

ON EQUICONTINUITY, TRANSITIVITY AND DISTALITY OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. In this paper, equicontinuity, transitivity, minimality, sensitivity, and distality of iterated function systems (IFS) have been discussed. The equicontinuity, almost equicontinuity, and distality of an IFS have been defined and some relevant results have been introduced and proved. The transitivity, minimality, and sensitivity of an IFS $\mathcal{F}$ have been investigated when each of the constituent maps f_λ has these properties and vice versa. It has been found that the IFS has these properties if at least one of the constituent maps f_λ has these properties but the converse statements are not true. We give counterexamples to support that the converse statements are not true. It has also been shown that an IFS $\mathcal{F}$ is distal if and only if the constituent maps f_λ are distal.

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1. INTRODUCTION

John Hutchinson first developed iterated function systems in the year 1981 [14] and later Michael Barnsley and others popularized it [3]. The mathematics of iterated functions is embedded in the general theory of dynamical systems. Important

concepts in dynamics like minimality, transitivity, attractors, shadowing, etc., can be extended to iterated function systems [1, 2, 6, 7, 8, 9, 10, 11, 13, 17, 18]. Also, the main objective of this paper is to extend the dynamical properties namely, equicontinuity, transitivity, minimality, sensitivity, and distality to iterated function systems. Here, we check whether the iterated function systems have these properties if the constituent maps have these properties and vice versa. When the converse statements are not true, we give counterexamples.

A dynamical system is a pair (X, f) , where X is a compact metric space and f is a continuous self map on X . Let $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. The set $O(x, f) = \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the orbit of a point $x \in X$ under the dynamical system (X, f) . Let (X, f) be a dynamical system. The set $\mathcal{E} \subset X$ is said to be the set of equicontinuous points of (X, f) if for each $x \in \mathcal{E}$ and for each $\epsilon > 0$, $\exists \delta > 0$ such that $\forall y \in B_\delta(x)$, $\forall n \geq 0$, $d(f^n(y), f^n(x)) < \epsilon$. If $\mathcal{E} = X$, then the dynamical system (X, f) is said to be equicontinuous. A dynamical system (X, f) is said to be almost equicontinuous if $\mathcal{E}$ is a residual set. A dynamical system (X, f) is said to be distal if for $x, y \in X$, $\inf\{d(f^n(x), f^n(y)) : n \in \mathbb{Z}_+\} = 0$, then $x = y$.

An iterated function system (IFS), $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is a family of continuous mappings $f_\lambda : X \rightarrow X$, $\lambda \in \Lambda$, where Λ is a non-empty finite set and (X, d) is a compact metric space. Let $\Lambda^{\mathbb{Z}_+}$ denote the set of all infinite sequences $\{\lambda_n\}_{n \in \mathbb{Z}_+}$ of symbols belonging to Λ . Thus, we use the shorted notation $\mathcal{F}_{\sigma_n} = f_{\lambda_{n-1}} \circ f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_1} \circ f_{\lambda_0}$. Let $\sigma = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be a typical element of $\Lambda^{\mathbb{Z}_+}$. A sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in X is said to be an orbit of $\mathcal{F}$ if there exists $\sigma \in \Lambda^{\mathbb{Z}_+}$, such that $x_n = \mathcal{F}_{\sigma_n}(x_0)$ for any $n \in \mathbb{Z}_+$, where $\mathcal{F}_{\sigma_n}(x_0) = f_{\lambda_{n-1}} \circ f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_1} \circ f_{\lambda_0}(x_0)$ and $\mathcal{F}_{\sigma_0}(x_0) = x_0$. Thus, for every $\sigma \in \Lambda^{\mathbb{Z}_+}$, the orbit of $x \in X$ is defined as $O_\sigma(x) = \{\mathcal{F}_{\sigma_n}(x) : n \in \mathbb{Z}_+\}$.

