP-0-chain to RP-1-chain by using following rule:

- (a) δ is linear ;
- (b) If $\delta v = \sum \varepsilon_i x_i$, where $\varepsilon_i = 1$ whenever x_i is incident with v .

7. 1-CHAIN AND 0-CHAIN OF AN OPTIMAL DRAWING ON A MANIFOLD WHICH INCLUDE VERTICES CORRESPONDING TO CROSSINGS.

In section5, we have seen the 0-chain and 1-chain of an optimal drawing of a graph on a manifold. In this section, we have consider the 0-chain and 1-chain of an optimal drawing on a manifold which include vertices corresponding to crossings. Let $G = (V(G), E(G))$ be a graph. Let $V(D) = \{v_1, \dots, v_n\}$. Let D_M be an optimal drawing of G on a manifold M with a crossing number k . Give labels $v_{n+1}, \dots, v_{n+k}$ to the crossing of D_M . Therefore $V(D_M) = \{v_1, \dots, v_n, \dots, v_{n+k}\}$. Let $E(D_M) = \{e_{ij} = \{v_i, v_j\} : v_i, v_j \in V(D_M) \text{ and there is an edge between } v_i \text{ and } v_j\}$.

The RPC-0-chain of D_M is a formal linear combination $\sum \alpha_i v_i$ of vertices of $V(D_M)$ such that $v_i \in \{v_{n+1}, \dots, v_{n+k}\}$.

The RPC-1-chain of D_M is a formal linear combination $\sum \alpha_i x_i$ of edges of D_M such that $v_i \in \{v_{n+1}, \dots, v_{n+k}\}$.

Lemma 7.1. *$RPCCV_0(D_M) = \{\sum \alpha_i v_i : \alpha_i \in \mathbb{F}_2 \text{ and } v_i \in V(D_M) \text{ such that } v_i \in \{v_{n+1}, \dots, v_{n+k}\}\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $u = \sum \alpha_i v_i$ and $v = \sum \beta_i v_i$ then $u + v = \sum (\alpha_i + \beta_i) v_i$. and scalar multiplication $\gamma u = \sum \gamma \alpha_i v_i$, where $\gamma \in \mathbb{F}_2$.*

Lemma 7.2. *$RPCCV_1(D_M) = \{\sum \alpha_i x_i : \alpha_i \in \mathbb{F}_2 \text{ and } x_i \in E(D_M) \text{ such that } v_i \notin \{v_{n+1}, \dots, v_{n+k}\}\}$ form a vector space over a finite field $\mathbb{F}_2$ with respect to addition $x = \sum \alpha_i x_i$ and $y = \sum \beta_i x_i$ then $x + y = \sum (\alpha_i + \beta_i) x_i$. and scalar multiplication $\gamma x = \sum \gamma \alpha_i x_i$, where $\gamma \in \mathbb{F}_2$.*

RPC-boundary operator ∂ on an optimal drawing D_M send RPvertex-1-chain to RP-0-chain by using following rule:

- (a) ∂ is linear ;
- (b) If $x = uv$ then $\partial x = u + v$

.

RPC-co-boundary operator δ on an optimal drawing D_M send RP-0-chain to RP-1-chain by using following rule:

- (a) δ is linear ;
- (b) If $\delta v = \sum \varepsilon_i x_i$, where $\varepsilon_i = 1$ whenever x_i is incident with v .

8. CONCLUSION

As we know that there is no decent set of invariants of a graph is known. In this paper, by using graph G , we have introduced graph $G_{p,q}$. The invariant associated with $G_{p,q}$ are also an invariant associated with G . In this paper, we have studied the boundary operator and co-boundary operator associated with the graph $G_{p,q}$.

We have also introduced an optimal drawing of a graph on a manifold. Then we have studied the boundary operator on an optimal drawing. we have also observed that invariant associated with an optimal drawing of a graph on a manifold may be useful to classify the manifold.

There are lot of thing remain to investigate to the theory of an optimal drawing on a manifold.

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