

ON THE $\mathcal{D}$ -HOMOTHETIC BI-WARPING AND BIHARMONIC MAPS

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ABSTRACT. The purpose of this paper is to study the biharmonicity of maps to or from almost contact manifolds. It also gives some results on the $\mathcal{D}$ -homothetic bi-warping. We establish necessary and sufficient conditions under which a map of the product of a Riemannian manifold and an almost contact metric manifold is harmonic or biharmonic and we have constructed several examples.

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1. INTRODUCTION

The notion of $\mathcal{D}$ -homothetic warping was introduced by D. E. Blair [4] where he gave some basic properties and examples of various structures. Recently, Beldjilali and Belkhefha [2] introduced the notion of $\mathcal{D}$ -homothetic bi-warping and proved some results of certain geometric structures, which is a generalization of Blair's results. This paper is organized as follows. In section 2, we give a basic definition

about almost contact manifolds and some properties of Kenmotsu manifolds; in the same section, we recall definitions of the harmonic and biharmonic maps among Riemannian manifolds. In section 3, we consider the product of a Riemannian manifold (M^m, g) and an almost contact metric manifold $(N^{2n+1}, \varphi, \xi, \eta, h)$ where we prove some results about the $\mathcal{D}$ -homothetic bi-warping metric and Levi-Civita connections (Proposition 1 and 2). Next, we study the particular case when $(N^{2n+1}, \varphi, \xi, \eta, h)$ is a Kenmotsu manifold and this is an important class of almost contact Riemannian manifolds introduced by K. Kenmotsu in [8], and we give the relations among the Riemannian curvature tensors (Theorem 3.5, Corollary 3.6 and 3.7). In [9], the author studies the harmonic maps and harmonic morphisms on Kenmotsu manifolds and she gives some results on the spectral theory of harmonic maps for which the target manifold is a Kenmotsu space-form. Moreover, in [11] the author gives some extensions of the results in [9] and some concrete examples and he characterizes the harmonicity between two almost Kenmotsu manifolds. Recently, in [12] the authors have studied the biharmonicity between Kählerian and Kenmotsu manifolds where they constructed an example for the biharmonic maps of a three-Kenmotsu manifolds. In the last section, using the $\mathcal{D}$ -homothetic bi-warping, we will study the biharmonicity of certain maps. We start with the first and the second projection (Theorem 4.1 and 4.9) where we give some concrete examples. Finally, we determine the conditions of the biharmonicity of the inclusion map (Theorem 4.11).

2. PRELIMINARIES.

2.1. Almost contact manifolds. Let $(N^{2n+1}, \varphi, \xi, \eta, h)$ be an almost contact Riemannian manifold, where φ is a $(1, 1)$ tensor field, ξ is the structure vector field, η is a 1-form and h is a Riemannian metric. We know that φ, ξ, η and h satisfy the following equations (see [8])

$$\varphi^2 = -Id + \eta \otimes \xi, \quad \eta(X) = h(X, \xi),$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0$$

and

$$h(\varphi X, \varphi Y) = h(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TN)$. If moreover

$$(\nabla_X \varphi)Y = -h(X, \varphi Y)\xi - \eta(Y)\varphi X$$

and

$$\nabla_X \xi = X - \eta(X) \xi,$$

$(N^{2n+1}, \varphi, \xi, \eta, h)$ is called a Kenmotsu manifold. In a Kenmotsu manifold, we have the following properties :

$$\eta(R^N(X, Y)Z) = \eta(Y)h(X, Z) - \eta(X)h(Y, Z),$$

$$R^N(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$R^N(\xi, X)Y = \eta(Y)X - h(X, Y)\xi$$

and

$$S^N(X, \xi) = -2n\eta(X),$$

where R^N is the Riemannian curvature tensor and S^N is the Ricci tensor.

2.2. Harmonic and biharmonic maps. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. ϕ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g.$$

By the Euler-Lagrange equation associated to the energy functional, ϕ is harmonic if and only if

$$\tau(\phi) = \text{Tr}_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ . The bi-energy functional of the map ϕ is defined by

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g.$$

The critical points of E_2 are called biharmonic maps. ϕ is biharmonic if and only if

$$\tau_2(\phi) = -\text{Tr}_g (\nabla^\phi)^2 \tau(\phi) - \text{Tr}_g R^N(\tau(\phi), d\phi)d\phi = 0.$$

$\tau_2(\phi)$ is called the bi-tension field of the map ϕ .

3. SOME PROPERTIES OF THE $\mathcal{D}$ -HOMOTHETIC BI-WARPING.

For the functions $\alpha, \beta \in C^\infty(M)$, we consider $(M \times N, \tilde{G})$ the product of Riemannian manifolds (M^m, g) and an almost contact metric manifold $(N^{2n+1}, \varphi, \xi, \eta, h)$, where

$$\tilde{G} = g + \alpha^2 h + \alpha^2 (\beta^2 - 1) \eta \otimes \eta.$$

$(M \times N, \tilde{G})$ is called a $\mathcal{D}$ -Homothetic Bi-Warping. Denote by ∇ , ∇' and $\tilde{\nabla}$ the Levi-Civita connections on (M^m, g) , $(N^{2n+1}, \varphi, \xi, \eta, h)$ and $(M \times N, \tilde{G})$ respectively. Using the Koszul formula for the Levi-Civita connection, we obtain the following proposition :

Proposition 3.1. *For all $X_1, Y_1, Z_1 \in \Gamma(TM)$ and $X_2, Y_2, Z_2 \in \Gamma(TN)$, we have*

$$\tilde{G} \left(\tilde{\nabla}_{(X_1,0)} (Y_1, 0), (Z_1, 0) \right) = g(\nabla_{X_1} Y_1, Z_1),$$

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(0,X_2)} (0, Y_2), (0, Z_2) \right) &= \alpha^2 h(\nabla'_{X_2} Y_2, Z_2) + \alpha^2 (\beta^2 - 1) \eta(Z_2) \eta(\nabla'_{X_2} Y_2) \\ &\quad + \frac{1}{2} \alpha^2 (\beta^2 - 1) \eta(X_2) \{ h(\nabla'_{Y_2} \xi, Z_2) - h(\nabla'_{Z_2} \xi, Y_2) \} \\ &\quad + \frac{1}{2} \alpha^2 (\beta^2 - 1) \eta(Y_2) \{ h(\nabla'_{X_2} \xi, Z_2) - h(\nabla'_{Z_2} \xi, X_2) \} \\ &\quad + \frac{1}{2} \alpha^2 (\beta^2 - 1) \eta(Z_2) \{ h(\nabla'_{X_2} \xi, Y_2) + h(\nabla'_{Y_2} \xi, X_2) \}, \end{aligned}$$

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(X_1,0)} (0, Y_2), (0, Z_2) \right) &= \alpha h(Y_2, Z_2) X_1(\alpha) + \alpha (\beta^2 - 1) \eta(Y_2) \eta(Z_2) X_1(\alpha) \\ &\quad + \alpha^2 \beta \eta(Y_2) \eta(Z_2) X_1(\beta), \end{aligned}$$

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(0,X_2)} (Y_1, 0), (0, Z_2) \right) &= \alpha h(X_2, Z_2) Y_1(\alpha) + \alpha (\beta^2 - 1) \eta(X_2) \eta(Z_2) Y_1(\alpha) \\ &\quad + \alpha^2 \beta \eta(X_2) \eta(Z_2) Y_1(\beta), \end{aligned}$$

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(0,X_2)} (0, Y_2), (Z_1, 0) \right) &= -\alpha h(X_2, Y_2) Z_1(\alpha) - \alpha (\beta^2 - 1) \eta(X_2) \eta(Y_2) Z_1(\alpha) \\ &\quad - \alpha^2 \beta \eta(X_2) \eta(Y_2) Z_1(\beta) \end{aligned}$$

and

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(X_1,0)} (Y_1, 0), (0, Z_2) \right) &= \tilde{G} \left(\tilde{\nabla}_{(X_1,0)} (0, Y_2), (Z_1, 0) \right) \\ &= \tilde{G} \left(\tilde{\nabla}_{(0,X_2)} (Y_1, 0), (Z_1, 0) \right) = 0 \end{aligned}$$

Based on the equations of Proposition 3.1, we obtain the following result.

Proposition 3.2. *If we consider $\{e_i\}_{1 \leq i \leq m}$ and $\{f_j, \varphi f_j, \xi\}_{1 \leq j \leq n}$ two orthonormal frames on (M, g) and (N, h) respectively, then $\left\{(e_i, 0), \frac{1}{\alpha}(0, f_j), \frac{1}{\alpha}(0, \varphi f_j), \frac{1}{\alpha\beta}(0, \xi)\right\}$ is an orthonormal frame on $(M \times N, \tilde{G})$. The following equations give us the relation between $\tilde{\nabla}$, ∇ and ∇'*

$$(1) \quad \tilde{\nabla}_{(X_1, 0)}(Y_1, 0) = (\nabla_{X_1} Y_1, 0),$$

$$(2) \quad \begin{aligned} \tilde{\nabla}_{(0, X_2)}(0, Y_2) &= (0, \nabla'_{X_2} Y_2) \\ &+ \frac{(\beta^2 - 1)}{2} \{ \eta(X_2)(0, \nabla'_{Y_2} \xi) + \eta(Y_2)(0, \nabla'_{X_2} \xi) \} \\ &- \frac{(\beta^2 - 1)}{2} \{ \eta(X_2)(0, Tr_h h(\nabla' \xi, Y_2) \cdot) + \eta(Y_2)(0, Tr_h h(\nabla' \xi, X_2) \cdot) \} \\ &+ \frac{(\beta^2 - 1)^2}{2\beta^2} \{ \eta(X_2)h(\nabla'_{\xi} \xi, Y_2) + \eta(Y_2)h(\nabla'_{\xi} \xi, X_2) \} (0, \xi) \\ &+ \frac{(\beta^2 - 1)}{2\beta^2} \{ h(\nabla'_{X_2} \xi, Y_2) + h(\nabla'_{Y_2} \xi, X_2) \} (0, \xi) \\ &- \alpha h(X_2, Y_2)(grad \alpha, 0) - \alpha(\beta^2 - 1)\eta(X_2)\eta(Y_2)(grad \alpha, 0) \\ &- \alpha^2 \beta \eta(X_2)\eta(Y_2)(grad \beta, 0), \end{aligned}$$

$$(3) \quad \tilde{\nabla}_{(X_1, 0)}(0, Y_2) = \frac{1}{\alpha} X_1(\alpha)(0, Y_2) + \frac{1}{\beta} \eta(Y_2) X_1(\beta)(0, \xi)$$

and

$$(4) \quad \tilde{\nabla}_{(0, X_2)}(Y_1, 0) = \frac{1}{\alpha} Y_1(\alpha)(0, X_2) + \frac{1}{\beta} \eta(X_2) Y_1(\beta)(0, \xi).$$

For all $X_1, Y_1 \in \Gamma(TM)$ and $X_2, Y_2 \in \Gamma(TN)$.

