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WEIGHTED EQUILIBRIUM STATES AT ZERO TEMPERATURE

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Abstract. Weighted thermodynamic and multifractal formalism for finite alphabet subshifts have been presented by Barral and Feng (*Asian J. Math,* 16, 319–352, 2012). Here we analyze the problem of finding weighted equilibrium states for infinite countable shifts. Also we study the problem of "zero temperature" which consists into consider a sequence of equilibrium states depending of a parameter, interpreted in a Statistical Mechanics context as "the inverse of the temperature", and prove the existence of accumulation points of such a sequence.

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1. Introduction

Thermodynamic formalism, developed by Ruelle, is a theory which emerged as a mathematical rigorization of physical thermodynamic laws. This theory studies the relationships among several quantities like the topological pressure $P(\varphi)$ of a potential φ and the measure-theoretic entropy h_μ of a measure μ. One of the main problems is to analyze the existence and uniqueness of *equilibrium states,* i.e. measures μ which maximize the expression $h_\mu + \int \varphi d\mu$. Thus, by the variational principle, an equilibrium state μ_φ, associated to a potential φ, is that which verifies $P(\varphi) = h_{\mu_\varphi} + \int \varphi d\mu_\varphi$. This leads to the analysis of existence or absence of phase transitions. Another related problem refers to the existence and properties of sequences $\{\mu_{q\varphi}\}_{q>1}$ of equilibrium states associated to potentials $q\varphi$. The issue is to analyze the existence of accumulation points and the behavior for $q \to \infty$ of the sequence $\{\mu_{q\varphi}\}$. Accumulation points are called states at zero temperature. The name is due to the interpretation, within the field of Statistical Mechanics, of the parameter q as the inverse of the temperature. Thus, in this ambit, accumulation points are considered as ground states. The existence of states at zero temperature μ_∞ was established, for finitely primitive shifts with countable alphabet and adequate potentials, by Jenkinson, Mauldin and Urbanski[8] and by Freire and Vargas[6] for countable topologically transitive shifts (that in Statistical Mechanics means infinite spins) and potential of summable variation. In particular, Jenkinson, Mauldin and Urbanski have proved that the accumulation point μ_∞ of the sequence of states is a maximizing measure, i.e. $\int \varphi d\mu \leq \int \varphi d\mu_\infty$, for any map φ[8]. Freire and Vargas, for their part, proved the existences of equilibrium states for each potential $q\varphi$ and that μ_∞ maximizes the measure-theoretic entropy[6]. In particular, in that article, the big image and preimage (BIP) property is not imposed, so the results obtained do not depend on the existence of Gibbs states. The problem of zero temperature states is also considered in the works of Kempton[9], Morris[13] and Leplaideur[10] which show that, in order to ensure the existence of equilibrium or Gibbs states, conditions on the dynamics and the potential should be imposed. We must mention that, in the compact case, equilibrium and Gibbs states agree.

Lately the problem of studying the thermodynamics properties with a vector "weight" for finite sequences of systems has received some attention. Weighted versions of invariants like for instance entropy, were introduced by Feng and Huang[4] who in this way extended the Dimension Theory of sets on the torus. Further were established weighted versions of results in thermodynamics and Dynamical Systems. For instance a weighed version of Shannon-McMillan theorem and a variational principle have been proved in ref.[4] whereas a weighted extension of the result of Bowen[3], which relates weighted measure-theoretic entropy with weighted topological entropy of generic points, was given by Zhao et al.[15]. Recently we have proved that, under a property of weighted specification, weighted saturation holds for any measure[12]. In ref.[1] Barral and Feng developed a weighted thermodynamic formalism for shifts with finite alphabet (compact case) continuing the studies of Feng in ref. [4].

In this article we analyze the case of countable alphabet shifts. Thermodynamic formalism for countable shifts were developed independently by Mauldin-Urbanski[11] and Sarig[14]. We study the possibility of getting weighted equilibrium states for each potential $q\varphi$ and consequently accumulation points of the sequence $\{\mu_{q\varphi}\}_{q>1}$. The strategy is similar to that considered in [6], which essentially consists into approximate the states of countable shift by equilibrium states of finite alphabet shifts, taking advantage of the fact that we have to our disposal the formalism developed by Barral and Feng.

2. Preliminary definitions and statement of the results

Let X be a subshift with alphabet Ω, finite or countable, and a transition or transference matrix A. Let the string $i_1 i_2 ... i_n$, where $i_j \in \Omega$. For this string we define the cylinder

$$\mathcal{C}\left(i_1 i_2 ... i_n\right) = \left\{x \in X,\ x = x_1 x_2 ...,\ \text{with } x_\ell = i_\ell,\ \ell = 1, ..., n\right\}.$$

The symbolic space X is endowed with the product topology of the discrete topology in $\mathbf{\Omega}$, or equivalently the topology generated by cylinders $\mathcal{C}\left(i_1 i_2 ... i_n\right)$. If $i = i_1 i_2 ... i_n \in \Omega^n$, its length is denoted by$|\ i| = n$. A string $\underline{i} = i_1 i_2 ... i_n$ is called *admissible* if $\mathcal{C}\left(i_1 i_2 ... i_n\right) \neq \varnothing$. By $\mathcal{L}_n(X)$ is denoted the set of admissible sequences of length n and $\mathcal{L}(X) = \bigcup\limits_{n \geq 1} \mathcal{L}_n(X)$. The set $\mathcal{C} = \left\{\mathcal{C}\left(1\right), \mathcal{C}\left(2\right), ...\right\}$ formed by the cylinders in X of length 1 is a partition of X, called the natural partition.

In the space X is defined the metric $d(x, y) = t^{-N(x,y)}$ where $N(x, y) =$ first coordinate in which differ x and y, and $0 < t < 1$.

If $\sigma : X \to X$ is the shift map then by $\mathcal{M}_{inv}(X, \sigma)$ is denoted the set of $\sigma-$invariant measures on X, endowed with the weak topology.

