

FURTHER REFINEMENTS OF CERTAIN PACHPATTE- GAMIDOV-TYPE INEQUALITIES IN TWO INDEPENDENT VARIABLES AND THEIR APPLICATIONS

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ABSTRACT. The main objective in this paper is to investigate some Pachpatte-Gamidov-type inequalities in two independent variables. The obtained inequalities can be used as handy tools to study the properties of certain differential and integral equations.

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1. INTRODUCTION

Integral inequalities play a dominant role in the study of qualitative and quantitative properties of differential and integral equations, see [1, 10, 11, 13 – 16, 19 – 22] and the references given therein. During the past few years an enormous amount of effort has been devoted to the discovery of new types of inequalities and their applications in various branches of ordinary and partial differential and integral equations see [2 – 9, 18].

In [11], Sh.G.Gamidov discussed the following integral inequality :

$$(1.1) \quad x(t) \leq \alpha + \int_a^t h(s)x(s)ds + \int_a^b m(s)x(s)ds,$$

for $t \in [a, b]$, under some suitable conditions on the functions involved in (1.1).

Pachpatte in [17], established the following inequality :

$$(1.2) \quad x(t) \leq \alpha(t) + \int_a^t \Psi(t, s)x(s)ds + \int_a^b m(s)x(s)ds.$$

K. Cheng, C. Guo in [8] also has a result in two independent variables as follows

:

$$(1.3) \quad x(s, t) \leq \alpha(s, t) + \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau)x(\eta, \tau)d\eta d\tau + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau)x(\eta, \tau)d\eta d\tau,$$

for $(s, t) \in [0, A] \times [0, B]$.

In this paper, we derive some new results on Pachpatte-Gamidov-type inequalities, which can be used in the analysis of differential and integral equations. To show the feasibility of the obtained inequalities, some illustrative examples are also introduced. We mention that the results obtained in this paper are extensions of the paper [6].

Now, we give some Lemmas which are used in our work.

Lemma 1. [8] Assume $x(s, t)$, $\alpha(s, t)$, $\gamma(s, t)$, $m(s, t) \in C([0, A] \times [0, B], [0, \infty))$ and

$$x(s, t) \leq \alpha(s, t) + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau)x(\eta, \tau)d\eta d\tau,$$

for (s, t) in $[0, A] \times [0, B]$.

If $\int_0^A \int_0^B \gamma(\eta, \tau)g(\eta, \tau)d\eta d\tau < 1$, then the following estimate

$$x(s, t) \leq \alpha(s, t) + \frac{\gamma(s, t) \int_0^A \int_0^B \alpha(\eta, \tau)m(\eta, \tau)d\eta d\tau}{1 - \int_0^A \int_0^B \gamma(\eta, \tau)m(\eta, \tau)d\eta d\tau},$$

hold for $(s, t) \in [0, A] \times [0, B]$.

Lemma 2. [12] Assume that $\sigma \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then

$$(1.4) \quad \sigma^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} \sigma + \frac{p-q}{p} K^{\frac{q}{p}},$$

for any $K > 0$.

2. MAIN RESULTS

In what follows, $\mathbb{R}$ denotes the set of real numbers $\mathbb{R}_+ = [0, \infty)$, $J_1 = [0, A]$, and $J_2 = [0, B]$ are given subsets of $\mathbb{R}$. Let $\Gamma = J_1 \times J_2$, $C(W, \Sigma)$ denotes the collection of continuous functions from W to Σ . Now we state the main results of this work.

Lemma 3. Let $x(s, t)$, $\alpha(s, t)$, $\gamma(s, t)$, $m(s, t) \in C(\Gamma, \mathbb{R}_+)$ and $S : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ be a continuous function which satisfies the condition

$$(2.1) \quad 0 \leq S(s, t, x) - S(s, t, y) \leq R(s, t, y)(x - y),$$

for $x \geq y \geq 0$, where $R(s, t, y)$ is a nonnegative continuous function defined for $s, t, y \in \mathbb{R}_+$.

