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YAMABE SOLITONS ON $(LCS)_n$-MANIFOLDS

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Abstract. The object of the present paper is to study some properties of $(LCS)_n$-manifolds whose metric is Yamabe soliton. We establish some characterization of $(LCS)_n$-manifolds when the soliton becomes steady. Next we have studied some certain curvature conditions of $(LCS)_n$-manifolds admitting Yamabe solitons. Lastly we construct a 3-dimensional $(LCS)_n$-manifold satisfying the results.

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1. Introduction

In 2003, the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) was introduced by Shaikh [12] with an example, that generalize the notion of LP-Sasakian manifolds which was introduced by Matsumoto [7] and also by Mihai and Rosca [8]. In 2005 and 2006, the application of $(LCS)_n$-manifolds to the general theory of relativity and cosmology was studied by Shaikh and Baishya [13], [14].

Many other authors M. Ateceken[1], S. K. Hui[5], D. Narain[9], S. Yadav([19], [20], [21], [22]), A. Shaikh([15], [16]) also studied the $(LCS)_n$-manifolds.

The concept of Yamabe flow was first introduced by Hamilton [4] to construct Yamabe metrics on compact Riemannian manifolds. On a Riemannian or pseudo-Riemannian manifold M, a time-dependent metric $g(\cdot, t)$ is said to evolve by the Yamabe flow if the metric g satisfies the given equation,

$$\frac{\partial}{\partial t} g(t) = -r g(t), \qquad g(0) = g_0, \tag{1}$$

where r is the scalar curvature of the manifold M.

In 2-dimension the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t} g(t) = -2S(g(t))$, where S denotes the Ricci tensor). But in dimension > 2 the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not in general.

A Yamabe soliton correspond to self-similar solution of the Yamabe flow, is defined on a Riemannian or pseudo-Riemannian manifold (M, g) by a vector field ξ satisfying the equation [2],

$$\frac{1}{2} \pounds_\xi g = (r - \lambda) g, \tag{2}$$

where $\pounds_\xi g$ denotes the Lie derivative of the metric g along the vector field ξ, r is the scalar curvature and λ is a constant. Moreover a Yamabe soliton is said to be expanding if $\lambda > 0$, steady if $\lambda = 0$ and shrinking if $\lambda < 0$.
Yamabe solitons on a three-dimensional Sasakian manifold were studied by R. Sharma [17].

Again from [18],

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{3}$$

$$\begin{aligned} H(X, Y)Z &= R(X, Y)Z - \frac{1}{(n-2)}[g(Y, Z)QX - g(X, Z)QY \\ &+ S(Y, Z)X - S(X, Z)Y], \end{aligned} \tag{4}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(QY,Z)X - g(QX,Z)Y], \tag{5}$$

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{6}$$

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \tag{7}$$

are the Riemannian-Christoffel curvature tensor R [10], the conharmonic curvature tensor H [6], the projective curvature tensor P [23], the concircular curvature tensor $\tilde{C}$ [11] and the W_2-curvature tensor [11] respectively in a Riemannian manifold (M^n, g), where Q is the Ricci operator, defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, $r = tr(S)$ is the scalar curvature, where $tr(S)$ is the trace of S and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

In the present paper we study Yamabe soliton on $(LCS)_n$-manifolds. The paper is organized as follows:
After introduction, section 2 is devoted for preliminaries on $(LCS)_n$-manifolds. In section 3, we have studied Yamabe soliton on $(LCS)_n$-manifolds. Here we examined when $(LCS)_n$-manifold admits Yamabe soliton, then the manifold becomes $K-(LCS)_n$-manifold and Ricci symmetric. In this section we have also shown that, $(LCS)_n$-manifold admits Yamabe soliton is ξ-projectively flat, ξ-concircularly flat and ξ-conharmonically flat iff the soliton becomes steady. Section 4 deals with curvature properties on $(LCS)_n$-manifold admitting Yamabe soliton. Here we obtained some results on Yamabe soliton satisfying the conditions of the following type:
$S(\xi, X)\cdot R = 0$, $S(\xi, X)\cdot W_2 = 0$, where W_2 is the W_2- curvature tensor and S is the Ricci tensor. Also we have found that if the manifold admits Yamabe soliton then $R(\xi, X)\cdot S = 0$ and $W_2(\xi, X)\cdot S = 0$.
In the last section, we gave an example of 3-dimensional $(LCS)_n$-manifold on which we can easily verify our results.

