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MATHEMATICAL MODELLING OF BURIED ELECTRODE RESISTIVITY RESPONSE FROM A THREE-LAYERED CONDUCTIVE MEDIUM CONTAINING TRANSITIONAL LAYERS

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Abstract. In this work, we study the depth of a buried current electrode that constitutes an effect on the electric potential in a transition multilayer earth. A three-layered earth model is applied to simulate the data of electric potential. The model assumes that an overburden with the thickness has a constant conductivity over the layers of material having the feather of specific conductivity. The model is composed of a set of partial differential equations defined on the spatial domain. The Hankel transform is applied to derive equations defined on the wave number domain from the original problem. On the wave number domain, the analytic solution is determined by solving a boundary value problem. The achieved equations are then transformed back to get the equations defined on the spatial domain. The results of electric potential generated by the three-layered earth model are plotted and compared to indicate the behavior in response to different positions of current source while the parameters in the model are approximately assigned.

AMS Classification: 86A20.
Keywords: Resistivity method; Buried electrode; Transitional layer.

1. Introduction

In exploration geophysics, the electrical resistivity method involves the electrical properties of the ground layers in the term of potential difference. During resistivity surveys, a pair of current electrodes is buried and the direct current is then administered into the ground through the electrodes. Resulting from current injection, the direct current flow between two current electrodes occurs hence the potential difference can be measured between a pair of potential electrodes. With these processes, the apparent resistivity of subsurface can be determined as a function of depth by using the mathematical model of electric potential for a multilayered earth.

Mallick and Roy [9] proposed the model of a two-layer earth with transitional boundary. In this model, the electrical conductivity of transitional boundary was assigned to be linearly dependent upon depth for the purpose of continuity of the conductivity between two layers having difference in the electrical property. The similar problems consisting of transitional layer were presented by Jain [5], Koefoed [7] and Lal [8]. Stoyer and Wait [15] introduced the electrical conductivity varying exponentially with depth for the earth's layer under homogeneous overburden. Banerjee et al. [2] studied the problem concerning multilayered earth with a layer having exponential varying of conductivity. Kim and Lee [6] proposed more complicated earth's structure with layers having exponential resistivities. Our model study of the vertical electrical sounding is performed by assuming that the earth's structure is horizontally stratified. The uppermost of ground surface, called overburden, is the homogeneous material, such as the nature rock and soil, lying above the specific geologic features. The second layer represents the transitional boundary between first and third layer. The last layer is the main medium that may stand for area of subsurface where the degree of temperature depends on the vertical depth. Under these assumptions, the three-layered earth consists of homogeneous overburden of constant conductivity while the conductivity of the second layer varies linearly with depth and changes continuously at the boundaries, and the third layer has conductivity changing according to the exponential law of depth.

For the conventional resistivity surveys, the current electrodes are positioned onto the surface. In some cases, these surveys give the unsatisfied resolution resulting from a very small amplitude response on account of conductive overburden. To solve this difficulty, the use of current electrodes down a drill hole into the overburden layer is introduced to enhance the response and supply valuable data. In this work, we show how the result for shifting electrode positions from surface to underground in aspect of potential differences, similar to the approach used by Sato [12] and Sripanya [13]. First, a mathematical model of the electric potential generated by a direct point source is proposed and the analytical solution of the electric potential is then expressed by application of the Hankel transform. The data sets of the electric potential by our model are numerically simulated with the difference of vertical position of the source and the specification of appeared parameters of the model. Finally, the simulated data are compared to reveal the responsiveness of the electric potential with given conditions.

2. Potential from a Buried Source

Consider two conductive half-spaces formed by a region of air above a ground structure (see Figure 1). The conductivity of air region is assigned to be a constant. The lower half-space is assumed to consist of a series of three layers of different conductivity uniform and thickness. The uppermost layer has a constant conductivity while the middle layer and lowermost layer have linear and exponential conductivity, respectively. The earth ground is stratified at the depths h_1 and h_2, and the thickness of lowermost layer extents to infinity. A point source of direct current is located underground at $z = h_1$.