If $x(s, t)$ satisfies

$$(2.2) \quad x(s, t) \leq \alpha(s, t) + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau,$$

for $(s, t) \in \Gamma$, then

$$(2.3) \quad x(s, t) \leq \alpha(s, t) + \frac{\gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, \alpha(\eta, \tau)) d\eta d\tau}{1 - \int_0^A \int_0^B \gamma(\eta, \tau) m(\eta, \tau) R(\eta, \tau, \alpha(\eta, \tau)) d\eta d\tau},$$

holds for $(s, t) \in \Gamma$, when $\int_0^A \int_0^B \gamma(\eta, \tau) m(\eta, \tau) R(\eta, \tau, \alpha(\eta, \tau)) d\eta d\tau < 1$.

Proof. It is clear that $\int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau$ is a constant.

Setting

$$(2.4) \quad \phi = \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau,$$

from (2.2), we obtain

$$(2.5) \quad x(s, t) \leq \alpha(s, t) + \gamma(s, t)\phi.$$

and

$$(2.6) \quad 0 \leq S(s, t, x(s, t)) \leq S(s, t, \alpha(s, t) + \gamma(s, t))\phi,$$

using (2.1), we have

$$(2.7) \quad 0 \leq S(s, t, x(s, t)) \leq S(s, t, \alpha(s, t)) + R(s, t, \alpha(s, t))\gamma(s, t)\phi.$$

As $m(s, t)$ is positive function, then

$$(2.8) \quad m(s, t)S(s, t, x(s, t)) \leq m(s, t) \{S(s, t, \alpha(s, t)) + R(s, t, \alpha(s, t))\gamma(s, t)\phi\}.$$

Integrating both sides of (2.8) on Γ , we get

$$(2.9) \quad \begin{aligned} \phi &= \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau \\ &\leq \phi \int_0^A \int_0^B \gamma(\eta, \tau) m(\eta, \tau) R(\eta, \tau, \alpha(\eta, \tau)) d\eta d\tau + \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, \alpha(\eta, \tau)) d\eta d\tau. \end{aligned}$$

It is easy to see that

$$(2.10) \quad \phi \leq \frac{\int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, \alpha(\eta, \tau)) d\eta d\tau}{1 - \int_0^A \int_0^B \gamma(\eta, \tau) m(\eta, \tau) R(\eta, \tau, \alpha(\eta, \tau)) d\eta d\tau}.$$

Substituting the inequality (2.10) into (2.5), we get the desired inequality (2.3). This completes the proof. $\square$

Remark 1. Setting $S(s, t, x(s, t)) = x(s, t)$, we see that the obtained inequality is as seen in Lemma 1 and if we replace $S(s, t, x(s, t))$ by $n(x(s, t))$ where n is a differentiable increasing function on $]0, +\infty[$ with continuous decreasing first derivative on $]0, +\infty[$, one can easily derive Lemma 2.1 in [6].

Theorem 1. Suppose that $\alpha(s, t), \beta(s, t), \gamma(s, t), \psi(s, t), m(s, t) \in C(\Gamma, \mathbb{R}_+)$ and $\alpha(s, t), \beta(s, t), \gamma(s, t)$ are nondecreasing in s and t . If $x(s, t) \in C(\Gamma, \mathbb{R}_+)$ satisfies

$$(2.11) \quad \begin{aligned} x(s, t) &\leq \alpha(s, t) + \beta(s, t) \int_0^s \int_0^t \psi(\eta, \tau) x(\eta, \tau) d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau \end{aligned}$$

then

(2.12)

$$x(s, t) \leq M^*(s, t) + N^*(s, t) \times \frac{\int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau}{1 - \int_0^A \int_0^B N^*(\eta, \tau) m(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau},$$

 for $(s, t) \in \Gamma$, where

$$(2.13) \quad \begin{aligned} M^*(s, t) &= \alpha(s, t) \exp \left\{ \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) d\eta d\tau \right\}, \\ N^*(s, t) &= \gamma(s, t) \exp \left\{ \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) d\eta d\tau \right\}, \end{aligned}$$

 and S (also R) is introduced as in Lemma 3, with

$$(2.14) \quad \int_0^A \int_0^B N^*(\eta, \tau) m(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau < 1.$$