Definition 1: Let $\mu \in \mathcal{M}_{inv}(X,\sigma)$, let $\mathcal{C} = \{\mathcal{C}(1), \mathcal{C}(2), ...\}$ be the natural partition and let $\mathcal{C}^n$ be the partition whose members are cylinders $\mathcal{C}(i_1 i_2 ... i_n)$. Thus set

$$h_\mu(\sigma, \mathcal{C}^n) = \sum_{i_1 i_2 ... i_n \in \mathcal{L}_n(X).} \mu(\mathcal{C}(i_1 i_2 ... i_n)) \log \mu(\mathcal{C}(i_1 i_2 ... i_n)).$$

Then the *measure-theoretic entropy* of μ with respect to the shift (X,σ) is

$$h_\mu(\sigma) = \lim_{n\to\infty} -\frac{1}{n} h_\mu(\sigma, \mathcal{C}^n).$$

Let $(X_i, \sigma_i, \Omega_i)$ with $i = 1, 2, ..., k$ and $k \geq 2$, be a finite family of dynamical systems in which X_i is a subshift over a countable alphabet Ω_i, and $\sigma_i : X_i \to\to X_i$ is the shift map. Let us assume that each (X_{i+1}, σ_{i+1}) is a factor of (X_i, σ_i), where factor map is denoted by $\pi_i : X_i \to X_{i+1}$(which is assumed to be a one-block factor map), i.e. $\sigma_{i+1} \circ \pi_i = \pi_i \circ \sigma_i$, $i = 1, 2, ..., k-1$. Then define the maps $\tau_i : X_1 \to X_{i+1}$, by $\tau_i = \pi_i \circ ... \circ \pi_1$. If $\mu \in \mathcal{M}_{inv}(X_1, \sigma_1)$ by $(\tau_{i-1})_*(\mu)$ is denoted the push-forward of the measure μ, then $(\tau_{i-1})_*(\mu)(E) = \mu\left(\tau_{i-1}^{-1}(E)\right)$ for any $E \subset X_i$.

Definition 2: Let $\mathbf{a} = (a_1, ..., a_k) \in \mathbf{R}^k$, the $\mathbf{a}-$*weighted measure-theoretic entropy* of μ with respect to (X_1, σ_1) is

$$h_\mu^{\mathbf{a}}(\sigma_1) = \sum_{i=1}^{k} a_i h_{(\tau_{i-1})_*(\mu)}(\sigma_i). \tag{1}$$

By a potential is understood a map $\varphi : X \to \mathbf{R}$. Thus for a potential φ let

$$V_n(\varphi) = \sup_{i_1 i_2 ... i_n \in \mathbf{N}^n} \sup_{x,y \in \mathcal{C}(i_1 i_2 ... i_n)} \{|\varphi(x) - \varphi(y)|\},$$

which is called the $n-$variation of φ. The potential φ is of *summable variation* if $S(\varphi) := \sum_{n=1}^{\infty} V_n(\varphi) < \infty$. A potential φ is called *summable if* $\sum_{n=1}^{\infty} \exp(\sup \varphi \mid \mathcal{C}(i)) < \infty$. A topological Markov shift is *topologically transitive* if for any symbols $i_1, i_2 \in \Omega$ there is an admissible string $\underline{a} = a_1 a_2 ... a_n$ with $a_1 = i_1$, $a_n = i_2$.

Let us recall the definition of the Gurevich pressure [7],[14] for a shift (X,σ) and for a potential φ of summable variation: let j_1 be a symbol in the alphabet of the shift, the "partition function" is defined as

$$Z_n(\varphi, \sigma) = Z_n(\varphi, \sigma, j_1) = \sum_{x \in P_n(\sigma)} \exp(S_n(\varphi)(x)) I_{C(j_1)},$$

where $S_n(\varphi)(x) = \sum_{i=0}^{n} \varphi\left(\sigma^i(x)\right)$ (called the *statistical sum*), I_E is the characteristic function of the set E, by $P_n(\sigma)$ is denoted the set of $n-$periodic points of σ, i.e. $P_n(\sigma) = \{x : \sigma^n(x) = x\}$. If the shift is topologically transitive the expression does not depend on the symbol considered, then

$$P(\varphi) = P(\varphi, \sigma, j_1) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, \sigma).$$

The writing of the symbol may be omitted for topologically transitive spaces: it is understood that some symbol is assumed.

The variational principle[14] in this case states: let φ be a potential of summable variation and (X, σ) a topologically transitive, then

$$P(\varphi) = \sup\left\{h_\mu(\sigma) + \int \varphi d\mu : \int \varphi d\mu > -\infty \text{ and } \mu \in \mathcal{M}_{inv}(X, \sigma)\right\}. \tag{2}$$

A measure μ_φ such that

$$h_{\mu_{\varphi.}}(\sigma) + \int \varphi d\mu = \sup\left\{h_\mu(\sigma) + \int \varphi d\mu : \int \varphi d\mu > -\infty \text{ and } \mu \in \mathcal{M}_{inv}(X, \sigma)\right\}$$

is called an *equilibrium state associated to the potential* $\varphi \in$ By $\mathcal{I}(\varphi)$ is denoted the set of equilibrium sates associated to the potential φ.

In the version of the variational principle considered in ref.[11] is not required summable variation for the potential, but it is imposed that the shift to be finitely irreducible.

Another result is the *approximation property*: under the same hypothesis as for the variational principle

$$P(\varphi) = \sup\{P(\varphi \mid Y) : Y \text{ is a compact (finite alphabet) subshift of } (X, \sigma)\}. \tag{3}$$

The approximation property can be presented in the following equivalent form: let $Y_m = \{x \in X : x_n \leq m\}$, then

$$P(\varphi) = \sup\{P(\varphi \mid Y_m) : m \in \mathbf{Z}\}. \tag{4}$$

Definition 3: Let $(X_1, \sigma_1, \Omega_1), (X_2, \sigma_2, \Omega_2)$ be countable shifts and let $\pi : X_1 \to X_2$ be the factor map, a measure $\mu \in \mathcal{M}_{inv}(X_1, \sigma_1)$ is a conditional equilibrium state for a potential φ with respect to a measure $\nu = \pi_*(\mu) \in \mathcal{M}_{inv}(X_2, \sigma_2)$ if

$$h_{\mu.}(\sigma_1) + \int \varphi d\mu = \sup \left\{ h_m(\sigma_1) + \int \varphi dm : \int \varphi d\mu > -\infty \text{ and } \nu = \pi_*(m) \right\}.$$

The set of conditional equilibrium sates of φ with respect to a measure ν is denoted by $\mathcal{I}_\nu(\varphi)$.