Proof. The proof of the equations (1), (3) and (4) is not complicated. For equation (2), we have

$$\begin{aligned} \tilde{\nabla}_{(0, X_2)}(0, Y_2) &= \tilde{G} \left(\tilde{\nabla}_{(0, X_2)}(0, Y_2), (e_i, 0) \right) (e_i, 0) \\ &+ \frac{1}{\alpha^2} \tilde{G} \left(\tilde{\nabla}_{(0, X_2)}(0, Y_2), (0, f_j) \right) (0, f_j) + \frac{1}{\alpha^2} \tilde{G} \left(\tilde{\nabla}_{(0, X_2)}(0, Y_2), (0, \varphi f_j) \right) (0, \varphi f_j) \\ &+ \frac{1}{\alpha^2 \beta^2} \tilde{G} \left(\tilde{\nabla}_{(0, X_2)}(0, Y_2), (0, \xi) \right) (0, \xi). \end{aligned}$$

We will study this last equation term by term. From Proposition 3.1, we have

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(0, X_2)} (0, Y_2), (e_i, 0) \right) (e_i, 0) &= -\alpha h(X_2, Y_2) e_i(\alpha) (e_i, 0) \\ &\quad - \alpha (\beta^2 - 1) \eta(X_2) \eta(Y_2) e_i(\alpha) (e_i, 0) - \alpha^2 \beta \eta(X_2) \eta(Y_2) e_i(\beta) (e_i, 0) \\ &= -\alpha h(X_2, Y_2) (\text{grad} \alpha, 0) - \alpha (\beta^2 - 1) \eta(X_2) \eta(Y_2) (\text{grad} \alpha, 0) \\ &\quad - \alpha^2 \beta \eta(X_2) \eta(Y_2) (\text{grad} \beta, 0), \end{aligned}$$

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(0, X_2)} (0, Y_2), (0, f_j) \right) (0, f_j) &= \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(Y_2) h(\nabla'_{X_2} \xi, f_j) (0, f_j) \\ &\quad + \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(X_2) h(\nabla'_{Y_2} \xi, f_j) (0, f_j) - \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(X_2) h(\nabla'_{f_j} \xi, Y_2) (0, f_j) \\ &\quad - \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(Y_2) h(\nabla'_{f_j} \xi, X_2) (0, f_j) + \alpha^2 h(\nabla'_{X_2} Y_2, f_j) (0, f_j), \end{aligned}$$

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(0, X_2)} (0, Y_2), (0, \varphi f_j) \right) (0, \varphi f_j) &= \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(Y_2) h(\nabla'_{X_2} \xi, \varphi f_j) (0, \varphi f_j) \\ &\quad + \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(X_2) h(\nabla'_{Y_2} \xi, \varphi f_j) (0, \varphi f_j) + \alpha^2 h(\nabla'_{X_2} Y_2, \varphi f_j) (0, \varphi f_j) \\ &\quad - \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(X_2) h(\nabla'_{\varphi f_j} \xi, Y_2) (0, \varphi f_j) \\ &\quad - \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(Y_2) h(\nabla'_{\varphi f_j} \xi, X_2) (0, \varphi f_j) \end{aligned}$$

and

$$\begin{aligned} \tilde{G} \left(\tilde{\nabla}_{(0, X_2)} (0, Y_2), (0, \xi) \right) (0, \xi) &= \alpha^2 h(\nabla'_{X_2} Y_2, \xi) (0, \xi) + \alpha^2 (\beta^2 - 1) \eta(\nabla'_{X_2} Y_2) (0, \xi) \\ &\quad + \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(X_2) \{ h(\nabla'_{Y_2} \xi, \xi) - h(\nabla'_{\xi} \xi, Y_2) \} (0, \xi) \\ &\quad + \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(Y_2) \{ h(\nabla'_{X_2} \xi, \xi) - h(\nabla'_{\xi} \xi, X_2) \} (0, \xi) \\ &\quad + \frac{\alpha^2 (\beta^2 - 1)}{2} \{ h(\nabla'_{X_2} \xi, Y_2) + h(\nabla'_{Y_2} \xi, X_2) \} (0, \xi) \\ &= \frac{\alpha^2 (\beta^2 - 1)}{2} \{ h(\nabla'_{X_2} \xi, Y_2) + h(\nabla'_{Y_2} \xi, X_2) \} (0, \xi) \\ &\quad - \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(X_2) h(\nabla'_{\xi} \xi, Y_2) (0, \xi) \\ &\quad - \frac{\alpha^2 (\beta^2 - 1)}{2} \eta(Y_2) h(\nabla'_{\xi} \xi, X_2) (0, \xi) \\ &\quad + \alpha^2 \beta^2 h(\nabla'_{X_2} Y_2, \xi) (0, \xi) \end{aligned}$$

It follows that

$$\begin{aligned}\tilde{\nabla}_{(0,X_2)}(0,Y_2) &= (0,\nabla'_{X_2}Y_2) + \frac{(\beta^2-1)}{2} \{ \eta(X_2)(0,\nabla'_{Y_2}\xi) + \eta(Y_2)(0,\nabla'_{X_2}\xi) \} \\ &\quad - \frac{(\beta^2-1)}{2} \{ \eta(X_2)(0,Tr_h(\nabla'\xi,Y_2)\cdot) + \eta(Y_2)(0,Tr_h(\nabla'\xi,X_2)\cdot) \} \\ &\quad + \frac{(\beta^2-1)^2}{2\beta^2} \{ \eta(X_2)h(\nabla'_\xi\xi,Y_2) + \eta(Y_2)h(\nabla'_\xi\xi,X_2) \} (0,\xi) \\ &\quad + \frac{(\beta^2-1)}{2\beta^2} \{ h(\nabla'_{X_2}\xi,Y_2) + h(\nabla'_{Y_2}\xi,X_2) \} (0,\xi) - \alpha h(X_2,Y_2)(grad\alpha,0) \\ &\quad - \alpha(\beta^2-1)\eta(X_2)\eta(Y_2)(grad\alpha,0) - \alpha^2\beta\eta(X_2)\eta(Y_2)(grad\beta,0).\end{aligned}$$

Note that if $\beta = \pm 1$, we obtain the notion of the warped product used in [3] and [10]. As a particular case of Proposition 3.2, we have :

Lemma 3.3.

$$\begin{aligned}\tilde{\nabla}_{(e_i,0)}(e_i,0) &= (\nabla_{e_i}e_i,0), \\ \tilde{\nabla}_{(0,f_j)}(0,f_j) &= (0,\nabla'_{f_j}f_j) - \frac{\beta^2-1}{\beta^2}\eta(\nabla'_{f_j}f_j)(0,\xi) - n\alpha(grad\alpha,0), \\ \tilde{\nabla}_{(0,\varphi f_j)}(0,\varphi f_j) &= (0,\nabla'_{\varphi f_j}\varphi f_j) - \frac{\beta^2-1}{\beta^2}\eta(\nabla'_{\varphi f_j}\varphi f_j)(0,\xi) - n\alpha(grad\alpha,0), \\ \tilde{\nabla}_{(0,\xi)}(0,\xi) &= \beta^2(0,\nabla'_\xi\xi) - \alpha\beta^2(grad\alpha,0) - \alpha^2\beta(grad\beta,0), \\ \tilde{\nabla}_{(e_i,0)}(0,f_j) &= \tilde{\nabla}_{(0,f_j)}(e_i,0) = \frac{1}{\alpha}e_i(\alpha)(0,f_j), \\ \tilde{\nabla}_{(e_i,0)}(0,\varphi f_j) &= \tilde{\nabla}_{(0,\varphi f_j)}(e_i,0) = \frac{1}{\alpha}e_i(\alpha)(0,\varphi f_j)\end{aligned}$$

and

$$\tilde{\nabla}_{(e_i,0)}(0,\xi) = \tilde{\nabla}_{(0,\xi)}(e_i,0) = \frac{1}{\alpha}e_i(\alpha)(0,\xi) + \frac{1}{\beta}e_i(\beta)(0,\xi)$$

Moreover, if $(N^{2n+1}, \varphi, \xi, \eta, h)$ is a Kenmotsu manifold, Proposition 3.2 gives the following remark.

Remark 3.4. Let $(N^{2n+1}, \varphi, \xi, \eta, h)$ be a Kenmotsu manifold, then we have

$$\nabla'_{X_2}\xi = X_2 - \eta(X_2)\xi, \quad \nabla'_{Y_2}\xi = Y_2 - \eta(Y_2)\xi \quad \text{and} \quad \nabla'_\xi\xi = 0,$$

which gives us

$$\begin{aligned}\eta(X_2)(0, \nabla'_{Y_2}\xi) + \eta(Y_2)(0, \nabla'_{X_2}\xi) &= \eta(X_2)(0, Y_2 - \eta(Y_2)\xi) \\ &\quad + \eta(Y_2)(0, X_2 - \eta(X_2)\xi) \\ &= \eta(X_2)(0, Y_2) + \eta(Y_2)(0, X_2) \\ &\quad - 2\eta(X_2)\eta(Y_2)(0, \xi),\end{aligned}$$

$$\begin{aligned}Tr_h h(\nabla'\xi, Y_2) \cdot &= h(\nabla'_{f_j}\xi, Y_2)f_j + h(\nabla'_{\varphi f_j}\xi, Y_2)\varphi f_j + h(\nabla'_\xi\xi, Y_2)\xi \\ &= h(f_j, Y_2)f_j + h(\varphi f_j, Y_2)\varphi f_j \\ &= Y_2 - \eta(Y_2)\xi, \\ Tr_h h(\nabla'\xi, X_2) \cdot &= X_2 - \eta(X_2)\xi\end{aligned}$$

and

$$\begin{aligned}h(\nabla'_{X_2}\xi, Y_2) + h(\nabla'_{Y_2}\xi, X_2) &= h(X_2 - \eta(X_2)\xi, Y_2) + h(Y_2 - \eta(Y_2)\xi, X_2) \\ &= 2h(X_2, Y_2) - 2\eta(X_2)\eta(Y_2).\end{aligned}$$

Then the equation (2) becomes

$$\begin{aligned}\tilde{\nabla}_{(0, X_2)}(0, Y_2) &= (0, \nabla'_{X_2}Y_2) - \frac{(\beta^2 - 1)}{\beta^2}\eta(X_2)\eta(Y_2)(0, \xi) \\ &\quad + \frac{(\beta^2 - 1)}{\beta^2}h(X_2, Y_2)(0, \xi) - \alpha h(X_2, Y_2)(grad\alpha, 0) \\ &\quad - \alpha(\beta^2 - 1)\eta(X_2)\eta(Y_2)(grad\alpha, 0) \\ &\quad - \alpha^2\beta\eta(X_2)\eta(Y_2)(grad\beta, 0).\end{aligned}$$

We also get the following formulas

$$\begin{aligned}\tilde{\nabla}_{(0, f_j)}(0, f_j) &= (0, \nabla'_{f_j}f_j) - n\alpha(grad\alpha, 0) + \frac{n(\beta^2 - 1)}{\beta^2}(0, \xi), \\ \tilde{\nabla}_{(0, \varphi f_j)}(0, \varphi f_j) &= (0, \nabla'_{\varphi f_j}\varphi f_j) - n\alpha(grad\alpha, 0) + \frac{n(\beta^2 - 1)}{\beta^2}(0, \xi)\end{aligned}$$

and

$$\tilde{\nabla}_{(0, \xi)}(0, \xi) = -\alpha\beta^2(grad\alpha, 0) - \alpha^2\beta(grad\beta, 0).$$

As a first result, we will give the expression of the curvature tensor $\tilde{R}$ of the $\mathcal{D}$ -homothetic bi-warped in the case where $(N^{2n+1}, \varphi, \xi, \eta, h)$ is a Kenmotsu manifold.

Theorem 3.5. *Let $(N^{2n+1}, \varphi, \xi, \eta, h)$ be a Kenmotsu manifold; the relations between $\tilde{R}$, R^M and R^N are given by the following formulas:*

$$\tilde{R}((X_1, 0), (Y_1, 0))(Z_1, 0) = (R^M(X_1, Y_1)Z_1, 0),$$

$$\begin{aligned} \tilde{R}((X_1, 0), (0, Y_2))(0, Z_2) &= -\alpha\beta\eta(Y_2)\eta(Z_2)X_1(\beta)(grad\alpha, 0) \\ &\quad - \alpha\beta\eta(Y_2)\eta(Z_2)X_1(\alpha)(grad\beta, 0) - \alpha(\beta^2 - 1)\eta(Y_2)\eta(Z_2)(\nabla_{X_1}grad\alpha, 0) \\ &\quad - \alpha^2\beta\eta(Y_2)\eta(Z_2)(\nabla_{X_1}grad\beta, 0) - \alpha h(Y_2, Z_2)(\nabla_{X_1}grad\alpha, 0) \\ &\quad - \frac{1}{\beta^3}\eta(Y_2)\eta(Z_2)X_1(\beta)(0, \xi) + \frac{1}{\beta^3}h(Y_2, Z_2)X_1(\beta)(0, \xi) \\ &\quad - \frac{1}{\beta}\eta(Z_2)X_1(\beta)(0, Y_2) + \frac{1}{\beta}\eta(Y_2)\eta(Z_2)X_1(\beta)(0, \xi), \end{aligned}$$

$$\begin{aligned} \tilde{R}((0, X_2), (Y_1, 0))(Z_1, 0) &= -\frac{1}{\alpha\beta}\eta(X_2)Z_1(\alpha)Y_1(\beta)(0, \xi) \\ &\quad - \frac{1}{\alpha\beta}\eta(X_2)Y_1(\alpha)Z_1(\beta)(0, \xi) - \frac{1}{\beta}\eta(X_2)g(Z_1, \nabla_{Y_1}grad\beta)(0, \xi) \\ &\quad - \frac{1}{\alpha}g(Z_1, \nabla_{Y_1}grad\alpha)(0, X_2). \end{aligned}$$

and

$$\begin{aligned} \tilde{R}((0, X_2), (0, Y_2))(0, Z_2) &= (0, R^N(X_2, Y_2)Z_2) + \frac{\alpha^2}{\beta}h(Y_2, Z_2)\eta(X_2)(grad\beta, 0) \\ &\quad - \frac{\alpha^2}{\beta}h(X_2, Z_2)\eta(Y_2)(grad\beta, 0) + \frac{(\beta^2 - 1)}{\beta^2}h(Y_2, Z_2)(0, X_2) \\ &\quad - \frac{(\beta^2 - 1)}{\beta^2}\eta(Y_2)\eta(Z_2)(0, X_2) - h(Y_2, Z_2)|grad\alpha|^2(0, X_2) \\ &\quad - \frac{(\beta^2 - 1)}{\beta^2}h(X_2, Z_2)(0, Y_2) + \frac{(\beta^2 - 1)}{\beta^2}\eta(X_2)\eta(Z_2)(0, Y_2) \\ &\quad + h(X_2, Z_2)|grad\alpha|^2(0, Y_2) - (\beta^2 - 1)\eta(Y_2)\eta(Z_2)|grad\alpha|^2(0, X_2) \\ &\quad - \alpha\beta\eta(Y_2)\eta(Z_2)d\alpha(grad\beta)(0, X_2) + (\beta^2 - 1)\eta(X_2)\eta(Z_2)|grad\alpha|^2(0, Y_2) \\ &\quad + \alpha\beta\eta(X_2)\eta(Z_2)d\alpha(grad\beta)(0, Y_2) - \frac{\alpha}{\beta}\eta(X_2)h(Y_2, Z_2)d\alpha(grad\beta)(0, \xi) \\ &\quad + \frac{\alpha}{\beta}\eta(Y_2)h(X_2, Z_2)d\alpha(grad\beta)(0, \xi). \end{aligned}$$