Proof. Fixing any arbitrary $(S, T) \in \Gamma$, then for $(s, t) \in \Gamma_1 = [0, S] \times [0, T]$, from (2.11) and taking into account the fact that $\alpha(s, t)$, $\beta(s, t)$, and $\gamma(x, y)$ are nondecreasing functions in s and t , we have

$$(2.15) \quad \begin{aligned} x(s, t) &\leq \alpha(S, T) + \beta(S, T) \int_0^s \int_0^t \Psi(\eta, \tau) x(\eta, \tau) d\eta d\tau \\ &\quad + \gamma(S, T) \int_0^A \int_0^B m(s, t) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau. \end{aligned}$$

Define a function $z(s, t)$, $(s, t) \in \Gamma_1$ by the right side of (2.15). It is easy to see that $z(s, t)$ is positive and nondecreasing in s and t , then

$$(2.16) \quad x(s, t) \leq z(s, t),$$

further, one gets

$$(2.17) \quad z(0, t) = \alpha(S, T) + \gamma(S, T) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau,$$

$$(2.18) \quad \begin{aligned} \frac{\partial}{\partial s} z(s, t) &= \beta(S, T) \int_0^t \Psi(s, \tau) x(s, \tau) d\tau \\ &\leq \beta(S, T) \int_0^t \Psi(s, \tau) z(s, \tau) d\tau \\ &\leq \beta(S, T) \int_0^t \Psi(s, \tau) d\tau z(s, t). \end{aligned}$$

Since $z(s, t)$ is nondecreasing in t , from (2.18), we have

$$(2.19) \quad \frac{(\partial/\partial s)z(s, t)}{z(s, t)} \leq B(S, T) \int_0^t \Psi(s, \tau) d\tau,$$

keeping t fixed in (2.19), letting $s = \tau$, and integrating it from 0 to s , we obtain

$$(2.20) \quad z(s, t) \leq z(0, t) \exp \left\{ B(S, T) \int_0^s \int_0^t \Psi(\eta, \tau) d\eta d\tau \right\}.$$

Combining (2.16) and (2.17), we have

$$(2.21) \quad \begin{aligned} x(s, t) &\leq \left[\alpha(S, T) + \gamma(S, T) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau \right] \\ &\times \exp \left\{ \beta(S, T) \int_0^s \int_0^t \Psi(\eta, \tau) ds dt \right\} \\ &= \alpha(S, T) \exp \left\{ \beta(S, T) \int_0^s \int_0^t \Psi(\eta, \tau) d\eta d\tau \right\} \\ &+ \gamma(S, T) \exp \left\{ \beta(S, T) \int_0^s \int_0^t \Psi(\eta, \tau) d\eta d\tau \right\} \\ &\times \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau \\ &= M_1(s, t, S, T) + N_1(s, t, S, T) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau, \end{aligned}$$

where

$$(2.22) \quad \begin{aligned} M_1(s, t, S, T) &= \alpha(S, T) \exp \left\{ \beta(S, T) \int_0^s \int_0^t \Psi(\eta, \tau) d\eta d\tau \right\}, \\ N_1(x, y, X, Y) &= \gamma(S, T) \exp \left\{ \beta(S, T) \int_0^s \int_0^t \Psi(\eta, \tau) d\eta d\tau \right\}. \end{aligned}$$

Using Lemma 3 from (2.21), one gets

$$(2.23) \quad \begin{aligned} x(s, t) &\leq M_1(s, t, S, T) + N_1(s, t, S, T) \\ &\times \frac{\int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, M_1(\eta, \tau, S, T)) d\eta d\tau}{1 - \int_0^A \int_0^B M_1(\eta, \tau, S, T) m(\eta, \tau) R(\eta, \tau, M_1(\eta, \tau, S, T)) d\eta d\tau}, \end{aligned}$$

since the inequality (2.23) holds for all $(s, t) \in \Gamma_1$, taking $s = S$ and $t = T$, we have

$$(2.24) \quad \begin{aligned} x(S, T) &\leq M_1(S, T, S, T) + N_1(S, T, S, T) \\ &\times \frac{\int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, M_1(\eta, \tau, S, T)) d\eta d\tau}{1 - \int_0^A \int_0^B m(\eta, \tau) N_1(\eta, \tau, S, T) R(\eta, \tau, M_1(\eta, \tau, S, T)) d\eta d\tau} \\ &= M^*(S, T) + N^*(S, T) \times \frac{\int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau}{1 - \int_0^A \int_0^B N^*(\eta, \tau) m(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau}, \end{aligned}$$

for $(S, T) \in \Gamma$, where $M^*(S, T), N^*(S, T)$ are defined as in (2.13).