2. **Preliminaries**

Let(M, g) be an n-dimensional Lorentzian manifold [18] admitting a unit timelike concircular vector field ξ. Then the vector field satisfying $g(\xi, \xi) = -1$. Since ξ is a unit concircular vector field, it follows that there exists a nonzero 1-form η

such that, $g(X,\xi)=\eta(X)$. Also ξ satisfies $\nabla\xi=\alpha(I+\eta\otimes\xi)$ with a nowhere zero smooth function α on M verifying the equation $\nabla_X\alpha=(X\alpha)=d\alpha(X)=\rho\eta(X)$, for $\rho\in C^\infty(M)$, where ∇ is the Levi-Civita connection of g and X is a vector field. Here also ϕ is the (1,1) tensor field, denoted by, $\phi:=\frac{1}{\alpha}\nabla\xi$.
The notion of Lorentzian para-Sasakian manifold was introduced by K. Matsumoto [7]. More general the Lorentzian manifold M together with the unit timelike concircular vector field ξ, an 1-form η, and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold $(M,g,\xi,\eta,\phi,\alpha)$ (briefly, $(LCS)_n$-manifold), which was introduced by A. A. Shaikh [12].
In an n-dimensional $(LCS)_n$-manifold the following relations hold:

$$\phi^2=I+\eta\otimes\xi,\quad \eta(\xi)=-1,\quad \phi\xi=0,\quad \eta\circ\phi=0, \tag{8}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y)\ \ and\ \ g(\phi X,Y)=g(X,\phi Y), \tag{9}$$

$$(\nabla_X\phi)Y=\alpha[g(X,Y)\xi+2\eta(X)\eta(Y)\xi+\eta(Y)X], \tag{10}$$

for any $X,Y\in\chi(M)$.

$$\phi X=\frac{1}{\alpha}\nabla_X\xi, \tag{11}$$

$$\eta(\nabla_X\xi)=0,\quad \nabla_\xi\xi=0, \tag{12}$$

$$R(X,Y)Z=(\alpha^2-\rho)[g(Y,Z)X-g(X,Z)Y], \tag{13}$$

$$R(X,Y)\xi=(\alpha^2-\rho)[\eta(Y)X-\eta(X)Y], \tag{14}$$

$$R(\xi,X)Y=(\alpha^2-\rho)[g(X,Y)\xi-\eta(Y)X], \tag{15}$$

$$\eta(R(X,Y)Z)=(\alpha^2-\rho)[\eta(X)g(Y,Z)-\eta(Y)g(X,Z)], \tag{16}$$

for any $X,Y,Z\in\chi(M)$.

$$\eta(R(X,Y)\xi)=0, \tag{17}$$

$$S(X,Y)=(\alpha^2-\rho)(n-1)g(X,Y), \tag{18}$$

$$r=n(n-1)(\alpha^2-\rho), \tag{19}$$

$$\nabla \eta = \alpha(g + \eta \otimes \eta), \quad \nabla_\xi \eta = 0, \tag{20}$$

$$\pounds_\xi \phi = 0, \quad \pounds_\xi \eta = 0, \quad \pounds_\xi g = 2\nabla \eta = 2\alpha(g + \eta \otimes \eta), \tag{21}$$

where R is the Riemannian Curvature tensor, S is the Ricci tensor, r is the scalar curvature, ∇ is the Levi-Civita connection associated with g and $\pounds_\xi$ denotes the Lie derivative along the vector field ξ.