Proof. Let $X_1, Y_1, Z_1 \in \Gamma(TM)$ and $X_2, Y_2, Z_2 \in \Gamma(TN)$. The equality

$$\tilde{R}((X_1, 0), (Y_1, 0))(Z_1, 0) = (R^M(X_1, Y_1)Z_1, 0)$$

is deduced directly from equation (1). For the term $\tilde{R}((X_1, 0), (0, Y_2))(0, Z_2)$, we have

$$\tilde{R}((X_1, 0), (0, Y_2))(0, Z_2) = \tilde{\nabla}_{(X_1, 0)} \tilde{\nabla}_{(0, Y_2)}(0, Z_2) - \tilde{\nabla}_{(0, Y_2)} \tilde{\nabla}_{(X_1, 0)}(0, Z_2).$$

Using equation (2), we obtain

$$\begin{aligned} \tilde{\nabla}_{(0, Y_2)}(0, Z_2) &= (0, \nabla'_{Y_2} Z_2) + \frac{(\beta^2 - 1)}{\beta^2} h(Y_2, Z_2)(0, \xi) \\ &\quad - \alpha h(Y_2, Z_2)(grad\alpha, 0) - \alpha(\beta^2 - 1)\eta(Y_2)\eta(Z_2)(grad\alpha, 0) \\ &\quad - \alpha^2\beta\eta(Y_2)\eta(Z_2)(grad\beta, 0) - \frac{(\beta^2 - 1)}{\beta^2}\eta(Y_2)\eta(Z_2)(0, \xi). \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{\nabla}_{(X_1, 0)} \tilde{\nabla}_{(0, Y_2)}(0, Z_2) &= -\eta(Y_2)\eta(Z_2)\tilde{\nabla}_{(X_1, 0)}\alpha(\beta^2 - 1)(grad\alpha, 0) \\ &\quad - \eta(Y_2)\eta(Z_2)\tilde{\nabla}_{(X_1, 0)}\alpha^2\beta(grad\beta, 0) - h(Y_2, Z_2)\tilde{\nabla}_{(X_1, 0)}\alpha(grad\alpha, 0) \\ &\quad + \tilde{\nabla}_{(X_1, 0)}(0, \nabla'_{Y_2} Z_2) + h(Y_2, Z_2)\tilde{\nabla}_{(X_1, 0)}\frac{(\beta^2 - 1)}{\beta^2}(0, \xi) \\ &\quad - \eta(Y_2)\eta(Z_2)\tilde{\nabla}_{(X_1, 0)}\frac{(\beta^2 - 1)}{\beta^2}(0, \xi). \end{aligned}$$

Then

$$\begin{aligned} \tilde{\nabla}_{(X_1, 0)} \tilde{\nabla}_{(0, Y_2)}(0, Z_2) &= -h(Y_2, Z_2)X_1(\alpha)(grad\alpha, 0) - (\beta^2 - 1)\eta(Y_2)\eta(Z_2)X_1(\alpha)(grad\alpha, 0) \\ &\quad - 2\alpha\beta\eta(Y_2)\eta(Z_2)X_1(\beta)(grad\alpha, 0) - \alpha^2\eta(Y_2)\eta(Z_2)X_1(\beta)(grad\beta, 0) \\ &\quad - 2\alpha\beta\eta(Y_2)\eta(Z_2)X_1(\alpha)(grad\beta, 0) - \alpha h(Y_2, Z_2)(\nabla_{X_1} grad\alpha, 0) \\ &\quad - \alpha(\beta^2 - 1)\eta(Y_2)\eta(Z_2)(\nabla_{X_1} grad\alpha, 0)\alpha^2\beta\eta(Y_2)\eta(Z_2)(\nabla_{X_1} grad\beta, 0) \\ &\quad - \frac{(\beta^2 - 1)}{\alpha\beta^2}\eta(Y_2)\eta(Z_2)X_1(\alpha)(0, \xi) - \frac{(\beta^2 + 1)}{\beta^3}\eta(Y_2)\eta(Z_2)X_1(\beta)(0, \xi) \\ &\quad + \frac{(\beta^2 - 1)}{\alpha\beta^2}h(Y_2, Z_2)X_1(\alpha)(0, \xi) + \frac{(\beta^2 + 1)}{\beta^3}h(Y_2, Z_2)X_1(\beta)(0, \xi) \\ &\quad + \frac{1}{\alpha}X_1(\alpha)(0, \nabla'_{Y_2} Z_2) + \frac{1}{\beta}\eta(\nabla'_{Y_2} Z_2)X_1(\beta)(0, \xi). \end{aligned}$$

Now calculate the second term $\tilde{\nabla}_{(0, Y_2)} \tilde{\nabla}_{(X_1, 0)}(0, Z_2)$; we have

$$\tilde{\nabla}_{(X_1, 0)}(0, Z_2) = \frac{1}{\alpha}X_1(\alpha)(0, Z_2) + \frac{1}{\beta}\eta(Z_2)X_1(\beta)(0, \xi),$$

which gives us

$$\widetilde{\nabla}_{(0,Y_2)} \widetilde{\nabla}_{(X_1,0)} (0, Z_2) = \frac{1}{\alpha} X_1 (\alpha) \widetilde{\nabla}_{(0,Y_2)} (0, Z_2) + \frac{1}{\beta} X_1 (\beta) \widetilde{\nabla}_{(0,Y_2)} \eta (Z_2) (0, \xi).$$

A similar calculation leads us to

$$\begin{aligned} & \widetilde{\nabla}_{(0,Y_2)} \widetilde{\nabla}_{(X_1,0)} (0, Z_2) \\ &= -h(Y_2, Z_2) X_1 (\alpha) (\text{grad}\alpha, 0) - (\beta^2 - 1) \eta(Y_2) \eta(Z_2) X_1 (\alpha) (\text{grad}\alpha, 0) \\ & \quad - \alpha\beta\eta(Y_2) \eta(Z_2) X_1 (\beta) (\text{grad}\alpha, 0) - \alpha\beta\eta(Y_2) \eta(Z_2) X_1 (\alpha) (\text{grad}\beta, 0) \\ & \quad - \alpha^2\eta(Y_2) \eta(Z_2) X_1 (\beta) (\text{grad}\beta, 0) + \frac{(\beta^2 - 1)}{\alpha\beta^2} h(Y_2, Z_2) X_1 (\alpha) (0, \xi) \\ & \quad + \frac{1}{\beta} \eta(Z_2) X_1 (\beta) (0, Y_2) - \frac{1}{\beta} \eta(Y_2) \eta(Z_2) X_1 (\beta) (0, \xi) + \frac{1}{\beta} X_1 (\beta) \eta(\nabla'_{Y_2} Z_2) (0, \xi) \\ & \quad + \frac{1}{\alpha} X_1 (\alpha) (0, \nabla'_{Y_2} Z_2) - \frac{(\beta^2 - 1)}{\alpha\beta^2} \eta(Y_2) \eta(Z_2) X_1 (\alpha) (0, \xi) \\ & \quad + \frac{1}{\beta} h(Y_2, Z_2) X_1 (\beta) (0, \xi) - \frac{1}{\beta} \eta(Y_2) \eta(Z_2) X_1 (\beta) (0, \xi). \end{aligned}$$

Finally, we deduce that

$$\begin{aligned} & \widetilde{R}((X_1, 0), (0, Y_2)) (0, Z_2) \\ &= -\alpha\beta\eta(Y_2) \eta(Z_2) X_1 (\beta) (\text{grad}\alpha, 0) - \alpha\beta\eta(Y_2) \eta(Z_2) X_1 (\alpha) (\text{grad}\beta, 0) \\ & \quad - \alpha(\beta^2 - 1) \eta(Y_2) \eta(Z_2) (\nabla_{X_1} \text{grad}\alpha, 0) - \alpha^2\beta\eta(Y_2) \eta(Z_2) (\nabla_{X_1} \text{grad}\beta, 0) \\ & \quad - \alpha h(Y_2, Z_2) (\nabla_{X_1} \text{grad}\alpha, 0) - \frac{1}{\beta^3} \eta(Y_2) \eta(Z_2) X_1 (\beta) (0, \xi) \\ & \quad + \frac{1}{\beta^3} h(Y_2, Z_2) X_1 (\beta) (0, \xi) - \frac{1}{\beta} \eta(Z_2) X_1 (\beta) (0, Y_2) \\ & \quad + \frac{1}{\beta} \eta(Y_2) \eta(Z_2) X_1 (\beta) (0, \xi). \end{aligned}$$

For the term $\widetilde{R}((0, X_2), (Y_1, 0)) (Z_1, 0)$, we have

$$\widetilde{R}((0, X_2), (Y_1, 0)) (Z_1, 0) = \widetilde{\nabla}_{(0,X_2)} \widetilde{\nabla}_{(Y_1,0)} (Z_1, 0) - \widetilde{\nabla}_{(Y_1,0)} \widetilde{\nabla}_{(0,X_2)} (Z_1, 0).$$

Using the same calculation methods, we obtain

$$\widetilde{\nabla}_{(0,X_2)} \widetilde{\nabla}_{(Y_1,0)} (Z_1, 0) = \frac{1}{\alpha} (\nabla_{Y_1} Z_1) (\alpha) (0, X_2) + \frac{1}{\beta} \eta(X_2) (\nabla_{Y_1} Z_1) (\beta) (0, \xi)$$

and

$$\begin{aligned}
& \tilde{\nabla}_{(Y_1,0)} \tilde{\nabla}_{(0,X_2)} (Z_1, 0) \\
&= \frac{1}{\alpha\beta} \eta(X_2) Z_1(\alpha) Y_1(\beta)(0, \xi) + \frac{1}{\alpha\beta} \eta(X_2) Y_1(\alpha) Z_1(\beta)(0, \xi) \\
&+ \frac{1}{\beta} \eta(X_2) (\nabla_{Y_1} Z_1)(\beta)(0, \xi) + \frac{1}{\beta} \eta(X_2) g(Z_1, \nabla_{Y_1} \text{grad}\beta)(0, \xi) \\
&+ \frac{1}{\alpha} (\nabla_{Y_1} Z_1)(\alpha)(0, X_2) + \frac{1}{\alpha} g(Z_1, \nabla_{Y_1} \text{grad}\alpha)(0, X_2).
\end{aligned}$$

Then, we conclude that

$$\begin{aligned}
& \tilde{R}((0, X_2), (Y_1, 0))(Z_1, 0) \\
&= -\frac{1}{\alpha\beta} \eta(X_2) Z_1(\alpha) Y_1(\beta)(0, \xi) - \frac{1}{\alpha\beta} \eta(X_2) Y_1(\alpha) Z_1(\beta)(0, \xi) \\
&- \frac{1}{\beta} \eta(X_2) g(Z_1, \nabla_{Y_1} \text{grad}\beta)(0, \xi) - \frac{1}{\alpha} g(Z_1, \nabla_{Y_1} \text{grad}\alpha)(0, X_2).
\end{aligned}$$