We introduce here the weighted pressure in a variational way; in the next section we prove that the variational expression can be given by an average of a partition function.

Let $(X_i, \sigma_i, \Omega_i)$ with $i = 1, 2, ..., k$ and $k \geq 2$ be a finite family of countable shifts and $\mathbf{a} = (a_1, ..., a_k) \in \mathbf{R}^k$. Also, let $\varphi : X_1 \to \mathbf{R}$ be a potential of summable variation and (X_1, σ_1) be a topologically transitive shift, thus the *weighted pressure of* φ and σ_1 is

$$(5) \quad P^{\mathbf{a}}(\varphi, \sigma_1) = \sup \left\{ h_\mu^{\mathbf{a}}(\sigma_1) + \int \varphi d\mu : \int \varphi d\mu > -\infty \text{ and } \mu \in \mathcal{M}_{inv}(X_1, \sigma_1) \right\}.$$

A measure μ_φ such that

$$h_{\mu_\varphi}^{\mathbf{a}}(\sigma) + \int \varphi d\mu_\varphi = \sup \left\{ h_\mu(\sigma) + \int \varphi d\mu : \int \varphi d\mu > -\infty \text{ and } \mu \in \mathcal{M}_{inv}(X_1, \sigma_1) \right\}$$

is called *weighted equilibrium state, or* $\mathbf{a}-$*equilibrium* state, associated to the potential φ. By $\mathcal{I}(\varphi, \mathbf{a})$ we denote the set of weighted equilibrium states or $\mathbf{a}-$*equilibrium* states associated to the potential φ.

Definition 4: Let $\{\mu_{q\varphi}\}_{q>1}$ be a sequence of equilibrium states for the potentials $q\varphi$. If the space of measures is endowed with the weak topology, then the limit for $q \to \infty$ of the sequence $\{\mu_{q\varphi}\}_{q>1}$, if it exists, or else an accumulation point of this sequence is called a *weighted equilibrium state at zero temperature.*

Theorem A ([6]): Let X be a topologically transitive countable shift and let $\varphi : X \to \mathbf{R}$ be a summable potential with $S(\varphi) < \infty$ and $P(\varphi) < \infty$. Then, for any $q > 1$, there is an unique equilibrium state $\mu_{q\varphi}$ associated to each potential $q\varphi$ and the sequence $\{\mu_{q\varphi}\}_{q>1}$ has an accumulation point.

We introduce a sort of weighted summability condition of a potential $\varphi : X_1 \to \mathbf{R}$ and a family of countable shifts $(X_i, \sigma_i, \Omega_i)$, $i = 1, 2$. Let $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2$ and $\pi : X_1 \to X_2$ be the factor map, the mentioned condition is

$$\sum_{n=1}^{\infty} \left[\exp\left(\frac{1}{a_1} \sup(\varphi \mid \mathcal{C}(i) \right) + \left(\frac{1}{a_1 + a_2} - \frac{1}{a_1} \right) \sum_{r\,:\,\pi(i)=\pi(r)} (\sup(\varphi \mid \mathcal{C}(r)) \right] < \infty.$$

We may call to this class of potentials potentials weighted summable.

Our objective is to obtain a weighted version of the theorem A.

Theorem 1: Let $(X_i, \sigma_i, \Omega_i)$, $i = 1, 2$ be a pair of countable shifts, with X_1, X_2 be topologically transitive shifts. If X_1, is a full, or Bernoulli, shift and $\varphi : X_1 \to \mathbf{R}$ is a weighted summable potential, then for any $q > 1$, each potential $q\varphi$ has an unique weighted equilibrium state $\mu_{q\varphi}$. Besides the sequence $\{\mu_{q\varphi}\}_{q>1}$ has an accumulation point , or a state at zero temperature.

3. Proof of the theorem

In all cases is assumed that the potential is of weighted summable variation and the shifts are topologically transitive.

Next proposition shows how the weighted and non-weighted pressures for $k = 2$ are related, its proof being similar to that given in ref.[1]. Before we introduce some notation.

Let $Sum(X_i\ )$ be the set of potentials of summable variation defined on X_i $(i = 1, 2)$ and $\eta : Sum(X_1\ ) \to Sum(X_2\ )$ such that $\varphi \mapsto \psi = \eta(\varphi)$, with the potential ψ characterized by the fact that its statistical sum is given, for any $y \in X_2$, by

$$S_n(\psi)(y) = \sum_{y \in P_n(\sigma_2)} \sum_{x \in P_n(\sigma_1) \cap \pi^{-1}(t)} \exp(S_n(\varphi)(x)),$$

where π is the factor map between X_1 and X_2.