The equation of a vector of the electric field $\mathbf{E}$ in each layer in the form of a partial derivative operator, called the del operator, of the scalar voltage (electric potential) field V can be expressed as

$$\mathbf{E} = -\nabla V. \tag{1}$$

The vector of the current density $\mathbf{J}$ is directly proportional to the vector of the electric field through the Ohm's law, hence

$$\mathbf{J} = \sigma \mathbf{E}, \tag{2}$$

where σ is the conductivity of interested medium. Except at the source, the divergence operator of the current density vector is equal to zero, i.e.,

$$\nabla \cdot \mathbf{J} = 0. \tag{3}$$

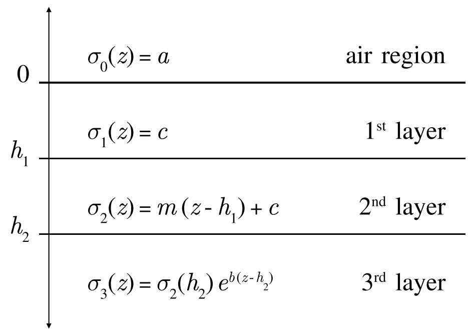

FIGURE 1. Geometric model of three-layered conductive medium containing transitional layers.

Substituting equations (1) and (2) in equation (3), we get

$$\nabla \cdot (\sigma \nabla V) = 0. \tag{4}$$

To find the solution, this equation can be expressed in cylindrical coordinate (r, ϕ, z), where r and ϕ are polar coordinates in two-dimensional plan and z is the coordinate perpendicular to that plan, then we have

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\left(\sigma r \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\frac{\sigma}{r}\frac{\partial V}{\partial \phi}\right) + r\frac{\partial}{\partial z}\left(\sigma \frac{\partial V}{\partial z}\right)\right) = 0. \tag{5}$$

The value of the voltage field V depends only on r and z in cylindrical coordinate. Therefore, equation (5) is reduced the term into

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\sigma r \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial z}\left(\sigma \frac{\partial V}{\partial z}\right) = 0. \tag{6}$$

This yields

$$\frac{1}{r}\left(\sigma \frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + r\frac{\partial \sigma}{\partial r}\frac{\partial V}{\partial r}\right) + \frac{\sigma}{r}\frac{\partial V}{\partial r} + \sigma \frac{\partial^2 V}{\partial z^2} + \frac{\partial \sigma}{\partial z}\frac{\partial V}{\partial z} = 0. \tag{7}$$

In our model, σ is assigned to be a function of depth z in each layer, thus we have

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial z}\frac{\partial V}{\partial z} = 0. \tag{8}$$

To solve this partial derivative equation defined on the spatial domain, the Hankel transform (Ali and Kalla [1]) is applied by defining the new variables on the wave number domain as

$$\widetilde{V}(\lambda, z) = \int_0^\infty \lambda r V(r, z) J_0(\lambda r)\, dr \tag{9}$$

and

$$V(r, z) = \int_0^\infty \tilde{V}(\lambda, z) J_0(\lambda r)\, d\lambda, \tag{10}$$

where J_0 is the zero order Bessel function of the first kind and λ is the scaling factor. Applying the transformation for equation (8), we obtain

$$\int_0^\infty \lambda r \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial z}\frac{\partial V}{\partial z} \right) J_0(\lambda r)\, dr = \int_0^\infty \lambda r \cdot 0 \cdot J_0(\lambda r)\, dr. \tag{11}$$

By the mathematical solving and rearrangement, the important partial derivative equation can be written as

$$\frac{\partial^2 \widetilde{V}}{\partial z^2} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial z}\frac{\partial \widetilde{V}}{\partial z} - \lambda^2 \widetilde{V} = 0. \tag{12}$$

When the solution of equation (12) is acquired, the inverse Hankel transform is taken for finding the electric potential in each layer as equation (10).

Consider the three-layered earth model. For the conductivity profile in each layer, we assign σ_0, σ_1, σ_2 and σ_3 to denote the functions of conductivity inducing the electric potential V_0, V_1, V_2, and V_3 in the air region, overburden, middle layer and lowermost layer, respectively. The variations of conductivity are defined to satisfy our assumption and these functions are then substituted in equation (10) to derive the partial derivative equations of $\widetilde{V}_0$, $\widetilde{V}_1$, $\widetilde{V}_2$ and $\widetilde{V}_3$ that are the Hankel transform of the electric potential V_0, V_1, V_2 and V_3, respectively. By separating with the kind of conductivity in medium, the variables of the Hankel transform can be determined as follows.

2.1. **Homogeneous Layer.** We first consider the air region assumed to have a positive constant conductivity

$$\sigma_0(z) = a. \tag{13}$$

Hence, the equation (12) can be written in the form

$$\frac{\partial^2 \widetilde{V}_0}{\partial z^2} - \lambda^2 \widetilde{V}_0 = 0. \tag{14}$$

So, the solution for $\widetilde{V}_0$ is

$$\widetilde{V}_0(\lambda, z) = A_0 e^{-\lambda z} + B_0 e^{\lambda z}, \tag{15}$$

where A_0 and B_0 are the arbitrary constant values.