To complete the proof, we will calculate the term $\tilde{R}((0, X_2), (0, Y_2))(0, Z_2)$, we have

$$\begin{aligned}
\tilde{R}((0, X_2), (0, Y_2))(0, Z_2) &= \tilde{\nabla}_{(0,X_2)} \tilde{\nabla}_{(0,Y_2)} (0, Z_2) - \tilde{\nabla}_{(0,Y_2)} \tilde{\nabla}_{(0,X_2)} (0, Z_2) \\
&- \tilde{\nabla}_{[(0,X_2),(0,Y_2)]} (0, Z_2).
\end{aligned}$$

Using the following formulas

$$\begin{aligned}
\tilde{\nabla}_{(0,X_2)} (0, Y_2) &= (0, \nabla'_{X_2} Y_2) + \frac{(\beta^2 - 1)}{\beta^2} \{h(X_2, Y_2) - \eta(X_2) \eta(Y_2)\}(0, \xi) \\
&- \alpha h(X_2, Y_2) (\text{grad}\alpha, 0) - \alpha(\beta^2 - 1) \eta(X_2) \eta(Y_2) (\text{grad}\alpha, 0) \\
&- \alpha^2 \beta \eta(X_2) \eta(Y_2) (\text{grad}\beta, 0),
\end{aligned}$$

$$X_2(\eta(Y_2)) = h(X_2, Y_2) - \eta(X_2) \eta(Y_2) + \eta(\nabla'_{X_2} Y_2),$$

$$\tilde{\nabla}_{(0,X_2)} (0, \xi) = (0, X_2) - \eta(X_2)(0, \xi) - \alpha\beta^2 \eta(X_2) (\text{grad}\alpha, 0) - \alpha^2 \beta \eta(X_2) (\text{grad}\beta, 0),$$

$$\tilde{\nabla}_{(0,X_2)} (\text{grad}\alpha, 0) = \frac{1}{\alpha} |\text{grad}\alpha|^2 (0, X_2) + \frac{1}{\beta} d\alpha (\text{grad}\beta) \eta(X_2)(0, \xi)$$

and

$$\tilde{\nabla}_{(0,X_2)} (\text{grad}\beta, 0) = \frac{1}{\alpha} d\alpha (\text{grad}\beta) (0, X_2) + \frac{1}{\beta} |\text{grad}\beta|^2 \eta(X_2)(0, \xi),$$

we get

$$\begin{aligned}
& \widetilde{R}((0, X_2), (0, Y_2))(0, Z_2) \\
&= (0, R^N(X_2, Y_2)Z_2) + \frac{\alpha^2}{\beta} h(Y_2, Z_2) \eta(X_2)(grad\beta, 0) \\
&- \frac{\alpha^2}{\beta} h(X_2, Z_2) \eta(Y_2)(grad\beta, 0) + \frac{(\beta^2 - 1)}{\beta^2} h(Y_2, Z_2)(0, X_2) \\
&- \frac{(\beta^2 - 1)}{\beta^2} \eta(Y_2) \eta(Z_2)(0, X_2) - h(Y_2, Z_2) |grad\alpha|^2(0, X_2) \\
&- \frac{(\beta^2 - 1)}{\beta^2} h(X_2, Z_2)(0, Y_2) + \frac{(\beta^2 - 1)}{\beta^2} \eta(X_2) \eta(Z_2)(0, Y_2) \\
&+ h(X_2, Z_2) |grad\alpha|^2(0, Y_2) - (\beta^2 - 1) \eta(Y_2) \eta(Z_2) |grad\alpha|^2(0, X_2) \\
&- \alpha\beta \eta(Y_2) \eta(Z_2) d\alpha(grad\beta)(0, X_2) + (\beta^2 - 1) \eta(X_2) \eta(Z_2) |grad\alpha|^2(0, Y_2) \\
&+ \alpha\beta \eta(X_2) \eta(Z_2) d\alpha(grad\beta)(0, Y_2) - \frac{\alpha}{\beta} \eta(X_2) h(Y_2, Z_2) d\alpha(grad\beta)(0, \xi) \\
&+ \frac{\alpha}{\beta} \eta(Y_2) h(X_2, Z_2) d\alpha(grad\beta)(0, \xi).
\end{aligned}$$

As consequences of Theorem 3.5, we get the following corollaries.

Corollary 3.6.

$$\widetilde{R}((X_1, 0), (e_i, 0))(e_i, 0) = (Ricci^M(X_1), 0),$$

$$\widetilde{R}((X_1, 0), (0, f_j))(0, f_j) = -n\alpha(\nabla_{X_1} grad\alpha, 0) + \frac{n}{\beta^3} X_1(\beta)(0, \xi),$$

$$\widetilde{R}((X_1, 0), (0, \varphi f_j))(0, \varphi f_j) = -n\alpha(\nabla_{X_1} grad\alpha, 0) + \frac{n}{\beta^3} X_1(\beta)(0, \xi)$$

and

$$\begin{aligned}
\widetilde{R}((X_1, 0), (0, \xi))(0, \xi) &= -\alpha\beta X_1(\beta)(grad\alpha, 0) - \alpha\beta X_1(\alpha)(grad\beta, 0) \\
&- \alpha\beta^2(\nabla_{X_1} grad\alpha, 0) - \alpha^2\beta(\nabla_{X_1} grad\beta, 0).
\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{Ricci}(X_1, 0) &= (Ricci^M(X_1), 0) - \frac{2n+1}{\alpha}(\nabla_{X_1} grad\alpha, 0) + \frac{2n}{\alpha^2\beta^3} X_1(\beta)(0, \xi) \\
&- \frac{1}{\alpha\beta} X_1(\beta)(grad\alpha, 0) - \frac{1}{\alpha\beta} X_1(\alpha)(grad\beta, 0) - \frac{1}{\beta}(\nabla_{X_1} grad\beta, 0).
\end{aligned}$$

Likewise, we have

$$\begin{aligned}\widetilde{R}((0, X_2), (e_i, 0))(e_i, 0) &= -\frac{1}{\alpha}(\Delta\alpha)(0, X_2) - \frac{1}{\beta}\eta(X_2)(\Delta\beta)(0, \xi) \\ &\quad - \frac{2}{\alpha\beta}\eta(X_2)d\alpha(grad\beta)(0, \xi),\end{aligned}$$

$$\begin{aligned}\widetilde{R}((0, X_2), (0, f_j))(0, f_j) &= (0, R^N(X_2, f_j)f_j) + \frac{n\alpha^2}{\beta}\eta(X_2)(grad\beta, 0) \\ &\quad + \frac{n(\beta^2 - 1)}{\beta^2}(0, X_2) - \frac{(\beta^2 - 1)}{\beta^2}h(X_2, f_j)(0, f_j) \\ &\quad + |grad\alpha|^2 h(X_2, f_j)(0, f_j) - n|grad\alpha|^2(0, X_2) \\ &\quad - \frac{n\alpha}{\beta}\eta(X_2)d\alpha(grad\beta)(0, \xi),\end{aligned}$$

$$\begin{aligned}\widetilde{R}((0, X_2), (0, \varphi f_j))(0, \varphi f_j) &= (0, R^N(X_2, \varphi f_j)\varphi f_j) + \frac{n\alpha^2}{\beta}\eta(X_2)(grad\beta, 0) \\ &\quad + \frac{n(\beta^2 - 1)}{\beta^2}(0, X_2) - \frac{(\beta^2 - 1)}{\beta^2}h(X_2, \varphi f_j)(0, \varphi f_j) \\ &\quad + |grad\alpha|^2 h(X_2, \varphi f_j)(0, \varphi f_j) - n|grad\alpha|^2(0, X_2) \\ &\quad - \frac{n\alpha}{\beta}\eta(X_2)d\alpha(grad\beta)(0, \xi)\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}((0, X_2), (0, \xi))(0, \xi) &= (0, R^N(X_2, \xi)\xi) - \alpha\beta d\alpha(grad\beta)(0, X_2) \\ &\quad - \beta^2|grad\alpha|^2(0, X_2) + \beta^2\eta(X_2)|grad\alpha|^2(0, \xi) \\ &\quad + \alpha\beta\eta(X_2)d\alpha(grad\beta)(0, \xi),\end{aligned}$$

which gives us

$$\begin{aligned}\widetilde{Ricci}(0, X_2) &= \frac{1}{\alpha^2}(0, Ricci^N(X_2)) - \frac{1}{\alpha}(\Delta\alpha)(0, X_2) - \frac{2n}{\alpha^2}|grad\alpha|^2(0, X_2) \\ &\quad + \frac{2n(\beta^2 - 1)}{\alpha^2\beta^2}(0, X_2) - \frac{1}{\alpha\beta}d\alpha(grad\beta)(0, X_2) - \frac{1}{\beta}\eta(X_2)(\Delta\beta)(0, \xi) \\ &\quad - \frac{2n+1}{\alpha\beta}\eta(X_2)d\alpha(grad\beta)(0, \xi) + \frac{2n}{\beta}\eta(X_2)(grad\beta, 0)\end{aligned}$$

Corollary 3.7.

$$\begin{aligned}
& \widetilde{Ricci}(X_1, X_2) \\
&= \left(Ricci^M(X_1), \frac{1}{\alpha^2} Ricci^N(X_2) \right) - \frac{2n+1}{\alpha} (\nabla_{X_1} grad\alpha, 0) - \frac{1}{\beta} (\nabla_{X_1} grad\beta, 0) \\
&+ \frac{2n}{\beta} \eta(X_2)(grad\beta, 0) - \frac{1}{\alpha\beta} X_1(\beta)(grad\alpha, 0) - \frac{1}{\alpha\beta} X_1(\alpha)(grad\beta, 0) \\
&- \frac{1}{\alpha} (\Delta\alpha)(0, X_2) - \frac{2n}{\alpha^2} |grad\alpha|^2(0, X_2) + \frac{2n(\beta^2 - 1)}{\alpha^2\beta^2}(0, X_2) \\
&- \frac{1}{\alpha\beta} d\alpha(grad\beta)(0, X_2) - \frac{2n+1}{\alpha\beta} \eta(X_2) d\alpha(grad\beta)(0, \xi) \\
&+ \frac{2n}{\alpha^2\beta^3} X_1(\beta)(0, \xi) - \frac{1}{\beta} \eta(X_2)(\Delta\beta)(0, \xi)
\end{aligned}$$

and

$$\begin{aligned}
& \widetilde{S}((X_1, X_2), (Y_1, Y_2)) \\
&= S^M(X_1, Y_1) + S^N(X_2, Y_2) - \frac{2n(\beta^2 - 1)}{\beta^2} \eta(X_2) \eta(Y_2) \\
&- \frac{2n+1}{\alpha} g(\nabla_{X_1} grad\alpha, Y_1) - \frac{1}{\beta} g(\nabla_{X_1} grad\beta, Y_1) + \frac{2n}{\beta} Y_1(\beta) \eta(X_2) \\
&+ \frac{2n}{\beta} X_1(\beta) \eta(Y_2) - \frac{1}{\alpha\beta} X_1(\beta) Y_1(\alpha) - \frac{1}{\alpha\beta} X_1(\alpha) Y_1(\beta) \\
&- \alpha(\beta^2 - 1)(\Delta\alpha) \eta(X_2) \eta(Y_2) - 2n(\beta^2 - 1) |grad\alpha|^2 \eta(X_2) \eta(Y_2) \\
&- 2(n+1)\alpha\beta d\alpha(grad\beta) \eta(X_2) \eta(Y_2) + \frac{\alpha}{\beta} d\alpha(grad\beta) \eta(X_2) \eta(Y_2) \\
&- \alpha^2\beta(\Delta\beta) \eta(X_2) \eta(Y_2) - \alpha(\Delta\alpha) h(X_2, Y_2) - 2n |grad\alpha|^2 h(X_2, Y_2) \\
&- \frac{\alpha}{\beta} d\alpha(grad\beta) h(X_2, Y_2) + \frac{2n(\beta^2 - 1)}{\beta^2} h(X_2, Y_2).
\end{aligned}$$

4. BIHARMONIC MAPS ON THE $\mathcal{D}$ -HOMOTHETIC BI-WARPING.

As a first result in this section, we will study the biharmonicity of the map $\Phi : (M \times N, \widetilde{G}) \longrightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$ where $\phi : (M^m, g) \longrightarrow (P^p, k)$ is a smooth map.