Proposition 1: Let $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2$, is valid P

$$^{\mathbf{a}}(\varphi, \sigma_1) = (a_1 + a_2) P(\psi = \eta(\varphi), \sigma_2).$$

Proof: We have

$$
\begin{aligned}
&P^{\mathbf{a}}(\varphi,\sigma_1)\\
&=\sup\left\{h_\mu^{\mathbf{a}}(\sigma_1)+\int\varphi d\mu:\int\varphi d\mu>-\infty \text{ and } \mu\in\mathcal{M}_{inv}(X_1,\sigma_1)\right\}\\
&=\sup_{\mu\in\mathcal{M}_{inv}(X_1,\sigma_1)}\ \sup_{\nu=\pi\hat{}_*(\mu)\in\mathcal{M}_{inv}(X_2,\sigma_2)}\{a_1h_\mu(\sigma_1)+a_2h_\nu(\sigma_2)+\\
&\qquad\qquad\int\varphi d\mu:\int\varphi d\mu>-\infty\Big\}\\
&=\frac{1}{a_1}\sup_{\mu\in\mathcal{M}_{inv}(X_1,\sigma_1)}\ \sup_{\nu=\pi\hat{}_*(\mu)\in\ \mathcal{M}_{inv}(X_2,\sigma_2)}\{h_\mu(\sigma_1)+(a_1+a_2)h_\nu(\sigma_2)-\\
&\qquad\qquad h_\nu(\sigma_2)+\int\varphi d\mu:\int\varphi d\mu>-\infty\Big\}\\
&=\sup_{\nu=\pi\hat{}_*(\mu)\in\mathcal{M}_{inv}(X_2,\sigma_2)}\left\{\int\psi d\nu+(a_1+a_2)h_\nu(\sigma_2):\int\psi d\nu>-\infty\right\},
\end{aligned}
$$

where the last equality follows from a result given in ref.[1] for the finite alphabet case and which can be proved, in a similar way, to be valid for countable alphabets too. Thus

$$P^{\mathbf{a}}(\varphi,\sigma_1)=(a_1+a_2)P(\psi,\sigma_2) \tag{6}$$

□

From the above proposition we have:

Corollary 1 $\mu\in\mathcal{I}(\varphi,\mathbf{a})$ if and only if $\nu=\pi_*(\mu)\in\mathcal{I}\left(\dfrac{\psi}{a_1+a_2}\right)$ and $\mu\in\mathcal{I}_\nu\left(\dfrac{\varphi}{a_1}\right)$.

□

Let $\left\{C^i(1),C^i(2),..\right\}$ be partitions by cylinders of each space X_i, $i=1,2,..,k$. Let $\varphi:X_1\to\mathbf{R}$ of summable variation, $n\in\mathbf{N}$ and let $P(\sigma_i)=\bigcup_{n\geq1}P_n(\sigma_i)\subset X_i$ for $i=1,2,..,k$, where $P_n(\sigma_i)$ is the set of periodic point in X_i with period n. For $\mathbf{a}=(a_1,...,a_k)$ we introduce the maps:

$$Z^{(0)}: P(\sigma_1) \to \mathbf{R}$$

$$Z^{(0)}(x) = \exp(S_n(\varphi)(x)), \text{ if } x \in P_n(\sigma_1) \tag{7}$$

Let us choose a symbol j_i in the alphabet of each space X_i. Then, for $i = 1, 2, .., k-1$, define

$$Z^{(i)}: P(\sigma_i) \to \mathbf{R}$$

(8)

$$Z^{(i)}(y) = Z^{(i)}(y, j_1, ..., , j_{k-1}) = \left[\sum_{x \in P_n(\sigma_i) \cap \pi_i^{-1}(y)} Z^{(i-1)}(x)^{\frac{1}{A_i}} \right]^{A_i}, \ y \in P_n(\sigma i)$$

where $A_i = a_1 + a_2 + ... + a_i$. Finally

$$Z_n^{(k)}(\varphi) = Z_n^{(k)}(\varphi, j_1, ..., j_k) = \sum_{y \in P_n(\sigma_k)} \left(Z^{(i)}(y)_{C(j_k)} \right)^{\frac{1}{A_i}}. \tag{9}$$

Proposition 2: For $k = 2$, the weighted topological pressure can be expressed as

$$P^{\mathbf{a}}(\varphi, \sigma_1, j_1, j_2) = A_2 \lim_{n \to \infty} \frac{1}{n} \log Z_n^{(2)}(\varphi, \sigma_1, \ j_1, j_2). \tag{10}$$

Proof: The proof follows by proposition 1 and the definition of pressure in the non-weighted case.

□

To prove the theorem 1 we must see that there is an unique weighted equilibrium state for each potential $q\varphi$ and any $q > 1$. This will done by using the theorem A which gives the condition for existence and uniqueness of equilibrium states for $q\varphi$ in the non-weighted case. Besides it must be justified the existence of weighted states at zero temperature. To this we prove that the sequence of weighted states is tight which ensures the existence of an accumulation point. To prove the tightness we proceed like in ref.[6], this is to consider a sequence of weighted equilibrium states for potentials on finite alphabet shifts which has a subsequence which converges to a weighted equilibrium of a potential on a countable alphabet shift. It will be proved the sequence of states in finite alphabet is tight and this will lead that the sequence $\{\mu_{q\varphi}\}_{q>1}$ is tight. Find accumulation points proving that the sequences of equilibrium sates for potentials is tight is a technique used in references [6] and [8]. Existence and uniqueness of weighted equilibrium states for finite alphabet subshifts are warranted by the thermodynamic formalism developed in ref.[1]. Most

of definitions are presented for arbitrary k, although for the proof of the theorem we consider $k = 2$, but some auxiliary results for the proof of the theorem will be shown for general k too. The potentials belong to weighted summable class. For the case $k = 2$ is just considered a factor map $\pi : X_1 \to X_2$.

Let $\varphi : X_1 \to \mathbf{R}$. The *weighted statistical sum of* φ for $\mathbf{a} = (a_1, ..., a_k)$ is defined in the following way: for any $n \in \mathbf{N}$ let $S_n^{(0)} : \mathcal{L}(X_1) \to \mathbf{R}$ be the map given by $S_n^{(0)}(i_1, ..., , i_n) = \sup_{y \in \mathcal{C}(i_1 i_2 ... i_n)} (\varphi(y))$ and let $S_n^{(i)} : \mathcal{L}(X_{i+1}) \to \mathbf{R}$ with $i = 1, 2, ..., k-1$ given by

$$S_n^{(i)}(j_1, ..., j_n) = \left[\sum_{\substack{i_1, ..., i_n : \\ \pi_i(i_1, ..., i_n) = j_1, ..., j_n}} S_n^{(i-1)}(i_1, ..., i_n)^{\frac{1}{A_i}} \right]^{A_i}.$$

Thus the *weighted statistical sum* for potential φ is

$$S_n^{\mathbf{a}}(\varphi)(x) = \frac{1}{a_1} S_n^{(0)}(i_1, .., i_n) + \sum_{i=1}^{k} \left(\frac{1}{A_{i+1}} - \frac{1}{A_i} \right) S_n^{(i)}(\tau_i(i_1, ..., i_n)), \tag{11}$$

with $x = i_1, ..., i_n$ and $A_i = a_1 + ... + a_i$.