Similarly, the equation of $\widetilde{V}_1$ in the overburden having a positive constant conductivity

$$\sigma_1(z) = c \tag{16}$$

is given by

$$\widetilde{V}_1(\lambda, z) = A_1 e^{-\lambda z} + B_1 e^{\lambda z}, \tag{17}$$

where A_1 and B_1 are also the arbitrary constant values.

2.2. **Linear Boundary.** For the middle layer, the variation of conductivity is performed by

$$\sigma_2(z) = m(z - h_1) + c, \tag{18}$$

where m, h_1 and c are constants and h_1 indicates the thickness of overburden material over the middle layer. We simplify equation (12) after substituting the term of conductivity by equation (18) as

$$\frac{\partial^2 \widetilde{V}_2}{\partial z^2} + \frac{m}{mz + c}\frac{\partial \widetilde{V}_2}{\partial z} - \lambda^2 \widetilde{V}_2 = 0. \tag{19}$$

The solution of equation (19) has been proposed by Banerjee et al. [3] in the form

$$\widetilde{V}_2(\lambda, z) = A_2 I_0\left(\frac{\lambda c}{m}\psi(z)\right) + B_2 K_0\left(\frac{\lambda c}{m}\psi(z)\right), \tag{20}$$

where $\psi(z) = 1 + m(z - h_1)/c$, A_2 and B_2 are constants, I_0 and K_0 are the zero order modified Bessel functions of the first and second kind, respectively.

2.3. **Exponential Boundary.** For the exponentially varying conductivity profile in the lowermost layer initiating at the depth h_2 with infinite thickness, the represented equation can be written as

$$\sigma_3(z) = \sigma_2(h_2)\, e^{b(z-h_2)}, \tag{21}$$

where b is a constant value. With substituting this function of conductivity in equation (12), we derive the partial derivative equation of $\widetilde{V}_3$ as follows

$$\frac{\partial^2 \widetilde{V}_3}{\partial z^2} + b\frac{\partial \widetilde{V}_3}{\partial z} - \lambda^2 \widetilde{V}_3 = 0. \tag{22}$$

The general solution of equation (22) can be written in the form

$$\widetilde{V}_3(\lambda, z) = A_3 e^{\upsilon_0^-(z-h_2)} + B_3 e^{\upsilon_0^+(z-h_2)}, \tag{23}$$

where $\upsilon_0^- = \left(-b - \sqrt{b^2 + 4\lambda^2}\right)/2$, $\upsilon_0^+ = \left(-b + \sqrt{b^2 + 4\lambda^2}\right)/2$ and A_3, B_3 are constants.

3. Boundary Conditions

The variables of the Hankel transforms of electric potential in each layer can be determined by solving the boundary value problem under the following assumptions and procedures:

(i) The electric potential converges to zero as the depth z tends to infinity or minus infinity. In this case, we assign that the half-space above the interface between air region and overburden has $z < 0$ and the below half-space has $z > 0$. Hence, we have

$$\widetilde{V}_0(\lambda, z) = B_0 e^{\lambda z} \tag{24}$$

and

$$\widetilde{V}_3(\lambda, z) = A_3 e^{\upsilon_0^-(z - h_2)}. \tag{25}$$

(ii) The electric potential is continuous on each of the connected plane between any two layers, i.e.,

$$\lim_{z \to 0^-} \widetilde{V}_0(\lambda, z) = \lim_{z \to 0^+} \widetilde{V}_1(\lambda, z), \tag{26}$$

$$\lim_{z \to h_1^-} \widetilde{V}_1(\lambda, z) = \lim_{z \to h_1^+} \widetilde{V}_2(\lambda, z) \tag{27}$$

and

$$\lim_{z \to h_2^-} \widetilde{V}_2(\lambda, z) = \lim_{z \to h_2^+} \widetilde{V}_3(\lambda, z). \tag{28}$$

(iii) The vertical component of the current density needs to be continuous on each of the boundary plane except on $z = h_1$, where the direct current source is located on the plane $z = h_1$. This leads to

$$\lim_{z \to 0^-} \sigma_0(z) \frac{\partial}{\partial z} \widetilde{V}_0(\lambda, z) = \lim_{z \to 0^+} \sigma_1(z) \frac{\partial}{\partial z} \widetilde{V}_1(\lambda, z) \tag{29}$$

and

$$\lim_{z \to h_2^-} \sigma_2(z) \frac{\partial}{\partial z} \widetilde{V}_2(\lambda, z) = \lim_{z \to h_2^+} \sigma_3(z) \frac{\partial}{\partial z} \widetilde{V}_3(\lambda, z). \tag{30}$$