3. Yamabe soliton on $(LCS)_n$ manifold

Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$ manifold. Consider the Yamabe soliton on M as:

$$\frac{1}{2}\pounds_\xi g = (r - \lambda)g. \tag{22}$$

Then from (21), we get,

$$\alpha[g(X, Y) + \eta(X)\eta(Y)] = (r - \lambda)g(X, Y), \tag{23}$$

for all vector fields X, Y on M.
which implies that,

$$(r - \lambda - \alpha)g(X, Y) - \alpha\eta(X)\eta(Y) = 0. \tag{24}$$

Taking $Y = \xi$ in the above equation and using (8), we get,

$$(r - \lambda - \alpha)\eta(X) + \alpha\eta(X) = 0. \tag{25}$$

Then we have,

$$(r - \lambda)\eta(X) = 0. \tag{26}$$

Since $\eta(X) \neq 0$, so we get,

$$r = \lambda. \tag{27}$$

Using the above equation in (22), we have,

$$\pounds_\xi g = 0. \tag{28}$$

Thus ξ is a Killing vector field. Moreover, since λ is constant, the scalar curvature r is constant.

So we can state the following theorem:

Theorem 3.1. *If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton*

(g,ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, then the scalar curvature is constant and ξ becomes a Killing vector field.

Now from (19) and (27), we get,

$$\lambda = n(n-1)(\alpha^2 - \rho). \tag{29}$$

Then using (18) and (29), we obtain,

$$S(X,Y) = \frac{\lambda}{n} g(X,Y), \tag{30}$$

for all vector fields X, Y on M.

This leads to the following:

Proposition 3.2. *If an $(LCS)_n$ manifold $(M,g,\xi,\eta,\phi,\alpha)$ admits a Yamabe soliton (g,ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, then the manifold becomes Einstein manifold.*

Now replacing the expression of S from (30) in

$$(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)$$

we get,

$$(\nabla_X S)(Y,Z) = \frac{\lambda}{n}(\nabla_X g)(Y,Z), \tag{31}$$

which implies that,

$$\nabla S = 0. \tag{32}$$

This leads to the following:

Proposition 3.3. *If an $(LCS)_n$ manifold $(M,g,\xi,\eta,\phi,\alpha)$ admits a Yamabe soliton (g,ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, then the manifold becomes Ricci symmetric.*

Again let the Ricci tensor S of the $(LCS)_n$ manifold is η-recurrent i.e

$$\nabla S = \eta \otimes S,$$

which implies that,

$$(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z), \tag{33}$$

for all vector fields X, Y, Z on M.
Then using (32) and (30), we get

$$\frac{\lambda}{n}\eta(x)g(Y,Z) = 0. \tag{34}$$

Taking $Y = \xi, Z = \xi$ in the above equation we obtain,

$$\lambda\eta(X) = 0. \tag{35}$$

As $\eta(X) \neq 0$, hence $\lambda = 0$. Also from (27), we get $r = 0$.

This leads to the following:

Proposition 3.4. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure. If the Ricci tensor S of the manifold is η-recurrent (i.e $\nabla S = \eta \otimes S$), then the soliton is steady and the manifold becomes flat.*

Let us assume that a symmetric (0, 2) tensor field $h = \pounds_\xi g - 2rg$ is parallel with respect to the Levi-Civita connection associated to g.
Then

$$h(\xi,\xi) = \pounds_\xi g(\xi,\xi) - 2rg(\xi.\xi) = 2\lambda, \tag{36}$$

implies that,

$$\lambda = \frac{1}{2}h(\xi,\xi). \tag{37}$$

Now as h is parallel with respect to g, then from [3], we get,

$$h(X,Y) = -h(\xi,\xi)g(X,Y) \tag{38}$$

for all vector fields X, Y on M.
which leads to,

$$\pounds_\xi g(X,Y) = 2(r-\lambda)g(X,Y). \tag{39}$$

So we can state the following theorem:

Theorem 3.5. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold. Assume that a symmetric (0, 2) tensor field $h = \pounds_\xi g - 2rg$ is parallel with respect to the Levi-Civita connection associated to g. Then (g, ξ) yields an Yamabe soliton on M.*

We know,

$$(\nabla_\xi Q)X = \nabla_\xi QX - Q(\nabla_\xi X), \tag{40}$$

and

$$(\nabla_\xi S)(X,Y) = \xi S(X,Y) - S(\nabla_\xi X, Y) - S(X, \nabla_\xi Y), \tag{41}$$

for any vector fields X, Y on M.
Now using (30) we obtain,

$$QX = \frac{\lambda}{n}X, \tag{42}$$

for any vector field X on M.
Then in view of (30) and (42), the equations (40) and (41) become

$$(\nabla_\xi Q)X = 0, \tag{43}$$

$$(\nabla_\xi S)(X,Y) = 0, \tag{44}$$

respectively, for any vector fields X, Y on M.