Let (X, Ω) be a finite alphabet shift, a potential $\varphi : X \to \mathbf{R}$ has the *bounded distortion property* if for any x, y belonging to the same cylinder of length n in X there exists a constant $C_0 > 0$, such that $|S(\varphi)(x) - S_n(\varphi)(y)| \leq C_0$.

Theorem 2[1]**:** Let $(X_i, \sigma_i, \Omega_i)$, $i = 1, 2, ..., k$, with $k \geq 2$, be a finite family of finite alphabet shifts, with X_1 a full, or Bernoulli, shift. If a potential $\varphi : X_1 \to \mathbf{R}$ verifies the bounded distortion property then φ has an unique weighted equilibrium state μ_φ. Besides this measure satisfies the following *Gibbs property*

$$C^{-1} \leq \frac{\mu_\varphi(\mathcal{C}(i_1 i_2 ... i_n))}{\exp(nP^{\mathbf{a}}(\varphi, \sigma_1)/A_k) \exp(S_n^{\mathbf{a}}(\varphi)(x))} \leq C, \tag{12}$$

for any $n \geq 1$, $x = i_1 i_2 ... \in X_1$ and for some constant $C > 0$.

Definition 5: Let (X, f) be a general dynamical system, let $\mathcal{M}_{inv}(X, f)$ the set of invariant measures on X endowed with the weak topology. A subset $\mathcal{M}_0$ of $\mathcal{M}_{inv}$ (X, f) is *tight* if for any $\varepsilon > 0$ there is a compact $K \subset \mathcal{M}_{inv}(X, f)$, such that $\mu(K^c) < \varepsilon$, for any $\mu \in \mathcal{M}_0$.

Let$\{Y_m\}_{m\geq 1}$ be a family of compact (finite alphabet) topologically transitive subshifts of X_1, with $Y_m \subset Y_{m+1}$ and such this sequence of shifts verify the variational approximation property. Each Y_m is also topologically transitive. Let $\varphi_m = \varphi \mid Y_m$ and let $I_m = \int \varphi_m d\nu$ and $I = \int \varphi d\nu$, for a given measure μ with

$-\infty < I < \infty$. Such a measure can be chosen as a empirical measure supported on $t-$periodic orbits of points of Y_0, i.e. $\nu = \dfrac{1}{t}\sum\limits_{i=0}^{t-1} \delta_{\sigma_1^i(\overline{x})}$, where δ is the point mass measure and $\overline{x} \in P_t(\sigma_1)$. The measure ν can be considered defined in X_1 and in Y_m, for any m. In a similar way as is done in ref. [6] can be proved that $I_m = I$ for any $m \in \mathbf{N}$.

Let $\mu_{q\varphi_m}$ be the weighted equilibrium state for the potential $q\varphi_m$. For each m we have from the definition of the weighted pressure:

$$P^{\mathbf{a}}(q(\varphi_m - I_m)) \geq h_\nu^{\mathbf{a}}(\sigma_1) + q\int \varphi_m d\nu - qI_m = h_\nu^{\mathbf{a}}(\sigma_1) \geq 0,$$

so that $P^{\mathbf{a}}(q\varphi_m) - q\int \varphi_m d\nu \geq 0$.

Proposition 3: The sequence $\{\mu_{q\varphi_m}\}_{m\in\mathbf{N}}$, as a subset of $\mathcal{M}_{inv}(X_1, \sigma_1)$, is tight.

Proof: Consider $k = 2$ and $\varepsilon > 0$. Let $\{n_\ell\}$ be an increasing sequence of natural numbers and set

$$K = \{x = x_1x_2... \in X_1 : 1 \leq x_\ell \leq n_\ell\},$$

thus we have

$$\begin{aligned}
\mu_{q\varphi_m}(K^c) &= \mu_{q\varphi_m}\left(\bigcup_{\ell\geq 1}\{x \in X_1 : x_\ell > n_\ell\}\right) \\
&\leq \sum_{\ell\geq 1}\sum_{i>n_\ell.} \mu_{q\varphi_m}(\{x_\ell = n_\ell\}) \\
&= \sum_{\ell\geq 1}\sum_{i>n_\ell.} \mu_{q\varphi_m}(\mathcal{C}(i)).
\end{aligned}$$

By the proposition 2 any member of the sequence satisfies the Gibbs property (Eq.12), so for any $x \in \mathcal{C}(i)$ and some constant C holds

$$\mu_{q\varphi_m}(\mathcal{C}(i)) \leq C \exp\left(P^{\mathbf{a}}(q\varphi_m)/A_2\right) \exp\left(qS_1^{\mathbf{a}}(\varphi_m)(x)\right).$$

Also we have

$$qS_1^{\mathbf{a}}(\varphi_m)(x) \leq \frac{q}{a_1} \sup \varphi_m \mid \mathcal{C}(i)) + \left(\frac{1}{a_1+a_2} - \frac{1}{a_1}\right) \sum_{r:\pi(i)=\pi(r)} (\sup(\varphi_m \mid \mathcal{C}(r)),$$

thus

$$\begin{aligned}
&\mu_{q\varphi_m}(\mathcal{C}(i)) \\
&\leq C \exp\left(\frac{q}{a_1} \sup \varphi_m \mid \mathcal{C}(i)) + \left(\frac{1}{a_1+a_2} - \frac{1}{a_1}\right)\right. \\
&\qquad\qquad \left.\sum_{r:\pi(i)=\pi(r)} (\sup(\varphi_m \mid \mathcal{C}(r)) - \frac{q}{a_1+a_2} I\right) \\
&\times \exp\left(-\frac{P^{\mathbf{a}}(q\varphi_m)}{a_1+a_2} + \frac{q}{a_1+a_2} I_m\right).
\end{aligned}$$