(iv) The constant current intensity must be equal to the total current flowing out from any cylindrical surface around the point source (Sato [12] and Sripanya [13]), i.e., for any radius ξ,

$$\lim_{h\to 0}\left(\int_{\phi=0}^{2\pi}\int_{r=0}^{\xi}\mathbf{J}_1|_{z=h_1-h}\cdot(-\hat{z}rdrd\phi)+\int_{\phi=0}^{2\pi}\int_{r=0}^{\xi}\mathbf{J}_2|_{z=h_1+h}\cdot(\hat{z}rdrd\phi)\right)=I. \tag{31}$$

Equation (31) can be written in the simplified form as

$$\lim_{z\to h_1^-}\sigma_1(z)\frac{\partial}{\partial z}\widetilde{V}_1(\lambda,z)-\lim_{z\to h_1^+}\sigma_2(z)\frac{\partial}{\partial z}\widetilde{V}_2(\lambda,z)=\frac{\lambda I}{2\pi}. \tag{32}$$

4. Solution of the Problem

In this part, we deal with finding the formulas for the unknown constants B_0, A_1, B_1, A_2, B_2 and A_3. Deriving values of these constants leads to the solution of variables presenting the Hankel transform of the electric potential in (15), (17), (20), (23) and we then can determine the numerical solution of the electric potential in each layer by using (10). The boundary conditions (26), (27) and (28) assigned for the continuity of the electrical potential across the interface layers at $z=0$, $z=h_1$ and $z=h_2$ give us the equations

$$B_0=A_1+B_1, \tag{33}$$

$$A_1e^{-\lambda h_1}+B_1e^{\lambda h_1}=A_2I_0\left(\frac{\lambda c}{m}\right)+B_2K_0\left(\frac{\lambda c}{m}\right) \tag{34}$$

and

$$A_2I_0\left(\frac{\lambda c}{m}\psi(h_2)\right)+B_2K_0\left(\frac{\lambda c}{m}\psi(h_2)\right)=A_3. \tag{35}$$

Similarly, using the boundary conditions in (29) and (30), we are able to write the equations

$$B_0a=(-A_1+B_1)\,c \tag{36}$$

and

$$\left(A_2\lambda I_1\left(\frac{\lambda}{m}\psi(h_2)\right)-B_2K_1\left(\frac{\lambda}{m}\psi(h_2)\right)\right)\lambda=A_3\upsilon_0^-, \tag{37}$$

where I_1 and K_1 are the modified Bessel functions of the first and second kind of order one, respectively. From the last boundary condition (32) relating with the total current, we obtain

$$\left(-A_1e^{-\lambda h_1} + B_2e^{\lambda h_1}\right) - \left(A_2I_1\left(\frac{\lambda c}{m}\right) - B_2K_1\left(\frac{\lambda c}{m}\right)\right) = \frac{I}{2\pi c}. \tag{38}$$

Finally, the system of (33) to (38) consists of six equations with six unknowns. These equations can be solved and we then obtain the solution for A_1, B_1, A_2 and B_2 as follows

$$A_1 = \frac{(-a+c)\,e^{\lambda h_1}If_5}{2\pi cf_1}, \tag{39}$$

$$B_1 = \frac{(a+c)\,e^{\lambda h_1}If_5}{2\pi cf_1}, \tag{40}$$

$$A_2 = \frac{\lambda If_5}{m\pi} \tag{41}$$

and

$$B_2 = \frac{I}{2\pi cf_1}\left(\upsilon_1^- - \lambda\kappa_0 f_5\right), \tag{42}$$

where $\upsilon_1^- = (-a+c) + (a+c)\,e^{2\lambda h_1}$, $\upsilon_1^+ = (a-c) + (a+c)\,e^{2\lambda h_1}$, $\kappa_0 = c/m$, $\kappa_1 = -h_1 + h_2 + \kappa_0$, $f_1 = \upsilon_1^+ K_0(\lambda\kappa_0) + \upsilon_1^- K_1(\lambda\kappa_0)$, $f_2 = \upsilon_0^- K_0(\lambda\kappa_1) + \lambda K_1(\lambda\kappa_1)$, $f_3 = \upsilon_1^+ I_0(\lambda\kappa_0) - \upsilon_1^- I_1(\lambda\kappa_0)$, $f_4 = \upsilon_0^- I_0(\lambda\kappa_1) - \lambda I_1(\lambda\kappa_1)$ and $f_5 = K_0(\lambda\kappa_0) + \left(\upsilon_1^-/\lambda\kappa_0\right)\left(f_2/\left(f_2f_3 - f_1f_4\right)\right)$. Substituting the solution for A_1 and B_1 into (33) and the solution for A_2 and B_2 into (35) yield the solution for B_0 and A_3, respectively.