This leads to the following:
Theorem 3.6. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure. Then Q and S are parallel along ξ, where Q is the Ricci operator, defined by $S(X,Y) = g(QX, Y)$ and S is the Ricci tensor of M.*

Also in view of (42), we obtain,

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y) = 0, \tag{45}$$

for any vector fields X, Y on M.

Then we have,
Corollary 3.7. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure. Then Q is parallel to any arbitrary vector field on M.*

Let a Yamabe soliton is defined on an n-dimensional $(LCS)_n$ manifold M as,

$$\frac{1}{2}\pounds_V g = (r - \lambda)g \tag{46}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along a vector field V and r, λ are defined as (2).
Let V be pointwise co-linear with ξ i.e. $V = b\xi$ where b is a function on M.
Then the equation (46) becomes,

$$\pounds_{b\xi} g(X,Y) = 2(r - \lambda)g(X,Y), \tag{47}$$

for any vector fields X, Y on M.
Applying the property of Lie derivative and Levi-Civita connection we have,

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) = 2(r - \lambda)g(X,Y). \tag{48}$$

Using (11), the above equation reduces to,

$$b\alpha g(\phi X, Y) + (Xb)\eta(Y) + b\alpha g(\phi Y, X) + (Yb)\eta(X) = 2(r - \lambda)g(X,Y). \tag{49}$$

Taking $Y = \xi$ in the above equation and using (8), we get,

$$-Xb + (\xi b)\eta(X) = 2(r - \lambda)\eta(X). \tag{50}$$

Again putting $X = \xi$ in the above equation, we obtain,

$$\xi b = r - \lambda. \tag{51}$$

Then using (51), (50) becomes,

$$Xb = -(r - \lambda)\eta(X). \tag{52}$$

Applying exterior differentiation in (52), we have,

$$(r - \lambda)d\eta = 0. \tag{53}$$

Now in an n-dimensional $(LCS)_n$ manifold we have,

$$(d\eta)(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]),$$

which implies

$$\begin{aligned}(d\eta)(X,Y) &= g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi)\\ &= \alpha g(Y,X) + \eta(X)\eta(Y) - \alpha g(X,Y) + \eta(X)\eta(Y)\\ &= 0. \qquad (54)\end{aligned}$$

Hence the 1-form η is closed.
Then using the above equation, (53) implies that, either $r \neq \lambda$ or $r = \lambda$.
Now if $r \neq \lambda$ then from (46), we have,

$$\pounds_V g = 2(r-\lambda)g, \qquad (55)$$

which implies V is a conformal Killing vector field.
Again if $r = \lambda$ then from (52), we get,

$$Xb = 0, \qquad (56)$$

implies that b is constant.

So we can state the following theorem:
Theorem 3.8. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton (g, V), V being a vector field on M. If V is pointwise co-linear with ξ then either V is a conformal Killing vector field, provided $r \neq \lambda$, or V is a constant multiple of ξ, where ξ being the Reeb vector field of the Lorentzian concircular structure, r is the scalar curvature and λ is a constant.*

Also if $r = \lambda$ then from (46), we obtain,

$$\pounds_V g = 0, \qquad (57)$$

implies that V is a Killing vector field.

Then we have,
Corollary 3.9. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$ manifold, admitting a Yamabe soliton (g, V), V being a vector field on M. If V is pointwise co-linear with ξ and $r = \lambda$ then V becomes a Killing vector field, where ξ being the Reeb vector field of the Lorentzian concircular structure, r is the scalar curvature and λ is a constant.*

From the definition of projective curvature tensor (5), defined on an n-dimensional $(LCS)_n$ manifold, we have,

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y], \tag{58}$$

for any vector fields X, Y, Z on M.
Putting $Z = \xi$, we get,

$$P(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-1)}[S(Y,\xi)X - S(X,\xi)Y]. \tag{59}$$