Theorem 4.1. *If we consider a smooth map $\phi : (M^m, g) \longrightarrow (P^p, k)$, the tension and bi-tension field of the map $\Phi : (M \times N, \widetilde{G}) \longrightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$ are given by*

$$\tau(\Phi) = \tau(\phi) + \frac{2n+1}{\alpha} d\phi(grad\alpha) + \frac{1}{\beta} d\phi(grad\beta)$$

and

$$\begin{aligned}
\tau_2(\Phi) &= \tau_2(\phi) - \frac{2n+1}{\alpha} \nabla_{grad\alpha} \tau(\phi) - \frac{1}{\beta} \nabla_{grad\beta} \tau(\phi) \\
&\quad - \frac{2n+1}{\alpha} \{Tr_g \nabla^2 d\phi(grad\alpha) + Tr_g R^P(d\phi(grad\alpha), d\phi) d\phi\} \\
&\quad - \frac{1}{\beta} \{Tr_g \nabla^2 d\phi(grad\beta) + Tr_g R^P(d\phi(grad\beta), d\phi) d\phi\} \\
&\quad + \frac{2n+1}{\alpha^2} (\Delta\alpha) d\phi(grad\alpha) + \frac{4n^2-1}{\alpha^3} |grad\alpha|^2 d\phi(grad\alpha) \\
&\quad + \frac{1}{\beta^2} (\Delta\beta) d\phi(grad\beta) - \frac{1}{\beta^3} |grad\beta|^2 d\phi(grad\beta) \\
&\quad + \frac{2n+1}{\alpha^2\beta} d\alpha(grad\beta) d\phi(grad\alpha) + \frac{2n+1}{\alpha\beta^2} d\alpha(grad\beta) d\phi(grad\beta) \\
&\quad - \frac{4n^2-1}{\alpha^2} \nabla_{grad\alpha} d\phi(grad\alpha) - \frac{2n+1}{\alpha\beta} \nabla_{grad\alpha} d\phi(grad\beta) \\
&\quad - \frac{2n+1}{\alpha\beta} \nabla_{grad\beta} d\phi(grad\alpha) + \frac{1}{\beta^2} \nabla_{grad\beta} d\phi(grad\beta).
\end{aligned}$$

Proof. By the definition of Φ , we obtain $d\Phi(X, Y) = d\phi(X)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$. The calculation of the tension field is given by

$$\begin{aligned}
\tau(\Phi) &= Tr_{\tilde{G}} \nabla d\Phi \\
&= \nabla_{(e_i, 0)} d\Phi(e_i, 0) - d\Phi\left(\tilde{\nabla}_{(e_i, 0)}(e_i, 0)\right) \\
&\quad + \frac{1}{\alpha^2} \nabla_{(0, f_j)} d\Phi(0, f_j) - \frac{1}{\alpha^2} d\Phi\left(\tilde{\nabla}_{(0, f_j)}(0, f_j)\right) \\
&\quad + \frac{1}{\alpha^2} \nabla_{(0, \varphi f_j)} d\Phi(0, \varphi f_j) - \frac{1}{\alpha^2} d\Phi\left(\tilde{\nabla}_{(0, \varphi f_j)}(0, \varphi f_j)\right) \\
&\quad + \frac{1}{\alpha^2 \beta^2} \nabla_{(0, \xi)} d\Phi(0, \xi) - \frac{1}{\alpha^2 \beta^2} d\Phi\left(\tilde{\nabla}_{(0, \xi)}(0, \xi)\right).
\end{aligned}$$

Using Lemma 3.3 and the fact that $\tau(\phi) = \nabla_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)$, we deduce that

$$\tau(\Phi) = \tau(\phi) + \frac{2n+1}{\alpha} d\phi(grad\alpha) + \frac{1}{\beta} d\phi(grad\beta).$$

Now calculate the bi-tension field; we have

$$\begin{aligned}
\tau_2(\Phi) &= -Tr_{\tilde{G}} \nabla^2 \tau(\Phi) - Tr_{\tilde{G}} R^P(\tau(\Phi), d\Phi) d\Phi \\
&= -Tr_{\tilde{G}} \nabla^2 \tau(\phi) - Tr_{\tilde{G}} R^P(\tau(\phi), d\Phi) d\Phi \\
&\quad - (2n+1) Tr_{\tilde{G}} \nabla^2 \frac{1}{\alpha} d\phi(grad\alpha) - \frac{2n+1}{\alpha} Tr_{\tilde{G}} R^P(d\phi(grad\alpha), d\Phi) d\Phi \\
&\quad - Tr_{\tilde{G}} \nabla^2 \frac{1}{\beta} d\phi(grad\beta) - \frac{1}{\beta} Tr_{\tilde{G}} R^P(d\phi(grad\beta), d\Phi) d\Phi.
\end{aligned}$$

A simple calculation gives us

$$\begin{aligned}
Tr_{\tilde{G}} \nabla^2 \tau(\phi) &= \nabla_{(e_i, 0)} \nabla_{(e_i, 0)} \tau(\phi) - \nabla_{\tilde{\nabla}_{(e_i, 0)}(e_i, 0)} \tau(\phi) \\
&+ \frac{1}{\alpha^2} \nabla_{(0, f_j)} \nabla_{(0, f_j)} \tau(\phi) - \frac{1}{\alpha^2} \nabla_{\tilde{\nabla}_{(0, f_j)}(0, f_j)} \tau(\phi) \\
&+ \frac{1}{\alpha^2} \nabla_{(0, \varphi f_j)} \nabla_{(0, \varphi f_j)} \tau(\phi) - \frac{1}{\alpha^2} \nabla_{\tilde{\nabla}_{(0, \varphi f_j)}(0, \varphi f_j)} \tau(\phi) \\
&+ \frac{1}{\alpha^2 \beta^2} \nabla_{(0, \xi)} \nabla_{(0, \xi)} \tau(\phi) - \frac{1}{\alpha^2 \beta^2} \nabla_{\tilde{\nabla}_{(0, \xi)}(0, \xi)} \tau(\phi) \\
&= \nabla_{e_i} \nabla_{e_i} \tau(\phi) - \nabla_{\nabla_{e_i} e_i} \tau(\phi) - \frac{1}{\alpha^2} \nabla_{\tilde{\nabla}_{(0, f_j)}(0, f_j)} \tau(\phi) \\
&- \frac{1}{\alpha^2} \nabla_{\tilde{\nabla}_{(0, \varphi f_j)}(0, \varphi f_j)} \tau(\phi) - \frac{1}{\alpha^2 \beta^2} \nabla_{\tilde{\nabla}_{(0, \xi)}(0, \xi)} \tau(\phi) \\
&= Tr_g \nabla^2 \tau(\phi) + \frac{2n+1}{\alpha} \nabla_{grad\alpha} \tau(\phi) + \frac{1}{\beta} \nabla_{grad\beta} \tau(\phi)
\end{aligned}$$

and

$$Tr_{\tilde{G}} R^P(\tau(\phi), d\Phi) d\Phi = Tr_g R^P(\tau(\phi), d\phi) d\phi.$$

Similarly, we obtain

$$\begin{aligned}
Tr_{\tilde{G}} \nabla^2 \frac{1}{\alpha} d\phi(grad\alpha) &= \frac{1}{\alpha} Tr_g \nabla^2 d\phi(grad\alpha) - \frac{1}{\alpha^2} (\Delta\alpha) d\phi(grad\alpha) \\
&- \frac{(2n-1)}{\alpha^3} |grad\alpha|^2 d\phi(grad\alpha) + \frac{(2n-1)}{\alpha^2} \nabla_{grad\alpha} d\phi(grad\alpha) \\
&+ \frac{1}{\alpha\beta} \nabla_{grad\beta} d\phi(grad\alpha) - \frac{1}{\alpha^2\beta} d\alpha(grad\beta) d\phi(grad\alpha),
\end{aligned}$$

$$\begin{aligned}
Tr_{\tilde{G}} \nabla^2 \frac{1}{\beta} d\phi(grad\beta) &= \frac{1}{\beta} Tr_g \nabla^2 d\phi(grad\beta) - \frac{1}{\beta^2} (\Delta\beta) d\phi(grad\beta) \\
&+ \frac{1}{\beta^3} |grad\beta|^2 d\phi(grad\beta) - \frac{1}{\beta^2} \nabla_{grad\beta} d\phi(grad\beta) \\
&+ \frac{2n+1}{\alpha\beta} \nabla_{grad\alpha} d\phi(grad\beta) - \frac{2n+1}{\alpha\beta^2} d\alpha(grad\beta) d\phi(grad\beta),
\end{aligned}$$

$$Tr_{\tilde{G}} R^P(d\phi(grad\alpha), d\Phi) d\Phi = Tr_g R^P(d\phi(grad\alpha), d\phi) d\phi$$

and

$$Tr_{\tilde{G}} R^P(d\phi(grad\beta), d\Phi) d\Phi = Tr_g R^P(d\phi(grad\beta), d\phi) d\phi.$$

Finally, we deduce that

$$\begin{aligned}
\tau_2(\Phi) = & \tau_2(\phi) - \frac{2n+1}{\alpha} \nabla_{grad\alpha} \tau(\phi) - \frac{1}{\beta} \nabla_{grad\beta} \tau(\phi) \\
& - \frac{2n+1}{\alpha} \{Tr_g \nabla^2 d\phi(grad\alpha) + Tr_g R^P(d\phi(grad\alpha), d\phi)d\phi\} \\
& - \frac{1}{\beta} \{Tr_g \nabla^2 d\phi(grad\beta) + Tr_g R^P(d\phi(grad\beta), d\phi)d\phi\} \\
& + \frac{2n+1}{\alpha^2} (\Delta\alpha) d\phi(grad\alpha) + \frac{4n^2-1}{\alpha^3} |grad\alpha|^2 d\phi(grad\alpha) \\
& + \frac{1}{\beta^2} (\Delta\beta) d\phi(grad\beta) - \frac{1}{\beta^3} |grad\beta|^2 d\phi(grad\beta) \\
& + \frac{2n+1}{\alpha^2\beta} d\alpha(grad\beta) d\phi(grad\alpha) + \frac{2n+1}{\alpha\beta^2} d\alpha(grad\beta) d\phi(grad\beta) \\
& - \frac{4n^2-1}{\alpha^2} \nabla_{grad\alpha} d\phi(grad\alpha) - \frac{2n+1}{\alpha\beta} \nabla_{grad\alpha} d\phi(grad\beta) \\
& - \frac{2n+1}{\alpha\beta} \nabla_{grad\beta} d\phi(grad\alpha) + \frac{1}{\beta^2} \nabla_{grad\beta} d\phi(grad\beta).
\end{aligned}$$

If $\beta = \pm 1$, we find the Proposition 2 and the Theorem 4 obtained in [10]. As a consequence of Theorem 4.1, we have

Corollary 4.2. *The first projection $P_1 : (M \times N, \tilde{G}) \longrightarrow (M, g)$ is biharmonic if and only if*

$$\begin{aligned}
& \frac{2n+1}{\alpha} grad\Delta\alpha + \frac{1}{\beta} grad\Delta\beta - \frac{2n+1}{\alpha^2} (\Delta\alpha) grad\alpha - \frac{1}{\beta^2} (\Delta\beta) grad\beta \\
& - \frac{4n^2-1}{\alpha^3} |grad\alpha|^2 grad\alpha + \frac{1}{\beta^3} |grad\beta|^2 grad\beta \\
& + \frac{4n^2-1}{2\alpha^2} grad(|grad\alpha|^2) - \frac{1}{2\beta^2} grad(|grad\beta|^2) \\
& - \frac{2n+1}{\alpha\beta} \left(\frac{1}{\alpha} d\alpha(grad\beta) grad\alpha + \frac{1}{\beta} d\alpha(grad\beta) grad\beta \right) \\
& + \frac{2n+1}{\alpha\beta} (\nabla_{grad\alpha} grad\beta + \nabla_{grad\beta} grad\alpha) \\
& + \frac{2(2n+1)}{\alpha} Ricci^M(grad\alpha) + \frac{2}{\beta} Ricci^M(grad\beta) = 0.
\end{aligned}$$

Example 4.3. *We consider the first projection $P_1 : (\mathbb{R} \times N, \tilde{G}) \longrightarrow (\mathbb{R}, dt^2)$ defined by $P_1(t, x) = t$. By Corollary 4.2, P_1 is biharmonic if and only if the functions*

α and β satisfy the following differential equation

$$\begin{aligned} & \frac{2n+1}{\alpha} \alpha''' + \frac{2(2n+1)(n-1)}{\alpha^2} \alpha' \alpha'' - \frac{4n^2-1}{\alpha^3} (\alpha')^3 + \frac{1}{\beta} \beta''' - \frac{2}{\beta^2} \beta' \beta'' + \frac{1}{\beta^3} (\beta')^3 \\ & - \frac{2n+1}{\alpha^2 \beta} (\alpha')^2 \beta' - \frac{2n+1}{\alpha \beta^2} \alpha' (\beta')^2 + \frac{2n+1}{\alpha \beta} \alpha' \beta'' + \frac{2n+1}{\alpha \beta} \alpha'' \beta' = 0. \end{aligned}$$

Looking for particular solutions of type $\alpha(t) = t^a$ and $\beta(t) = t^b$ ($a, b \in \mathbb{R}^*$), then P_1 is biharmonic if and only if a and b are solutions of the following algebraic equation

$$(2n+1)^2 a^2 + 2(2n+1)(b-1)a + b^2 - 2b = 0,$$

which is equivalent to

$$\left(a + \frac{b}{2n+1}\right) \left(a + \frac{b-2}{2n+1}\right) = 0.$$

For this equation, we can distinguish the following cases :

- (1) For $a = \frac{-b}{2n+1}$, P_1 is harmonic and thus biharmonic.
- (2) For $a = \frac{2-b}{2n+1}$, we get $\alpha(t) = t^{\frac{2-b}{2n+1}}$ and $\beta(t) = t^b$ for any $b \in \mathbb{R}^*$. In this case, the first projection P_1 is biharmonic non-harmonic.