Taking into account the fact that t , S and T are arbitrary, we replace S and T by s and t , respectively, one gets the desired inequality. This completes the proof. $\square$

Remark 2. If we take $S(s, t, x(s, t)) = x(s, t)$, Theorem 1 will be reduced to Theorem 2 in [8] and if we replace $S(s, t, x(s, t))$ by $n(x(s, t))$ where n is defined as in Remark 1, one can easily derive Theorem 2.1 in [6].

Theorem 2. Assume $\alpha(s, t)$, $\beta(s, t)$, $\gamma(s, t)$, $\Psi(s, t)$ and $m(s, t)$ are defined as in Theorem 1. If $x(s, t) \in C(\Gamma, \mathbb{R}_+)$ satisfies

$$(2.25) \quad \begin{aligned} x^p(s, t) &\leq \alpha(s, t) + \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) x^q(\eta, \tau) d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau, \end{aligned}$$

where $p \geq q \geq 0$, $p \geq 1$ and p, q are constants, then

$$(2.26) \quad x(s, t) \leq M^*(s, t) + N^*(s, t) \times \frac{\int_0^A \int_0^B m^*(\eta, \tau) S(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau}{1 - \int_0^A \int_0^B N^*(\eta, \tau) m^*(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau},$$

holds for $(s, t) \in \Gamma$, where

$$(2.27) \quad \begin{aligned} M^*(s, t) &= M_1(s, t) \exp \left\{ B_1(s, t) \int_0^s \int_0^t \Psi^*(\eta, \tau) d\eta d\tau \right\}, \\ N^*(s, t) &= N_1(s, t) \exp \left\{ B_1(s, t) \int_0^s \int_0^t \Psi^*(\eta, \tau) d\eta d\tau \right\}. \end{aligned}$$

and

$$(2.28) \quad \begin{aligned} M_1(s, t) &= \frac{1}{p} K^{\frac{1-p}{p}} \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) \left[\frac{q}{p} K^{(q-p)/p} \alpha(\eta, \tau) + \frac{p-q}{p} K^{q/p} \right] d\eta d\tau \\ &\quad + \frac{1}{p} K^{\frac{1-p}{p}} \alpha(s, t) + \frac{p-1}{p} K^{\frac{1}{p}}, \\ B_1(s, t) &= \frac{q}{p} K^{(q-p)/p} \beta(s, t), \quad N_1(s, t) = \frac{1}{p} K^{\frac{1-p}{p}} \gamma(s, t), \\ \Psi^*(s, t) &= \Psi(s, t), \quad m^*(s, t) = m(s, t), \end{aligned}$$

and S (also R) is introduced as in Lemma 3, with

$$\int_0^A \int_0^B N^*(\eta, \tau) m^*(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau < 1.$$

Proof. Define a function $\theta(s, t)$ as follows :

$$(2.29) \quad \begin{aligned} \theta(s, t) &= \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) x^q(\eta, \tau) d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, x(\eta, \tau)) d\eta d\tau, \end{aligned}$$

for $(s, t) \in \Gamma$, then, from (2.29), we obtain

$$(2.30) \quad x^p(s, t) \leq \alpha(s, t) + \theta(s, t).$$

Applying Lemma 2, we have

$$(2.31) \quad \begin{aligned} x(s, t) &\leq (\alpha(s, t) + \theta(s, t))^{1/p} \leq \frac{1}{p} K^{\frac{1-p}{p}} (\alpha(s, t) + \theta(s, t)) + \frac{p-1}{p} K^{\frac{1}{p}} \\ &= \lambda(s, t). \\ x^q(s, t) &\leq (\alpha(s, t) + \theta(s, t))^{q/p} \leq \frac{q}{p} K^{(q-p)/p} (\alpha(s, t) + \theta(s, t)) \\ &\quad + \frac{p-q}{p} K^{q/p}, \\ \frac{1}{p} K^{\frac{1-p}{p}} \theta(s, t) &\leq \lambda(s, t). \end{aligned}$$