Using (14) and (30), we obtain,

$$P(X,Y)\xi = [(\alpha^2 - \rho) - \frac{\lambda}{n(n-1)}][\eta(Y)X - \eta(X)Y]. \tag{60}$$

Again using (29), we get,

$$P(X,Y)\xi = 0. \tag{61}$$

This leads to the following:

Proposition 3.10. *An $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$, admitting a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, is ξ-projectively flat.*

From the definition of concircular curvature tensor (6), defined on an n-dimensional $(LCS)_n$ manifold, we have,

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{62}$$

for any vector fields X, Y, Z on M.
Putting $Z = \xi$, we get,

$$\tilde{C}(X,Y)\xi = R(X,Y)\xi - \frac{r}{n(n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \tag{63}$$

Using (14) and (27), we obtain,

$$\tilde{C}(X,Y)\xi = [(\alpha^2 - \rho) - \frac{\lambda}{n(n-1)}][\eta(Y)X - \eta(X)Y]. \tag{64}$$

Again using (29), we get,

$$\tilde{C}(X,Y)\xi = 0. \tag{65}$$

This leads to the following:

Proposition 3.11. *An $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$, admitting a Yamabe* *soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, is ξ-concircularly flat.*

From the definition of conharmonic curvature tensor (4), defined on an n-dimensional $(LCS)_n$ manifold, we have,

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y], \end{aligned} \tag{66}$$

for any vector fields X, Y, Z on M.
Putting $Z = \xi$, we get,

$$\begin{aligned} H(X,Y)\xi &= R(X,Y)\xi - \frac{1}{(n-2)}[g(Y,\xi)QX - g(X,\xi)QY \\ &+ S(Y,\xi)X - S(X,\xi)Y], \end{aligned} \tag{67}$$

Using (14), (30) and (42), we obtain,

$$H(X,Y)\xi = [(\alpha^2 - \rho) - \frac{2\lambda}{n(n-2)}][\eta(Y)X - \eta(X)Y]. \tag{68}$$

Again using (29), we get,

$$H(X,Y)\xi = -\frac{\lambda}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y]. \tag{69}$$

This implies that $H(X,Y)\xi = 0$ iff $\lambda = 0$.

This leads to the following:
Proposition 3.12. *An $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$, admitting a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, is ξ-conharmonically flat iff the soliton is steady.*

4. Curvature properties on $(LCS)_n$ manifold admitting Yamabe soliton

We know,

$$R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z), \tag{70}$$

for any vector fields X, Y, Z on M.
Using (15), we obtain,
(71)
$$R(\xi, X)\cdot S = S((\alpha^2-\rho)(g(X,Y)\xi-\eta(Y)X), Z)+S(Y, (\alpha^2-\rho)(g(X,Z)\xi-\eta(Z)X)).$$

Then using (30), we get,

$$\begin{aligned} R(\xi, X)\cdot S &= \frac{\lambda}{n}(\alpha^2-\rho)[g(X,Y)\eta(Z) \\ &- g(X,Z)\eta(Y)+g(X.Z)\eta(Y)-g(X,Y)\eta(Z)], \end{aligned} \tag{72}$$

which implies that,

$$R(\xi, X)\cdot S = 0.$$

So we can state the following theorem:
Theorem 4.1. *If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, then $R(\xi, X)\cdot S = 0$, i.e the manifold is ξ-semi symmetric, where R is the Riemannian curvature tensor and S is the Ricci tensor.*

Again the condition $S(\xi, X)\cdot R = 0$ implies that,

(73)
$$\begin{aligned} &S(X, R(Y,Z)W)\xi - S(\xi, R(Y,Z)W)X + S(X,Y)R(\xi,Z)W - S(\xi,Y)R(X,Z)W \\ &+ S(X,Z)R(Y,\xi)W - S(\xi,Z)R(Y,X)W + S(X,W)R(Y,Z)\xi - S(\xi,W)R(Y,Z)X \\ &= 0, \end{aligned}$$

for any vector fields X, Y, Z, W on M.
Taking the inner product with ξ, the above equation becomes,

$$\begin{aligned} &- S(X, R(Y,Z)W) - S(\xi, R(Y,Z)W)\eta(X) + S(X,Y)\eta(R(\xi,Z)W) \\ &- S(\xi,Y)\eta(R(X,Z)W) + S(X,Z)\eta(R(Y,\xi)W) - S(\xi,Z)\eta(R(Y,X)W) \\ &+ S(X,W)\eta(R(Y,Z)\xi) - S(\xi,W)\eta(R(Y,Z)X) = 0. \end{aligned} \tag{74}$$