Example 4.4. We consider the first projection $P_1 : (\mathbb{R}^m \setminus \{0\} \times N^{2n+1}, \tilde{G}) \rightarrow \mathbb{R}^m \setminus \{0\}$ ($m \geq 2$) when we suppose that the functions α and β are radial ($\alpha = \alpha(r)$, $\beta = \beta(r)$, $r = |x|$ for any $x \in \mathbb{R}^m \setminus \{0\}$). By Corollary 3.6, P_1 is biharmonic if and only if the functions α and β satisfy the following differential equation

$$\begin{aligned} & \frac{2n+1}{\alpha} \alpha''' + \frac{2(n-1)(2n+1)}{\alpha^2} \alpha' \alpha'' + \frac{(m-1)(2n+1)}{r\alpha} \alpha'' - \frac{(m-1)(2n+1)}{r^2 \alpha} \alpha' \\ & - \frac{(m-1)(2n+1)}{r\alpha^2} (\alpha')^2 - \frac{4n^2-1}{\alpha^3} (\alpha')^3 + \frac{1}{\beta} \beta''' - \frac{2}{\beta^2} \beta' \beta'' + \frac{m-1}{r\beta} \beta'' \\ & - \frac{m-1}{r^2 \beta} \beta' - \frac{m-1}{r\beta^2} (\beta')^2 + \frac{1}{\beta^3} (\beta')^3 + \frac{2n+1}{\alpha \beta} \alpha' \beta'' + \frac{2n+1}{\alpha \beta} \alpha'' \beta' \\ & - \frac{2n+1}{\alpha^2 \beta} (\alpha')^2 \beta' - \frac{2n+1}{\alpha \beta^2} \alpha' (\beta')^2 = 0. \end{aligned}$$

To solve this equation, we will present two cases.

- First case : We look at particular solutions of the form $\alpha(r) = \frac{a}{r}$ and $\beta(r) = \frac{b}{r}$ ($a, b \in \mathbb{R}^*$); then, P_1 is biharmonic non-harmonic if and only if

$$m = n + 3.$$

In this case, we deduce that P_1 is biharmonic non-harmonic if and only if $m = n + 3$ for all functions of type $\alpha(r) = \frac{a}{r}$ and $\beta(r) = \frac{b}{r}$ ($a, b \in \mathbb{R}^*$).

- *Second case* : Another method to solve this equation is to look at the solutions of the form $\alpha(r) = r^a$ and $\beta(r) = r^b$ ($a, b \in \mathbb{R}^*$); then, P_1 is biharmonic non-harmonic if and only if a and b are solutions of the following algebraic equation

$$(2n+1)^2 a^2 + 2(2n+1)(b+m-2)a + (b^2 + 2(m-2)b) = 0.$$

For this equation, the solutions are given by :

- (1) $a = \frac{-b}{2n+1}$, then P_1 is harmonic so it is biharmonic.
- (2) $a = -\frac{b+2m-4}{2n+1}$ ($m \geq 3$), so $\alpha(r) = r^{-\frac{b+2m-4}{2n+1}}$ and $\beta(r) = r^b$ for any $b \in \mathbb{R}^*$. In this case, the first projection P_1 is biharmonic non-harmonic.

Remark 4.5. By Corollary 3.6, we can give some particular cases :

- (1) If $\alpha = \pm\beta$, we get the $\mathcal{D}$ -homothetically warped metric (see [4]), and the first projection is biharmonic if and only if

$$\begin{aligned} & \alpha^2 \text{grad} \Delta \alpha - \alpha (\Delta \alpha) \text{grad} \alpha - 2n |\text{grad} \alpha|^2 \text{grad} \alpha \\ & + n \alpha \text{grad} (|\text{grad} \alpha|^2) + 2\alpha^2 \text{Ricci}^M (\text{grad} \alpha) = 0. \end{aligned}$$

- (2) If $\beta = \pm 1$, we obtain a warped product metric and in this case the first projection is biharmonic if and only if

$$\begin{aligned} & \alpha^2 \text{grad} \Delta \alpha - \alpha (\Delta \alpha) \text{grad} \alpha - (2n-1) |\text{grad} \alpha|^2 \text{grad} \alpha \\ & + \frac{(2n-1)}{2} \alpha \text{grad} (|\text{grad} \alpha|^2) + 2\alpha^2 \text{Ricci}^M (\text{grad} \alpha) = 0. \end{aligned}$$

- (3) If α is a constant function, then the first projection is biharmonic if and only if

$$\begin{aligned} & \beta^2 \text{grad} \Delta \beta - \beta (\Delta \beta) \text{grad} \beta + |\text{grad} \beta|^2 \text{grad} \beta \\ & - \frac{1}{2} \beta \text{grad} (|\text{grad} \beta|^2) + 2\beta^2 \text{Ricci}^M (\text{grad} \beta) = 0. \end{aligned}$$

Proposition 4.6. Let $\Psi : (M \times N, \tilde{G}) \longrightarrow (P^p, k)$ be a map defined by $\Psi(x, y) = \psi(y)$ where $\psi : (N^{2n+1}, h) \longrightarrow (P^p, k)$ is a smooth map. The tension field of the map $\Psi : (M \times N, \tilde{G}) \longrightarrow (P^p, k)$ is given by

$$\tau(\Psi) = \frac{1}{\alpha^2} \tau(\psi) - \frac{\beta^2 - 1}{\alpha^2 \beta^2} (\text{div} \xi) d\psi(\xi) + \frac{1 - \beta^2}{\alpha^2 \beta^2} \nabla_\xi d\psi(\xi).$$

Proof. By definition, we have

$$\begin{aligned}\tau(\Psi) &= Tr_{\tilde{G}} \nabla d\Psi \\ &= \nabla_{(e_i,0)} d\Psi(e_i, 0) - d\Psi\left(\tilde{\nabla}_{(e_i,0)}(e_i, 0)\right) \\ &\quad + \frac{1}{\alpha^2} \nabla_{(0,f_j)} d\Psi(0, f_j) - \frac{1}{\alpha^2} d\Psi\left(\tilde{\nabla}_{(0,f_j)}(0, f_j)\right) \\ &\quad + \frac{1}{\alpha^2} \nabla_{(0,\varphi f_j)} d\Psi(0, \varphi f_j) - \frac{1}{\alpha^2} d\Psi\left(\tilde{\nabla}_{(0,\varphi f_j)}(0, \varphi f_j)\right) \\ &\quad + \frac{1}{\alpha^2 \beta^2} \nabla_{(0,\xi)} d\Psi(0, \xi) - \frac{1}{\alpha^2 \beta^2} d\Psi\left(\tilde{\nabla}_{(0,\xi)}(0, \xi)\right).\end{aligned}$$

Lemma 3.3 gives us

$$\begin{aligned}\tau(\Psi) &= \frac{1}{\alpha^2} \nabla_{f_j} d\psi(f_j) - \frac{1}{\alpha^2} d\psi\left(\nabla'_{f_j} f_j\right) + \frac{\beta^2 - 1}{\alpha^2 \beta^2} \eta\left(\nabla'_{f_j} f_j\right) d\psi(\xi) \\ &\quad + \frac{1}{\alpha^2} \nabla_{\varphi f_j} d\psi(\varphi f_j) - \frac{1}{\alpha^2} d\psi\left(\nabla'_{\varphi f_j} \varphi f_j\right) + \frac{\beta^2 - 1}{\alpha^2 \beta^2} \eta\left(\nabla'_{\varphi f_j} \varphi f_j\right) d\psi(\xi) \\ &\quad + \frac{1}{\alpha^2 \beta^2} \nabla_{\xi} d\psi(\xi) - \frac{1}{\alpha^2} d\psi\left(\nabla'_{\xi} \xi\right) \\ &= \frac{1}{\alpha^2} \nabla_{f_j} d\psi(f_j) - \frac{1}{\alpha^2} d\psi\left(\nabla'_{f_j} f_j\right) + \frac{1}{\alpha^2} \nabla_{\varphi f_j} d\psi(\varphi f_j) - \frac{1}{\alpha^2} d\psi\left(\nabla'_{\varphi f_j} \varphi f_j\right) \\ &\quad + \frac{1}{\alpha^2} \nabla_{\xi} d\psi(\xi) - \frac{1}{\alpha^2} d\psi\left(\nabla'_{\xi} \xi\right) + \frac{\beta^2 - 1}{\alpha^2 \beta^2} \eta\left(\nabla'_{f_j} f_j\right) d\psi(\xi) \\ &\quad + \frac{\beta^2 - 1}{\alpha^2 \beta^2} \eta\left(\nabla'_{\varphi f_j} \varphi f_j\right) d\psi(\xi) + \frac{1 - \beta^2}{\alpha^2 \beta^2} \nabla_{\xi} d\psi(\xi)\end{aligned}$$

Finally, by using the definition of $div\xi$ and the fact that

$$\begin{aligned}\tau(\psi) &= \nabla_{f_j} d\psi(f_j) - d\psi\left(\nabla'_{f_j} f_j\right) + \nabla_{\varphi f_j} d\psi(\varphi f_j) \\ &\quad - d\psi\left(\nabla'_{\varphi f_j} \varphi f_j\right) + \nabla_{\xi} d\psi(\xi) - d\psi\left(\nabla'_{\xi} \xi\right),\end{aligned}$$

we conclude that

$$\tau(\Psi) = \frac{1}{\alpha^2} \tau(\psi) - \frac{\beta^2 - 1}{\alpha^2 \beta^2} (div\xi) d\psi(\xi) + \frac{1 - \beta^2}{\alpha^2 \beta^2} \nabla_{\xi} d\psi(\xi).$$

In particular if $\beta = \pm 1$, we find the expression of the tension field obtained in Theorem 5 (see [10]). The harmonicity of Ψ is given as follows.

Remark 4.7. *The map Ψ is harmonic if and only if*

$$\beta^2 \tau(\psi) - (\beta^2 - 1) \{(div\xi) d\psi(\xi) + \nabla_{\xi} d\psi(\xi)\} = 0.$$

Moreover, if ψ is harmonic, then Ψ is harmonic if and only if

$$\nabla_{\xi} d\psi(\xi) + (\operatorname{div}\xi) d\psi(\xi) = 0.$$

In particular, if ψ is the identity map, we obtain

Remark 4.8. The tension field of the second projection is given by

$$\tau(P_2) = \frac{1 - \beta^2}{\alpha^2 \beta^2} \{(\operatorname{div}\xi)\xi + \nabla_{\xi}\xi\},$$

then P_2 is harmonic if and only if

$$\nabla'_{\xi}\xi + (\operatorname{div}\xi)\xi = 0.$$

In the case where $(N^{2n+1}, \varphi, \xi, \eta, h)$ is a Kenmotsu manifold, the tension field of the second projection becomes

$$\tau(P_2) = -\frac{2n(\beta^2 - 1)}{\alpha^2 \beta^2} \xi.$$

Note that if $\beta \neq \pm 1$, then the second projection is never harmonic.

The following results give us the conditions of biharmonicity of the second projection and the inclusion map in the case where $(N^{2n+1}, \varphi, \xi, \eta, h)$ is a Kenmotsu manifold.