For the weighted summability condition we have that

$$\left(\frac{1}{a_1} \sup \varphi_m \mid \mathcal{C}(i)) + \left(\frac{1}{a_1+a_2} - \frac{1}{a_1}\right) \sum_{r:\pi(i)=\pi(r)} (\sup(\varphi_m \mid \mathcal{C}(r)) - \frac{1}{a_1+a_2} I\right) \leq 0$$

for i enough large. Since $\frac{1}{a_1+a_2}\left(-P^{\mathbf{a}}(q\varphi_m) + qI_m\right) \leq 0$ and $q > 1$ this with the weighted summability condition give

$$\begin{aligned}
&\sum_{i>n_{\ell.}} \mu_{q\varphi_m}\left(\mathcal{C}\left(i\right)\right) \\
&\leq C\sum_{i>n_{\ell.}}\left(\frac{1}{a_1}\sup\varphi_m \mid \mathcal{C}\left(i\right)\right)+\left(\frac{1}{a_1+a_2}-\frac{1}{a_1}\right) \\
&\qquad\qquad \left.\sum_{r:\pi(i)=\pi(r)}\left(\sup(\varphi_m \mid \mathcal{C}\left(r\right)\right)-\frac{1}{a_1+a_2}I\right) \\
&<\frac{\varepsilon}{2^{\ell+1}}.
\end{aligned}$$

Therefore for any $\ell\geq 1$ is $\mu_{q\varphi_m}\left(K^c\right)<\sum\limits_{\ell\geq 1}\sum\limits_{i>n_{\ell.}}\frac{\varepsilon}{2^{\ell+1}}=\varepsilon.$

□

By the Prohorov theorem[2] there exists a convergent subsequence $\left\{\mu_{q\varphi_{m_\ell}}\right\}_{m_\ell\in\mathbf{N}}$ of $\left\{\mu_{q\varphi_m}\right\}_{m\in\mathbf{N}}$. Let μ_q be the limit of this convergent sequence, we shall prove that μ_q is a weighted equilibrium state for the potential $q\varphi$.

For a given partition $\mathcal{A}=\{B_1,B_2,...\}$ of a general dynamical space (X,f) , the name of a point x of length N, with respect to $\mathcal{A}$ and (X,f) is the string $(i_1,...,i_N)$ with $f^j(x)\in B_{i_j}$. The partition by names of length N from $\mathcal{A}$ will be denoted by $\mathcal{A}^N$. If the space is a shift (X,σ,Ω) and the partition is the natural one $\mathcal{C}$, i.e. by cylinders, the members of $\mathcal{C}$ are $\mathcal{C}\left(i_1i_2...i_N\right)$, with $i_1i_2...i_N\in\Omega^N$ an admissible string.

The following results are obtained for general k. We can consider a "weighted quantity of information", let $\mathbf{A}=\{\mathcal{A}_1,\mathcal{A}_2,...,\mathcal{A}_k\}$ be a set of measurable partitions of the spaces X_1, $X_2,...,$ X_k respectively, then for $\mu\in\mathcal{M}_{inv}(X_1,\sigma_1)$ we set

$$H_\mu^{\mathbf{a}}\left(\mathbf{A},n\right)=\sum_{\ell=1}^{k}a_iH_{\left(\tau_{i-1}\right)_*(\mu)}\left(\bigvee_{i=1}^{k}\tau_{-1}{}_{i-1}(\mathcal{A}_i)^{s_i(n)-1}\right),$$

where $s_i(n)=\lceil(a_1+...+a_i)\,n\rceil$ and with $\lceil z\rceil$ is the least integer greater than z.

Recall that the non-weighted quantity of information of a measure μ with respect to a partition $\mathcal{A}$ is $H_\mu\left(\mathcal{A}\right)=-\sum\limits_{A\in\mathcal{A}}\mu\left(A\right)\log\mu\left(A\right).$

If each $\mathcal{A}_i$ is the natural partition by cylinders $\mathbf{C}=\{\mathcal{C}_1,\mathcal{C}_2,...,\mathcal{C}_k\}$, in the sense of the Kolmogorov-Sinai for the non-weighted case we can set

$$h_{\mu}^{\mathbf{a}}(\sigma_1) = \inf_n \left\{ \frac{1}{n} H_{\mu}^{\mathbf{a}}(\mathbf{C}{,}n) \right\}, \tag{13}$$

There exist a $N_0 \in \mathbf{N}$ such that the sequence $\left(H^{\mathbf{a}}_{\mu_{q\varphi m_\ell}}(\mathbf{C}{,}N_0)\right)_\ell$ is bounded by above. If not it should be a $k \in \mathbf{N}$ such that for any $n \geq k$ such that $\left\{H^{\mathbf{a}}_{\mu_{q\varphi m_\ell}}(\mathbf{C}{,}n)\right\}$ is not bounded by above. Let $\overline{n} \geq k$ and $K_0 > 0$, thus there exists a subsequence (m_s) of (m_s) such that

$$\frac{1}{\overline{n}} H^{\mathbf{a}}_{\mu_{q\varphi m_s}}(\mathbf{C}{,}\overline{n}) > \frac{K_0}{\overline{n}}, \tag{14}$$

so $K \geq h^{\mathbf{a}}_{\mu_{q\varphi m_s}}(\sigma_1) \geq \frac{1}{\overline{n}} H^{\mathbf{a}}_{\mu_{q\varphi m_s}}(\mathbf{C}{,}\overline{n}) > 2K$, a contradiction.