5. Numerical Experiments

The analytical solution on the wave number domain for the three-layered conductive medium containing transitional layers can be expressed as (15), (17), (20) and (23). The formulas for the constants in our solution are assigned from solving the boundary value problem. After substituting the constants in our solution, the Chave's algorithm [4] is applied for numerical evaluating the inverse Hankel transform to derive the electric potential. The special functions are computed by using the Numerical Recipes source codes (Press et al. [10]). The data sets of the electric potential from a DC source at the ground surface and a DC source buried in the ground with specific depth are plotted to show the behavior of the solution against source-receiver spacing. The Model A-1 and A-2 are obtained under the assumption

Table 1. Model parameters used in numerical simulation.

Model	a (S/m)	m (S/m^2)	b (m^{-1})	h_1 (m)	h_2 (m)	h_s^* (m)
A-1	0.05	0.05	0.075	5	15	5
B-1	0.05	0.05	0.075	5	15	0
A-2	0.75	-0.05	-0.075	5	15	5
B-2	0.75	-0.05	-0.075	5	15	0

$^*h_s$ is denoted as the depth of a buried point source.

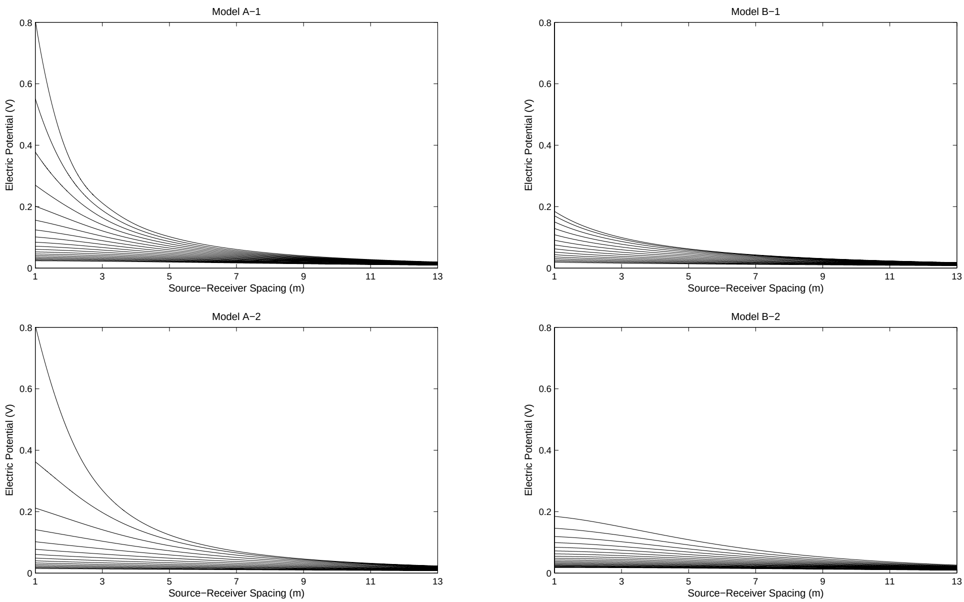

Figure 2. Electric potential against source-receiver spacing r from Model A-1, B-1, A-2 and B-2 at different depths.

of a buried DC source within a horizontally layered space at $z = 5$. The Model B-1 and B-2 are simulated from a DC source at the surface ($z = 0$), which are obtained by Raghuwanshi and Singh [11]. In data simulating, two sets of specific parameters a, m and b and the thickness of ground layers are applied as Table 1. Figure 2 shows the behavior of the electric potential against source-receiver spacing r from both of the two models at different depths.

6. Discussions and Conclusions

We now consider the electric potential's profiles from numerical simulation. Our model study is grouped in two categories. The Model A-1 and A-2 are constructed from underground DC source with different ground properties, and the Model B-1 and B-2 have the surface DC source. The parameters in the Model A-1 and A-2 are the same values as the parameters in the Model B-1 and B-2, respectively. The size of the electric potential from the Model A-1 and A-2 is remarkably larger than the size of the electric potential from the Model B-1 and B-2. As a result, the model simulation implies that using buried electrode for the ground investigation has the advantage of a more elevated electric potential (see also Sripanya [13], [14]).

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