Theorem 4.9. Let $(N^{2n+1}, \varphi, \xi, \eta, h)$ be a Kenmotsu manifold, the second projection $P_2 : (M \times N, \tilde{G}) \rightarrow (N, h)$ is biharmonic if and only if the functions α and β are solutions of the following equation

$$\begin{aligned} & \frac{2(\beta^2 - 1)}{\alpha^3 \beta^2} \Delta \alpha - \frac{2}{\alpha^2 \beta^3} (\Delta \beta) + \frac{4(n-1)(\beta^2 - 1)}{\alpha^4 \beta^2} |\operatorname{grad} \alpha|^2 + \frac{4}{\alpha^2 \beta^4} |\operatorname{grad} \beta|^2 \\ & - \frac{4(n-1) - 2\beta^2}{\alpha^3 \beta^3} d\alpha(\operatorname{grad} \beta) + \frac{4n(\beta^2 - 1)}{\alpha^4 \beta^2} = 0. \end{aligned}$$

Proof. By Remark 4.8, we obtain

$$\tau(P_2) = -\frac{2n(\beta^2 - 1)}{\alpha^2 \beta^2} \xi.$$

To simplify our calculations, we denote $f = \frac{\beta^2 - 1}{\alpha^2 \beta^2}$, then the second projection is biharmonic if and only if

$$\operatorname{Tr}_{\tilde{G}} \nabla^2 f \xi + f \operatorname{Tr}_{\tilde{G}} R^N(\xi, \cdot) \cdot = 0.$$

The calculation of the first term $Tr_{\tilde{G}} \nabla^2 f \xi$ gives us

$$\begin{aligned} Tr_{\tilde{G}} \nabla^2 f \xi &= \nabla_{(e_i, 0)} \nabla_{(e_i, 0)} f \xi - \nabla_{\tilde{\nabla}_{(e_i, 0)}(e_i, 0)} f \xi \\ &+ \frac{1}{\alpha^2} \nabla_{(0, f_j)} \nabla_{(0, f_j)} f \xi - \frac{1}{\alpha^2} \nabla_{\tilde{\nabla}_{(0, f_j)}(0, f_j)} f \xi \\ &+ \frac{1}{\alpha^2} \nabla_{(0, \varphi f_j)} \nabla_{(0, \varphi f_j)} f \xi - \frac{1}{\alpha^2} \nabla_{\tilde{\nabla}_{(0, \varphi f_j)}(0, \varphi f_j)} f \xi \\ &+ \frac{1}{\alpha^2 \beta^2} \nabla_{(0, \xi)} \nabla_{(0, \xi)} f \xi - \frac{1}{\alpha^2 \beta^2} \nabla_{\tilde{\nabla}_{(0, \xi)}(0, \xi)} f \xi. \end{aligned}$$

Using the equations of Lemma 3.3 and the fact that $\nabla'_\xi \xi = 0$, we get

$$\begin{aligned} Tr_{\tilde{G}} \nabla^2 f \xi &= e_i(e_i(f))\xi - (\nabla_{e_i} e_i)(f)\xi + \frac{f}{\alpha^2} \nabla'_{f_j} \nabla'_{f_j} \xi \\ &- \frac{f}{\alpha^2} \nabla'_{\nabla'_{f_j} f_j} \xi + \frac{2n+1}{\alpha} d\alpha(gradf)\xi + \frac{1}{\beta} d\beta(gradf)\xi \\ &+ \frac{f}{\alpha^2} \nabla'_{\varphi f_j} \nabla'_{\varphi f_j} \xi - \frac{f}{\alpha^2} \nabla'_{\nabla'_{\varphi f_j} \varphi f_j} \xi. \end{aligned}$$

As $(N^{2n+1}, \varphi, \xi, \eta, h)$ is a Kenmotsu manifold, a simple calculation gives us

$$\nabla'_{f_j} \nabla'_{f_j} \xi - \nabla'_{\nabla'_{f_j} f_j} \xi = \nabla'_{f_j} f_j - \nabla'_{f_j} f_j + \eta(\nabla'_{f_j} f_j)\xi = -n\xi$$

and

$$\nabla'_{\varphi f_j} \nabla'_{\varphi f_j} \xi - \nabla'_{\nabla'_{\varphi f_j} \varphi f_j} \xi = -n\xi.$$

It follows that

$$Tr_{\tilde{G}} \nabla^2 f \xi = \left(\Delta f - \frac{2nf}{\alpha^2} + \frac{2n+1}{\alpha} d\alpha(gradf) + \frac{1}{\beta} d\beta(gradf) \right) \xi.$$

For the term $fTr_{G_{\alpha, \beta}} R^N(\xi, \cdot) \cdot$, it is clear that

$$fTr_{G_{\alpha, \beta}} R^N(\xi, \cdot) \cdot = \frac{f}{\alpha^2} Ricci^N(\xi).$$

Then, we conclude that P_2 is biharmonic if and only if

$$\left(\Delta f - \frac{2nf}{\alpha^2} + \frac{2n+1}{\alpha} d\alpha(gradf) + \frac{1}{\beta} d\beta(gradf) \right) \xi + \frac{f}{\alpha^2} Ricci^N(\xi) = 0.$$

A simple calculation gives

$$\begin{aligned} \Delta f &= -\frac{2(\beta^2 - 1)}{\alpha^3 \beta^2} \Delta \alpha + \frac{6(\beta^2 - 1)}{\alpha^4 \beta^2} |grad\alpha|^2 + \frac{2}{\alpha^2 \beta^3} (\Delta \beta) \\ &- \frac{6}{\alpha^2 \beta^4} |grad\beta|^2 - \frac{8}{\alpha^3 \beta^3} d\alpha(grad\beta), \\ d\alpha(gradf) &= -\frac{2(\beta^2 - 1)}{\alpha^3 \beta^2} |grad\alpha|^2 + \frac{2}{\alpha^2 \beta^3} d\alpha(grad\beta) \end{aligned}$$

and

$$d\beta(\operatorname{grad} f) = -\frac{2(\beta^2 - 1)}{\alpha^3 \beta^2} d\alpha(\operatorname{grad} \beta) + \frac{2}{\alpha^2 \beta^3} |\operatorname{grad} \beta|^2.$$

We deduce that the second projection P_2 is biharmonic if and only if

$$\begin{aligned} & \left(-\frac{2(\beta^2 - 1)}{\alpha^3 \beta^2} \Delta \alpha - \frac{4(n-1)(\beta^2 - 1)}{\alpha^4 \beta^2} |\operatorname{grad} \alpha|^2 + \frac{2}{\alpha^2 \beta^3} (\Delta \beta) - \frac{4}{\alpha^2 \beta^4} |\operatorname{grad} \beta|^2 \right) \xi \\ & + \left(\frac{4(n-1) - 2\beta^2}{\alpha^3 \beta^3} d\alpha(\operatorname{grad} \beta) - \frac{2n(\beta^2 - 1)}{\alpha^4 \beta^2} \right) \xi + \frac{\beta^2 - 1}{\alpha^4 \beta^2} \operatorname{Ricci}^N(\xi) = 0. \end{aligned}$$

Finally, using the fact that $\operatorname{Ricci}^N(\xi) = -2n\xi$, then the last formula becomes

$$\begin{aligned} & \frac{2(\beta^2 - 1)}{\alpha^3 \beta^2} \Delta \alpha - \frac{2}{\alpha^2 \beta^3} (\Delta \beta) + \frac{4(n-1)(\beta^2 - 1)}{\alpha^4 \beta^2} |\operatorname{grad} \alpha|^2 + \frac{4}{\alpha^2 \beta^4} |\operatorname{grad} \beta|^2 \\ & - \frac{4(n-1) - 2\beta^2}{\alpha^3 \beta^3} d\alpha(\operatorname{grad} \beta) + \frac{4n(\beta^2 - 1)}{\alpha^4 \beta^2} = 0. \end{aligned}$$

Remark 4.10. By Theorem 4.9, we can give some particular cases :

(1) If $\alpha = \pm\beta$, we get the $\mathcal{D}$ -homothetically warped metric, and the second projection is biharmonic if and only if

$$\alpha(\alpha^2 - 2)\Delta\alpha + ((2n-1)\alpha^2 - 4n + 6)|\operatorname{grad} \alpha|^2 + 2n(\alpha^2 - 1) = 0.$$

(2) If $\beta = \pm 1$, the second projection is harmonic so it is biharmonic.

(3) If $\alpha = 1$, then the second projection is biharmonic if and only if

$$\beta(\Delta\beta) - 2|\operatorname{grad} \beta|^2 - 2n\beta^2(\beta^2 - 1) = 0.$$

Theorem 4.11. Let $(N^{2n+1}, \varphi, \xi, \eta, h)$ be a Kenmotsu manifold, the inclusion map $i_{x_0} : N \longrightarrow (M \times N, \tilde{G})$ defined by $i_{x_0}(y) = (x_0, y)$ is biharmonic if and only if the following condition is verified

$$\begin{aligned} & \left((\beta^2 + 2n)^2 |\operatorname{grad} \alpha|^2 + 2\alpha^2 \beta^2 |\operatorname{grad} \beta|^2 + 3\alpha\beta(\beta^2 + 2n) d\alpha(\operatorname{grad} \beta) \right) \operatorname{grad} \alpha \\ & + \alpha \left(\alpha^2 \beta |\operatorname{grad} \beta|^2 + \alpha(3\beta^2 + 2n) d\alpha(\operatorname{grad} \beta) + 2\beta(\beta^2 + 2n) |\operatorname{grad} \alpha|^2 \right) \operatorname{grad} \beta \\ & - \frac{4n^2(\beta^2 - 1)(\beta^2 + 1)}{\beta^2} \operatorname{grad} \alpha - \frac{4n^2\alpha(\beta^2 - 1)^2}{\beta^3} \operatorname{grad} \beta \\ & + \frac{\alpha(\beta^2 + 2n)^2}{2} \operatorname{grad}(|\operatorname{grad} \alpha|^2) + \frac{\alpha^3 \beta^2}{2} \operatorname{grad}(|\operatorname{grad} \beta|^2) \\ & + \alpha^2 \beta(\beta^2 + 2n) (\nabla_{\operatorname{grad} \alpha} \operatorname{grad} \beta + \nabla_{\operatorname{grad} \beta} \operatorname{grad} \alpha) = 0. \end{aligned}$$

and

$$\begin{aligned} & (\beta^2 - 1) (\beta^2 + 2n) |\operatorname{grad}\alpha|^2 + \alpha^2 \beta^2 |\operatorname{grad}\beta|^2 \\ & + \alpha\beta (2\beta^2 + n - 1) d\alpha (\operatorname{grad}\beta) + 2n (\beta^2 - 1) = 0. \end{aligned}$$

Proof. The tension field of i_{x_0} is given by

$$\begin{aligned} \tau(i_{x_0}) &= \operatorname{Tr}_h \tilde{\nabla} di_{x_0} \\ &= \tilde{\nabla}_{(0, f_j)} (0, f_j) - \nabla_{(0, f_j)} (0, f_j) + \tilde{\nabla}_{(0, \varphi f_j)} (0, \varphi f_j) \\ &\quad - \nabla_{(0, \varphi f_j)} (0, \varphi f_j) + \tilde{\nabla}_{(0, \xi)} (0, \xi) - \nabla_{(0, \xi)} (0, \xi) \end{aligned}$$

By Remark 3.4, we get

$$\tau(i_{x_0}) = -\alpha (2n + \beta^2) (\operatorname{grad}\alpha, 0) - \alpha^2 \beta (\operatorname{grad}\beta, 0) + \frac{2n (\beta^2 - 1)}{\beta^2} (0, \xi).$$

Then i_{x_0} is biharmonic if and only if

$$\begin{aligned} (5) \quad & \alpha (2n + \beta^2) \left(\operatorname{Tr}_h \left(\tilde{\nabla} \right)^2 (\operatorname{grad}\alpha, 0) + \operatorname{Tr}_h \tilde{R}((\operatorname{grad}\alpha, 0), di_{x_0}) di_{x_0} \right) \\ & + \alpha^2 \beta \left(\operatorname{Tr}_h \left(\tilde{\nabla} \right)^2 (\operatorname{grad}\beta, 0) + \operatorname{Tr}_h \tilde{R}((\operatorname{grad}\beta, 0), di_{x_0}) di_{x_0} \right) \\ & - \frac{2n (\beta^2 - 1)}{\beta^2} \left(\operatorname{Tr}_h \left(\tilde{\nabla} \right)^2 (0, \xi) + \operatorname{Tr}_h \tilde{R}((0, \xi), di_{x_0}) di_{x_0} \right) = 0. \end{aligned}$$