Replacing the expression of S from (30) and taking $Z = \xi$, $W = \xi$, we get,

$$\begin{aligned}
\frac{\lambda}{n}[&-g(X,R(Y,\xi)\xi) - \eta(R(Y,\xi)\xi)\eta(X) + g(X,Y)\eta(R(\xi,\xi)\xi) \\
&- \eta(Y)\eta(R(X,\xi)\xi) + \eta(X)\eta(R(Y,\xi)\xi) - \eta(\xi)\eta(R(Y,X)\xi) + \eta(X)\eta(R(Y,\xi)\xi) \\
&- \eta(\xi)\eta(R(Y,\xi)X)] = 0.
\end{aligned} \tag{75}$$

Now using (14), (16), (17), we get on simplification,

$$\frac{\lambda}{n}(\alpha^2 - \rho)[g(X,Y) + \eta(x)\eta(Y)] = 0. \tag{76}$$

Using (9), the above equation becomes,

$$\frac{\lambda}{n}(\alpha^2 - \rho)g(\phi X, \phi Y) = 0, \tag{77}$$

for any vector fields X, Y on M.
This implies that,

$$\frac{\lambda}{n}(\alpha^2 - \rho) = 0. \tag{78}$$

Then using (29), we get,

$$\frac{\lambda^2}{n^2(n-1)} = 0,$$

implies that $\lambda = 0$.
Hence using (27), $r = 0$.

So we can state the following theorem:
Theorem 4.2. *If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure and satisfies $S(\xi, X) \cdot R = 0$ then the manifold becomes flat and the soliton is steady, where R is the Riemannian curvature tensor and S is the Ricci tensor.*

We know,

$$W_2(\xi, X) \cdot S = S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z), \tag{79}$$

for any vector fields X, Y, Z on M.
Replacing the expression of S from (30) and using the definition of W_2-curvature

tensor from (7), we get,

$$
\begin{aligned}
W_2(\xi, X)\cdot S &= \frac{\lambda}{n} g(R(\xi, X)Y + \frac{1}{n-1}[g(\xi, Y)QX - g(X, Y)Q\xi], Z) \\
&+ \frac{\lambda}{n} g(Y, R(\xi, X)Z + \frac{1}{n-1}[g(\xi, Z)QX - g(X, Z)Q\xi]).
\end{aligned} \tag{80}
$$

Now using (15) and the property $g(QX, Y) = S(X, Y)$, we obtain on simplification,

$$
\begin{aligned}
W_2(\xi, X)\cdot S &= \frac{\lambda}{n(n-1)}[\eta(Y)S(X, Z) - S(\xi, Z)g(X, Y) \\
&+ \eta(Z)S(X, Y) - S(\xi, Y)g(X, Z)].
\end{aligned} \tag{81}
$$

Then using (30), the above equation becomes,

$$
W_2(\xi, X)\cdot S = 0. \tag{82}
$$

So we can state the following theorem:

Theorem 4.3. *If an $(LCS)_n$ manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, then $W_2(\xi, X)\cdot S = 0$, where W_2 is the W_2- curvature tensor and S is the Ricci tensor.*

Again the condition $S(\xi, X)\cdot W_2 = 0$ implies that,

$$
\begin{aligned}
&S(X, W_2(Y, Z)V)\xi - S(\xi, W_2(Y, Z)V)X + S(X, Y)W_2(\xi, Z)V \\
&\quad - S(\xi, Y)W_2(X, Z)V + S(X, Z)W_2(Y, \xi)V - S(\xi, Z)W_2(Y, X)V \\
&\qquad + S(X, V)W_2(Y, Z)\xi - S(\xi, V)W_2(Y, Z)X = 0,
\end{aligned} \tag{83}
$$

for any vector fields X, Y, Z, V on M.