Since the sequence $\left(H^{\mathbf{a}}_{\mu_{q\varphi m_\ell}}(\mathbf{C}{,}N_0)\right)_\ell$ is bounded by above, there is a subsequence (m_s) of (m_ℓ) such that the sequence $\left(H^{\mathbf{a}}_{\mu_{q\varphi m_s}}(\mathbf{C}{,}N_0)\right)_s$ converges. Let us denote $\mathcal{C}_i^n = \bigvee_{i=1}^{k} \tau_{-1}{}_{i-1}(\mathcal{C}_i)^{s_i(n)-1}$, for each partition by cylinders $\mathcal{C}_i$ of X_i . We have

$$H^{\mathbf{a}}_{\mu_{q\varphi m_s}}(\mathbf{C}{,}N_0) = \sum_{i=1}^{k} a_i \left(- \sum_{C \in \mathcal{C}_i^{N_0}} (\tau_{i-1})_* (\mu_{q\varphi_{ms}}) \log (\tau_{i-1})_* (\mu_{q\varphi_{ms}}) \right)$$

and

$$\sum_{i=1}^{k} a_i \left(- \sum_{C \in \mathcal{C}_i^{N_0}} \liminf_{s\to\infty} (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \log (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \right)$$
$$\leq \sum_{i=1}^{k} a_i \left(- \liminf_{s\to\infty} \sum_{C \in \mathcal{C}_i^{N_0}} (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \log (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \right).$$

This is justified by the application of the convergence dominated theorem to each sequence

$$F_\ell(x) = \inf_{s \geq \ell} \left\{ - (\tau_{i-1})_* (\mu_{q\varphi_{ms}}(x)) (\tau_{i-1})_* (\mu_{q\varphi_{ms}}(x)) \right\}, \quad i = 1, 2, ..., k$$

Let $\mu_q = \lim_{s\to\infty} \mu_{q\varphi_{ms}}$ then $\lim_{s\to\infty} (\tau_{i-1})_* (\mu_{q\varphi_{ms}}) = (\tau_{i-1})_* (\mu_q)$, in particular

$$\lim_{s\to\infty} (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) = (\tau_{i-1})_* (\mu_q)(C),$$

for each cylinder C. Therefore

$$\sum_{\ell=1}^{k} a_i \left(-\sum_{C\in\mathcal{C}_i^{N_0}} (\tau_{i-1})_* (\mu_q)(C) \log (\tau_{i-1})_* (\mu_{q\varphi})(C) \right)$$
$$\leq \sum_{\ell=1}^{k} a_i \left(-\liminf_{s\to\infty} \sum_{C\in\mathcal{C}_i^{N_0}} (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \log (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \right).$$

Analogously can be proved that

$$\sum_{\ell=1}^{k} a_i \left(-\sum_{C\in\mathcal{C}_i^{N_0}} (\tau_{i-1})_* (\mu_q)(C) \log (\tau_{i-1})_* (\mu_q)(C) \right)$$
$$\geq \sum_{i=1}^{k} a_i \left(-\limsup_{s\to\infty} \sum_{C\in\mathcal{C}_i^{N_0}} (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \log (\tau_{i-1})_* (\mu_{q\varphi_{ms}})(C) \right).$$

Finally we get

$$H^{\mathbf{a}}_{\mu_q}(\mathbf{C},N_0) = \lim_{s\to\infty} H^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\mathbf{C},N_0). \tag{15}$$

The next step is to prove that

i) $\lim_{s\to\infty} \sup h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1) \leq h^{\mathbf{a}}_{\mu_q}(\sigma_1)$

ii) $\int q\varphi d\mu_q = \lim_{s\to\infty} \int q\varphi d\mu_{q\varphi_{m_\ell}}$

Proof of *i)*

If $\lim_{s\to\infty} \sup h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1) > h^{\mathbf{a}}_{\mu_q}(\sigma_1)$, then there would be a $\varepsilon > 0$ such that $h^{\mathbf{a}}_{\mu_q}(\sigma_1) \leq \lim_{s\to\infty} . \sup h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1) - 3\varepsilon$. By eq.(13) we have

$$\frac{1}{n} H^{\mathbf{a}}_{\mu_q}(\mathbf{C},n) \leq h^{\mathbf{a}}_{\mu_q}(\sigma_1) + \varepsilon,$$

and by eq.(15) there is a number s_0 such that

$$h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1) \leq \frac{1}{N_0} H^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\mathbf{C},N_0).$$

Therefore

$$h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1) \leq \frac{1}{N_0} H^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\mathbf{C},N_0) + 2\varepsilon \leq \lim_{s\to\infty} \sup h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1) - \varepsilon,$$

so that

$$\lim_{s\to\infty}\sup h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1)\leq\lim_{s\to\infty}\sup h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1)-\varepsilon,$$

a contradiction. Hence is valid

$$\lim_{s\to\infty}\sup h^{\mathbf{a}}_{\mu_{q\varphi_{m_s}}}(\sigma_1)\leq h^{\mathbf{a}}_{\mu_q}(\sigma_1)\,. \tag{16}$$

Proof of *ii*)

The fact that a sequence of measures (μ_n) converges to a measure μ, in the weak topology, does not necessarily implies the convergence of the sequence $\int\varphi d\mu_n$ to $\int\varphi d\mu$ if the map is not bounded by below. By [6] for each $N\in\mathbf{N}$ can be found a map ψ such that $\varphi=\varphi^N+\psi^N$ with $\left|\psi^N(x)\right|\leq|\varphi(x)|$ for any $x\in X_1$. By [6] there is a natural n_0 such that $\psi^N(x)=0$ for any $x\in\mathcal{C}(i)$ with $i\leq n_0$ and N enough large. Thus, proceeding in a similar way than in ref.[6] for the non-weighted case, we have

$$\begin{aligned}\left|\int q\psi^N\ d\mu_{q\varphi_{m_\ell}}\right| &\leq\int -q\psi^N\ d\mu_{q\varphi_{m_\ell}}=\sum_{i>n_0}\int_{\mathcal{C}(i)}-q\psi^N\ d\mu_{q\varphi_{m_\ell}}\\ &\leq\sum_{i>n_0}\left(\sup(-q\psi^N\ \mid\mathcal{C}(i)\right)\mu_{q\varphi_{m_\ell}}(\mathcal{C}(i))\,.\end{aligned}$$