Using Remark 3.4, we obtain

$$\begin{aligned} \operatorname{Tr}_h \left(\tilde{\nabla} \right)^2 (\operatorname{grad}\alpha, 0) &= - (2n + \beta^2) |\operatorname{grad}\alpha|^2 (\operatorname{grad}\alpha, 0) - \alpha\beta |\operatorname{grad}\alpha|^2 (\operatorname{grad}\beta, 0) \\ &\quad - \alpha\beta d\alpha (\operatorname{grad}\beta) (\operatorname{grad}\alpha, 0) - \alpha^2 d\alpha (\operatorname{grad}\beta) (\operatorname{grad}\beta, 0) \\ &\quad + \frac{2n (\beta^2 - 1)}{\alpha\beta^2} |\operatorname{grad}\alpha|^2 (0, \xi) + \frac{2n}{\beta} d\alpha (\operatorname{grad}\beta) (0, \xi), \end{aligned}$$

$$\begin{aligned} \operatorname{Tr}_h \left(\tilde{\nabla} \right)^2 (\operatorname{grad}\beta, 0) &= - (2n + \beta^2) d\alpha (\operatorname{grad}\beta) (\operatorname{grad}\alpha, 0) - \alpha\beta |\operatorname{grad}\beta|^2 (\operatorname{grad}\alpha, 0) \\ &\quad - \alpha^2 |\operatorname{grad}\beta|^2 (\operatorname{grad}\beta, 0) - \alpha\beta d\alpha (\operatorname{grad}\beta) (\operatorname{grad}\beta, 0) \\ &\quad + \frac{2n (\beta^2 - 1)}{\alpha\beta^2} d\alpha (\operatorname{grad}\beta) (0, \xi) + \frac{2n}{\beta} |\operatorname{grad}\beta|^2 (0, \xi) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Tr}_h \left(\tilde{\nabla} \right)^2 (0, \xi) &= -2n\alpha (\beta^2 + 1) (\operatorname{grad}\alpha, 0) - 2n\alpha^2 \beta (\operatorname{grad}\beta, 0) \\ &\quad - \beta^2 |\operatorname{grad}\alpha|^2 (0, \xi) - \alpha^2 |\operatorname{grad}\beta|^2 (0, \xi) \\ &\quad - 2\alpha\beta d\alpha (\operatorname{grad}\beta) (0, \xi) - \frac{2n}{\beta^2} (0, \xi) \end{aligned}$$

By definition, we have

$$\begin{aligned} Tr_h \tilde{R}((grad\alpha, 0), di_{x_0}) di_{x_0} &= \tilde{R}((grad\alpha, 0), (0, f_j))(0, f_j) \\ &+ \tilde{R}((grad\alpha, 0), (0, \varphi f_j))(0, \varphi f_j) \\ &+ \tilde{R}((grad\alpha, 0), (0, \xi))(0, \xi). \end{aligned}$$

By Remark 3.4, a long calculation gives

$$\tilde{R}((grad\alpha, 0), (0, f_j))(0, f_j) = -n\alpha (\nabla_{grad\alpha} grad\alpha, 0) + \frac{n}{\beta^3} d\alpha (grad\beta)(0, \xi),$$

$$\tilde{R}((grad\alpha, 0), (0, \varphi f_j))(0, \varphi f_j) = -n\alpha (\nabla_{grad\alpha} grad\alpha, 0) + \frac{n}{\beta^3} d\alpha (grad\beta)(0, \xi)$$

and

$$\begin{aligned} \tilde{R}((grad\alpha, 0), (0, \xi))(0, \xi) &= -\alpha\beta d\alpha (grad\beta)(grad\alpha, 0) - \alpha\beta |grad\alpha|^2 (grad\beta, 0) \\ &- \alpha\beta^2 (\nabla_{grad\alpha} grad\alpha, 0) - \alpha^2\beta (\nabla_{grad\alpha} grad\beta, 0). \end{aligned}$$

Then

$$\begin{aligned} Tr_h \tilde{R}((grad\alpha, 0), di_{x_0}) di_{x_0} &= -\alpha\beta d\alpha (grad\beta)(grad\alpha, 0) - \alpha\beta |grad\alpha|^2 (grad\beta, 0) \\ &- \alpha(2n + \beta^2) (\nabla_{grad\alpha} grad\alpha, 0) - \alpha^2\beta (\nabla_{grad\alpha} grad\beta, 0) \\ &+ \frac{2n}{\beta^3} d\alpha (grad\beta)(0, \xi). \end{aligned}$$

Similarly, we get

$$\begin{aligned} Tr_h \tilde{R}((grad\beta, 0), di_{x_0}) di_{x_0} &= -\alpha\beta |grad\beta|^2 (grad\alpha, 0) - \alpha\beta d\alpha (grad\beta)(grad\beta, 0) \\ &- \alpha(2n + \beta^2) (\nabla_{grad\beta} grad\alpha, 0) - \alpha^2\beta (\nabla_{grad\beta} grad\beta, 0) \\ &+ \frac{2n}{\beta^3} |grad\beta|^2 (0, \xi) \end{aligned}$$

and

$$\begin{aligned} Tr_h \tilde{R}((0, \xi), di_{x_0}) di_{x_0} &= Tr_h \tilde{R}((0, \xi), \cdot) \cdot = \frac{2n\alpha^2}{\beta} (grad\beta, 0) - 2n |grad\alpha|^2 (0, \xi) \\ &- \frac{2n\alpha}{\beta} d\alpha (grad\beta)(0, \xi) - \frac{2n}{\beta^2} (0, \xi). \end{aligned}$$

It follows that

$$\begin{aligned}
 & Tr_h \left(\tilde{\nabla} \right)^2 (grad\alpha, 0) + Tr_h \tilde{R}((grad\alpha, 0), di_{x_0}) di_{x_0} \\
 &= - (2n + \beta^2) |grad\alpha|^2 (grad\alpha, 0) - 2\alpha\beta d\alpha (grad\beta) (grad\alpha, 0) \\
 (6) \quad & - 2\alpha\beta |grad\alpha|^2 (grad\beta, 0) - \alpha^2 d\alpha (grad\beta) (grad\beta, 0) \\
 & - \alpha (2n + \beta^2) (\nabla_{grad\alpha} grad\alpha, 0) - \alpha^2 \beta (\nabla_{grad\alpha} grad\beta, 0) \\
 & + \frac{2n(\beta^2 - 1)}{\alpha\beta^2} |grad\alpha|^2 (0, \xi) + \frac{2n(\beta^2 + 1)}{\beta^3} d\alpha (grad\beta) (0, \xi),
 \end{aligned}$$

$$\begin{aligned}
 & Tr_h \left(\tilde{\nabla} \right)^2 (grad\beta, 0) + Tr_h \tilde{R}((grad\beta, 0), di_{x_0}) di_{x_0} \\
 &= - (2n + \beta^2) d\alpha (grad\beta) (grad\alpha, 0) - 2\alpha\beta |grad\beta|^2 (grad\alpha, 0) \\
 (7) \quad & - \alpha^2 |grad\beta|^2 (grad\beta, 0) - 2\alpha\beta d\alpha (grad\beta) (grad\beta, 0) \\
 & - \alpha (2n + \beta^2) (\nabla_{grad\beta} grad\alpha, 0) - \alpha^2 \beta (\nabla_{grad\beta} grad\beta, 0) \\
 & + \frac{2n(\beta^2 - 1)}{\alpha\beta^2} d\alpha (grad\beta) (0, \xi) + \frac{2n(\beta^2 + 1)}{\beta^3} |grad\beta|^2 (0, \xi)
 \end{aligned}$$

and

$$\begin{aligned}
 & Tr_h \left(\tilde{\nabla} \right)^2 (0, \xi) + Tr_h \tilde{R}((0, \xi), di_{x_0}) di_{x_0} \\
 &= -2n\alpha (\beta^2 + 1) (grad\alpha, 0) - (\beta^2 + 2n) |grad\alpha|^2 (0, \xi) \\
 (8) \quad & - \alpha^2 |grad\beta|^2 (0, \xi) - \frac{2n\alpha^2 (\beta^2 - 1)}{\beta} (grad\beta, 0) \\
 & - \frac{2\alpha (\beta^2 + n)}{\beta} d\alpha (grad\beta) (0, \xi) - \frac{4n}{\beta^2} (0, \xi).
 \end{aligned}$$

By replacing (6), (7) and (8) in (5), we conclude that the inclusion map $i_{x_0} : N \longrightarrow (M \times N, \tilde{G})$ is biharmonic if and only if

$$\begin{aligned}
 & \left((\beta^2 + 2n) |grad\alpha|^2 + 2\alpha^2 \beta^2 |grad\beta|^2 + 3\alpha\beta (\beta^2 + 2n) d\alpha (grad\beta) \right) grad\alpha \\
 & + \alpha \left(\alpha^2 \beta |grad\beta|^2 + \alpha (3\beta^2 + 2n) d\alpha (grad\beta) + 2\beta (\beta^2 + 2n) |grad\alpha|^2 \right) grad\beta \\
 & - \frac{4n^2 (\beta^2 - 1) (\beta^2 + 1)}{\beta^2} grad\alpha - \frac{4n^2 \alpha (\beta^2 - 1)^2}{\beta^3} grad\beta \\
 & + \frac{\alpha (\beta^2 + 2n)^2}{2} grad(|grad\alpha|^2) + \frac{\alpha^3 \beta^2}{2} grad(|grad\beta|^2) \\
 & + \alpha^2 \beta (\beta^2 + 2n) (\nabla_{grad\alpha} grad\beta + \nabla_{grad\beta} grad\alpha) = 0
 \end{aligned}$$

and

$$(\beta^2 - 1) (\beta^2 + 2n) |\text{grad}\alpha|^2 + \alpha^2 \beta^2 |\text{grad}\beta|^2 \\ + \alpha\beta (2\beta^2 + n - 1) d\alpha (\text{grad}\beta) + 2n (\beta^2 - 1) = 0.$$

In the case where $\beta = \pm 1$, we find the results obtained in the two Corollaries 5 and 6 (see [3]).

Finally, we will study the harmonicity of the identity map $Id : (M \times N, \tilde{G}) \longrightarrow (M \times N, G)$.

Proposition 4.12. *The identity map $Id : (M \times N, \tilde{G}) \longrightarrow (M \times N, G)$ is harmonic if and only if*

$$\left\{ \begin{array}{l} (2n+1) \beta \text{grad}\alpha + \alpha \text{grad}\beta = 0 \\ \text{and} \\ \nabla'_\xi \xi + (\text{div}\xi) \xi = 0. \end{array} \right.$$

Proof. By definition, we have

$$\begin{aligned} \tau(Id) &= \nabla_{(e_i, 0)} (e_i, 0) - \tilde{\nabla}_{(e_i, 0)} (e_i, 0) \\ &+ \frac{1}{\alpha^2} \nabla_{(0, f_j)} (0, f_j) - \frac{1}{\alpha^2} \tilde{\nabla}_{(0, f_j)} (0, f_j) \\ &+ \frac{1}{\alpha^2} \nabla_{(0, \varphi f_j)} (0, \varphi f_j) - \frac{1}{\alpha^2} \tilde{\nabla}_{(0, \varphi f_j)} (0, \varphi f_j) \\ &+ \frac{1}{\alpha^2 \beta^2} \nabla_{(0, \xi)} (0, \xi) - \frac{1}{\alpha^2 \beta^2} \tilde{\nabla}_{(0, \xi)} (0, \xi). \end{aligned}$$

Using Lemma 3.3, we deduce that

$$\tau(Id) = \frac{2n+1}{\alpha} (\text{grad}\alpha, 0) + \frac{1}{\beta} (\text{grad}\beta, 0) + \frac{1-\beta^2}{\alpha^2 \beta^2} (0, \nabla'_\xi \xi) - \frac{\beta^2-1}{\alpha^2 \beta^2} (\text{div}\xi) (0, \xi).$$

It follows that the identity map $Id : (M \times N, \tilde{G}) \longrightarrow (M \times N, G)$ is harmonic if and only if

$$\left\{ \begin{array}{l} (2n+1) \beta \text{grad}\alpha + \alpha \text{grad}\beta = 0 \\ \text{and} \\ \nabla'_\xi \xi + (\text{div}\xi) \xi = 0. \end{array} \right.$$

Remark 4.13. *In particular, if $(N^{2n+1}, \varphi, \xi, \eta, h)$ is a Kenmotsu manifold, the tension field of the identity map Id is given by*

$$\tau(Id) = \frac{2n+1}{\alpha} (\text{grad}\alpha, 0) + \frac{1}{\beta} (\text{grad}\beta, 0) - \frac{2n(\beta^2-1)}{\alpha^2 \beta^2} (0, \xi).$$

We distinguish two cases :

- (1) If $\beta = \pm 1$, then the $Id : (M \times N, \widetilde{G}) \longrightarrow (M \times N, G)$ is harmonic if and only if α is the constant function.
- (2) If $\beta \neq \pm 1$, then the identity map is never harmonic.

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