Taking the inner product with ξ, the above equation becomes,

$$
\begin{aligned}
&- S(X, W_2(Y, Z)V) - S(\xi, W_2(Y, Z)V)\eta(X) + S(X, Y)\eta(W_2(\xi, Z)V) \\
&\quad - S(\xi, Y)\eta(W_2(X, Z)V) + S(X, Z)\eta(W_2(Y, \xi)V) - S(\xi, Z)\eta(W_2(Y, X)V) \\
&\qquad + S(X, V)\eta(W_2(Y, Z)\xi) - S(\xi, V)\eta(W_2(Y, Z)X) = 0.
\end{aligned} \tag{84}
$$

Replacing the expression of S from (30) and taking $Z=\xi$, $V=\xi$, we get,

$$\begin{aligned}(85)\quad &\frac{\lambda}{n}[-g(X,W_2(Y,\xi)\xi)-\eta(W_2(Y,\xi)\xi)\eta(X)+g(X,Y)\eta(W_2(\xi,\xi)\xi)\\ &-\eta(Y)\eta(W_2(X,\xi)\xi)+\eta(X)\eta(W_2(Y,\xi)\xi)-\eta(\xi)\eta(W_2(Y,X)\xi)+\eta(X)\eta(W_2(Y,\xi)\xi)\\ &-\eta(\xi)\eta(W_2(Y,\xi)X)]=0.\end{aligned}$$

Now using (7), (14), (16), (18), we obtain on simplification,

$$\frac{\lambda}{n}[g(X,Y)+\eta(X)\eta(Y)-(\alpha^2-\rho)g(X,Y)-(\alpha^2-\rho)\eta(X)\eta(Y)]=0, \tag{86}$$

implies that,

$$\frac{\lambda}{n}(1-\alpha^2+\rho)[g(X,Y)+\eta(x)\eta(Y)]=0. \tag{87}$$

Using (9), the above equation becomes,

$$\frac{\lambda}{n}(1-\alpha^2+\rho)g(\phi X,\phi Y)=0. \tag{88}$$

for any vector fields X,Y on M.
This implies that,

$$\lambda(1-\alpha^2+\rho)=0.$$

Then either $\lambda=0$, or $\alpha^2-\rho=1$.
Now if $\alpha^2-\rho=1$, then from (19), we have,

$$r=n(n-1).$$

So we can state the following theorem:
Theorem 4.4. *If an $(LCS)_n$ manifold $(M,g,\xi,\eta,\phi,\alpha)$ admits a Yamabe soliton (g,ξ), ξ being the Reeb vector field of the Lorentzian concircular structure and satisfies $S(\xi,X)\cdot W_2=0$ then either the soliton is steady, or $r=n(n-1)$, where W_2 is the W_2- curvature tensor and r is the scalar curvature.*

5. Example of an $(LCS)_3$ manifold satisfying a Yamabe soliton :

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, e_3 be a linearly independent system of vector fields on M given by,

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by,

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1,$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$,where $\chi(M)$ is the set of all differentiable vector fields on M and ϕ be the (1, 1)-tensor field defined by,

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Then, using the linearity of ϕ and g, we have

$$\eta(e_3) = -1, \phi^2(Z) = Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have,

$$[e_1, e_2] = 0, [e_2, e_3] = -e_2, [e_1, e_3] = -e_1.$$

The connection ∇ of the metric g is given by,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszuls formula.

Using Koszuls formula, we can easily calculate,

$$\nabla_{e_1} e_3 = -e_1, \nabla_{e_2} e_3 = -e_2, \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_1 = 0,$$

$$\nabla_{e_1} e_2 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_3} e_2 = 0.$$

Hence in this case $(g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_3$-structure on M, where $\alpha = -1$.
Also as $\alpha = -1$ then $\rho = 0$ and consequently, $(M, g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_n$ manifold of dimension 3.
Now from (19), $r = 6$.
Let us consider g defines a Yamabe soliton on M.
Then from (23) and by putting the value of α and r, we have,

$$(7 - \lambda)g(X, Y) + \eta(X)\eta(Y) = 0.$$

Taking $X = Y = \xi$, the above equation gives $\lambda = 6$.
Hence $\lambda = r = 6$, which satisfes (27).

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