In the proposition 3 was used the Gibbs property for the approximating potentials which read

$$\begin{aligned}&\mu_{q\varphi_{m_\ell}}(\mathcal{C}(i))\leq\\ &\qquad\leq C\exp\left(\frac{q}{a_1}\sup\varphi_{m_s}\mid\mathcal{C}(i))+\left(\frac{1}{a_1+a_2}-\frac{1}{a_1}\right)\right.\\ &\qquad\qquad\left.\sum_{r:\pi(i)=\pi(r)}(\sup(\varphi_{m_s}\mid\mathcal{C}(r))-\frac{q}{a_1+a_2}I\right)\\ &\qquad\times\exp\left(-\frac{P^{\mathbf{a}}\left(q\varphi_{m_{s_s}}\right)}{a_1+a_2}+\frac{q}{a_1+a_2}I_{m_s}\right).\end{aligned}$$

Since $\left|\psi^N(x)\right|\leq|\varphi(x)|$ the weighted condition of summability for φ and the above bound for $\mu_{q\varphi_{m_\ell}}(\mathcal{C}(i))$ allows to conclude that for any $\varepsilon>0$ is

$$\sum_{i>n_0}\left(\sup(-q\psi^N\ \mid\mathcal{C}(i)\right)\mu_{q\varphi_{m_\ell}}(\mathcal{C}(i))<\varepsilon/4$$

for enough large N.

Since μ_q is the limit of a subsequence $\{\mu_{q\varphi_m}\}$ then, by continuity, we can replace μ_q by $\mu_{q\varphi_{m_\ell}}$ and, in the same way to obtain

$$\sum_{i>n_0} \left(\sup(-q\psi^N \mid \mathcal{C}(i)\right) \mu_q(\mathcal{C}(i)) < \varepsilon/4$$

for enough large N. Then

$$\begin{aligned}
&\left|\int q\varphi d\mu_q - \int q\varphi d\mu_{q\varphi_{m_\ell}}\right| \\
&\le \left|\int q\psi^N \, d\mu_{q\varphi_{m_\ell}} + \int q\varphi^N \, d\mu_{q\varphi_{m_\ell}} - \left(\int q\psi^N \, d\mu_q + \int q\varphi^N \, d\mu_q\right)\right| \\
&\le \left|\int q\psi^N \, d\mu_{q\varphi_{m_\ell}} + \int q\varphi^N \, d\mu_{q\varphi_{m_\ell}} - \left(\int q\psi^N \, d\mu_q + \int q\varphi^N \, d\mu_q\right)\right| \\
&\le \left|\int q\varphi^N \, d\mu_{q\varphi_{m_\ell}} - \int q\varphi^N \, d\mu_q\right| + \left|\int q\psi^N \, d\mu_{q\varphi_{m_\ell}}\right| + \left|\int q\psi^N \, d\mu_q\right| \\
&< \left|\int q\varphi^N \, d\mu_{q\varphi_{m_\ell}} - \int q\varphi^N \, d\mu_q\right| + \varepsilon/2.
\end{aligned}$$

The fact that φ^N is continuous and bounded leads to the desired result.

Let us consider now a *weighted free energy* $T_\varphi^{\mathbf{a}}(q) = P^{\mathbf{a}}(\varphi, \sigma_1)$, so $T_\varphi^{\mathbf{a}}(q) \ge h_{\mu_q}^{\mathbf{a}}(\sigma_1) + \int \varphi d\mu_q$. Besides if $T_{m_s} = P^{\mathbf{a}}(q\varphi_{m_{ss}}, \sigma_1)$ then

$$\begin{aligned}
T_\varphi^{\mathbf{a}}(q) &= \limsup_{s\to\infty} T_{\varphi_{m_s}}^{\mathbf{a}}(q) \\
&= \limsup_{s\to\infty} h_{\mu_{q\varphi_{m_s}}}^{\mathbf{a}}(\sigma_1) + \limsup_{s\to\infty} \int q\varphi_{m_s} d\mu_{q\varphi_{m_{ss}}} \\
&\le h_{\mu_q}^{\mathbf{a}}(\sigma_1) + \int q\varphi d\mu_q.
\end{aligned}$$

Thus $T_\varphi^{\mathbf{a}}(q) = h_{\mu_q}^{\mathbf{a}}(\sigma_1) + \int q\varphi d\mu_q$, so that μ_q is a weighted equilibrium state for the potential $q\varphi$, for any. $q > 1$. Since by the thermodynamic formalism for finite alphabet shifts each potential $q\varphi_m$ has an unique weighted equilibrium state then μ_q is the unique weighted equilibrium state associated to the potential $q\varphi$.

To complete the proof of the theorem it remains to prove the existence of states at zero temperature. To this end we show that the sequence $\{\mu_q\}_{q>1}$ is tight. This can be done, in a similar way than for proposition 3, proving that the sequence $\{\mu_{q\varphi_m}\}_{m\in\mathbf{N}}$ is tight. In the proof of the proposition 3 is used Gibbs property for the potentials $q\varphi_m$ which leads to

$$\mu_q\varphi_m\left(\mathcal{C}\left(i\right)\right) \leq C\exp\left(\frac{q}{a_1}\sup\varphi_m \mid \mathcal{C}\left(i\right)\right) + \left(\frac{1}{a_1+a_2} - \frac{1}{a_1}\right)$$
$$\left.\sum_{r\colon\pi(i)=\pi(r)}\left(\sup(\varphi_m \mid \mathcal{C}\left(r\right)\right) - \frac{q}{a_1+a_2}I\right)$$
$$\times\exp\left(-\frac{P^{\mathbf{a}}\left(q\varphi_m\right)}{a_1+a_2} + \frac{q}{a_1+a_2}I_m\right).$$

Since μ_q is the limit of a subsequence $\{\mu_{q\varphi_m}\}$ then, by continuity, in the above expression we can put μ_q instead of $\mu_{q\varphi_m}$. Then, following the proof of proposition 3, can be proved that the sequence $\{\mu_q\}_{q>1}$ is tight. So by the Prohorov theorem there is an accumulation point of this sequence or a weighted state at zero temperature. Thus the theorem is proved.

□

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