Using (2.31), we easily obtain

$$(2.32) \quad \begin{aligned} \theta(s, t) &\leq \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) \times \left[\frac{q}{p} K^{(q-p)/p} (\alpha(\eta, \tau) + \theta(\eta, \tau)) + \frac{p-q}{p} K^{q/p} \right] d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, \lambda(\eta, \tau)) d\eta d\tau, \end{aligned}$$

from the last inequality of (2.31), we deduce that

$$(2.33) \quad \begin{aligned} \theta(s, t) &\leq \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) \left[\frac{q}{p} K^{(q-p)/p} \alpha(\eta, \tau) + \frac{p-q}{p} K^{q/p} \right] d\eta d\tau \\ &\quad + q K^{(q-1)/p} \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) \lambda(\eta, \tau) d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, \lambda(\eta, \tau)) d\eta d\tau, \end{aligned}$$

By a simple computation, we obtain

$$\begin{aligned}
 \lambda(s, t) &\leq M_1(s, t) + B_1(s, t) \int_0^s \int_0^t \Psi^*(\eta, \tau) \lambda(\eta, \tau) d\eta d\tau \\
 (2.34) \quad &+ N_1(s, t) \int_0^A \int_0^B m^*(\eta, \tau) S(\eta, \tau, \lambda(\eta, \tau)) d\eta d\tau,
 \end{aligned}$$

where $A_1(s, t)$, $B_1(s, t)$, $N_1(s, t)$, $\Psi^*(s, t)$ and $m^*(s, t)$ are defined as in (2.28).

Remarking that $A_1(s, t)$, $B_1(s, t)$ and $N_1(s, t)$ are nonnegative, continuous, and nondecreasing functions for $(s, t) \in \Gamma$. A suitable application of Theorem 1 to (2.34) gives

$$\begin{aligned}
 x(s, t) &\leq \lambda(s, t) \leq M^*(s, t) + N^*(s, t) \\
 (2.35) \quad &\times \frac{\int_0^A \int_0^B m^*(\eta, \tau) S(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau}{1 - \int_0^A \int_0^B N^*(\eta, \tau) m^*(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau},
 \end{aligned}$$

where $M^*(s, t)$ and $N^*(s, t)$ are defined as in (2.27). □

Remark 3. If we take $S(s, t, x(s, t)) = x(s, t)$, $p = 2$, $q = 1$, Theorem 2 will be reduced to corollary 8 in [8] and if we replace $S(s, t, x(s, t))$ by $n(x(s, t))$ where n is defined as in Remark 1, one can easily derive Theorem 2.2 in [6].

3. ILLUSTRATIVE EXAMPLES

In this section, we apply some inequalities obtained in the previous sections to investigate certain properties of the solution of integral equation in two independent variables.

Example 1. Consider the following integral equation:

$$\begin{aligned}
 (3.1) \quad x(s, t) &= \alpha(s, t) + \beta(s, t) \int_0^s \int_0^t \Phi(\eta, \tau, x) d\eta d\tau + \gamma(s, t) \int_0^A \int_0^B H(\eta, \tau, x) d\eta d\tau,
 \end{aligned}$$

for $(s, t) \in \Gamma$, where $x(s, t) \in C(\Gamma, \mathbb{R})$, $\alpha(s, t)$, $\beta(s, t)$, $\gamma(s, t) \in C(\Gamma, \mathbb{R}_+)$ such that $\alpha(x, y)$, $\beta(s, t)$, $\gamma(s, t)$ are nondecreasing functions in s and t and $\Phi(s, t, x)$, $H(s, t, x) \in C(\Gamma \times \mathbb{R}, \mathbb{R})$.

Theorem 3. Assume that the functions Φ and H in (3.1) satisfy the conditions

$$(3.2) \quad |\Phi(s, t, x)| \leq \Psi(s, t) |x|, \quad |H(s, t, x)| \leq m(s, t)S(s, t, x),$$

where $S, R : \mathbb{R}_+^2 \times \mathbb{R} \rightarrow \mathbb{R}_+$ are nonnegative continuous function satisfy the following properties

$$(3.3) \quad \begin{aligned} S(s, t, v) &\leq S(s, t, u), \quad (s, t) \in \mathbb{R}_+^2, v \leq u, u, v \in \mathbb{R} \\ S(s, t, u) - S(s, t, v) &\leq R(s, t, v)(u - v), \quad (s, t) \in \mathbb{R}_+^2, 0 \leq v \leq u. \end{aligned}$$

where $\Psi(s, t)$ and $m(s, t)$ are defined as in Theorem 1.

If $x(s, t)$ is the unique solution of (3.1), then

$$(3.4) \quad |x(s, t)| \leq M^*(s, t) + N^*(s, t) \times \frac{\int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau}{1 - \int_0^A \int_0^B N^*(\eta, \tau) m(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau},$$

for $(s, t) \in \Gamma$ where $M^*(s, t), N^*(s, t)$ are defined as in (2.13), with

$$(3.5) \quad \int_0^A \int_0^B N^*(\eta, \tau) m(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) d\eta d\tau < 1.$$

Proof. Assume that $x(s, t)$ is the unique solution of (3.1), from (3.2) and (3.3) we have

$$(3.6) \quad \begin{aligned} |x(s, t)| &\leq \alpha(s, t) + \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) |x(\eta, \tau)| d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, |x(\eta, \tau)|) d\eta d\tau. \end{aligned}$$

Now by applying Theorem 1 to (3.6), the required estimation (3.4) is obtained. $\square$

Proposition 1. Assume that the functions F and H in (3.1) satisfy the conditions

$$(3.7) \quad |\Phi(s, t, x) - \Phi(s, t, \bar{x})| \leq \Psi(s, t) |x - \bar{x}|, \quad |H(s, t, x) - H(s, t, \bar{x})| \leq m(s, t)S(s, t, x - \bar{x}),$$

where $\Psi(s, t), m(s, t)$ are defined as in Theorem 1 with $S(s, t, 0) = 0$. If

$$\int_0^A \int_0^B N^*(\eta, \tau) m(\eta, \tau) R(\eta, \tau, M^*(\eta, \tau)) ds dt < 1,$$

where $M^*(s, t)$ and $N^*(s, t)$ are defined as in Theorem 1, and $x(s, t)$ is a solution of (3.1), then (3.1) has at most one solution.

Proof. Let $x(s, t)$ and $\bar{x}(s, t)$ be two solutions of (3.1), then

$$\begin{aligned} (3.8) \quad x(s, t) &= \alpha(s, t) + \beta(s, t) \int_0^s \int_0^t \Phi(\eta, \tau, x) d\eta d\tau + \gamma(s, t) \int_0^A \int_0^B H(\eta, \tau, x) d\eta d\tau, \\ \bar{x}(s, t) &= \alpha(\eta, \tau) + \beta(s, t) \int_0^s \int_0^t \Phi(\eta, \tau, \bar{x}) d\eta d\tau + \gamma(s, t) \int_0^A \int_0^B H(\eta, \tau, \bar{x}) d\eta d\tau, \end{aligned}$$

From (3.8), we have

$$\begin{aligned} (3.9) \quad |x(s, t) - \bar{x}(s, t)| &\leq \beta(s, t) \int_0^s \int_0^t |\Phi(\eta, \tau, x) - \Phi(\eta, \tau, \bar{x})| d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B |H(\eta, \tau, x) - H(\eta, \tau, \bar{x})| d\eta d\tau \\ &\leq \beta(s, t) \int_0^s \int_0^t \Psi(\eta, \tau) |x - \bar{x}| d\eta d\tau \\ &\quad + \gamma(s, t) \int_0^A \int_0^B m(\eta, \tau) S(\eta, \tau, |x - \bar{x}|) d\eta d\tau. \end{aligned}$$

According to Theorem 1, we obtain that $|x(s, t) - \bar{x}(s, t)| \leq 0$, which implies $x(s, t) = \bar{x}(s, t)$ for $(s, t) \in \Gamma$.

□

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