The notions of transitivity and the minimality of an IFS have been discussed in [17]. A point $x \in X$ is transitive if its orbit $O_\sigma(x)$ is dense in X for some $\sigma \in \Lambda^{\mathbb{Z}_+}$. The iterated function system $\mathcal{F}$ is transitive if there exists a transitive point in X . This definition of a transitive system is called dense orbit definition. In [17], it is given as fiberwise transitivity. The iterated function system $\mathcal{F}$ is minimal if every point $x \in X$ is a transitive point. The iterated function system $\mathcal{F}$ is topologically transitive if for every pair of non-empty open sets $U, V \subset X$, there is $\sigma \in \Lambda^{\mathbb{Z}_+}$ such

that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \emptyset$ for some $n \geq 0$. This definition of transitivity is called the open set definition. Also, in [17], it has been shown that if (X, d) is a compact space without isolated points and if f_λ is a homeomorphism $\forall \lambda \in \Lambda$, then the dense orbit definition of transitivity implies topological transitivity but the converse is not true in general. The iterated function system $\mathcal{F}$ is topologically mixing if for every pair of non-empty open sets $U, V \subset X$, there is $N > 0$ such that $\forall n > N$, we can find $\sigma \in \Lambda^{\mathbb{Z}_+}$ where $\mathcal{F}_{\sigma_n}(U) \cap V \neq \emptyset$.

The iterated function system $\mathcal{F}$ is sensitive if $\exists \epsilon > 0$ such that $\forall x \in X, \forall \delta > 0, \exists y \in B_\delta(x)$ and $n \geq 0$ with $d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(x)) \geq \epsilon$ for some sequence $\sigma \in \Lambda^{\mathbb{Z}_+}$. The concept of sensitivity of an IFS have been discussed and several conditions for sensitivity of an IFS have been introduced in [12]. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ be two IFS, where (X, d) and (Y, d') are compact metric spaces. Then, we say that $\mathcal{F}$ is topologically conjugate to $\mathcal{G}$ if there is a homeomorphism $h : X \rightarrow Y$ such that $g_\lambda = h \circ f_\lambda \circ h^{-1}$, for all $\lambda \in \Lambda$. In such case, h is said to be a topological conjugacy. It was first defined by Mehdi in [8]. A set $A \subset X$ is said to be invariant in $\mathcal{F}$ if $f_\lambda(A) \subset A, \forall \lambda \in \Lambda$ and inversely invariant in $\mathcal{F}$ if $f_\lambda^{-1}(A) \subset A, \forall \lambda \in \Lambda$.

In this paper, we discuss equicontinuity, transitivity, minimality, sensitivity, and distality of iterated function systems. First, we discuss the notions of equicontinuity and almost equicontinuity in iterated function systems. In particular, we extend some results of equicontinuity from [16], [pp. 61,62] to iterated function systems in Theorems 2.4-2.6 and 2.8. Then, we investigate the notion of transitivity in iterated function systems. Here, we find that an iterated function system $\mathcal{F}$ is transitive if at least one of the constituent maps is transitive but the converse is not true. To support that the converse is not true, we give counterexamples in Examples 2.11-2.14 by considering several cases. Then, in Theorem 2.16, we consider the notion of topological conjugacy on transitive iterated function systems and show that if $\mathcal{F}$ and $\mathcal{G}$ are conjugated iterated function systems, then $\mathcal{F}$ is topologically transitive if and only if $\mathcal{G}$ does. Further, we find that an iterated function system $\mathcal{F}$ is minimal if at least one of the constituent maps is minimal but the converse is not true. To support that the converse is not true, we give a counterexample in Example 2.17. We further define the notion of distality on an iterated function system in 2.19. Then, in Theorem 2.21, we show that an iterated

function system is distal if and only if the constituent maps f_λ are distal. In Theorem 2.22, we show that an equicontinuous iterated function system $\mathcal{F}$ is distal if the constituent maps are homeomorphisms. It is a generalization of the result [prop. 2.7.2, pp. 46] in [4] to iterated function systems. Lastly, in Theorem 2.23, we consider topological conjugacy on distal iterated function systems and show that if $\mathcal{F}$ and $\mathcal{G}$ are conjugated iterated function systems, then $\mathcal{F}$ is distal if and only if $\mathcal{G}$ does.

2. DEFINITIONS & THEOREMS

Theorem 2.1. *Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an iterated function system. A set $A \subset X$ is said to be invariant if and only if $\mathcal{F}_{\sigma_n}(A) \subset A$ for every n and for every sequence $\sigma \in \Lambda^{\mathbb{Z}+}$.*

Proof. Suppose $A \subset X$ is invariant. Then, $f_\lambda(A) \subset A \forall \lambda \in \Lambda$. This follows that $(f_{\lambda_{n-1}} \circ f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_0})(A) = f_{\lambda_{n-1}}((f_{\lambda_{n-2}} \circ \cdots \circ f_{\lambda_0})(A)) \subset A$. Thus, $\mathcal{F}_{\sigma_n}(A) \subset A \forall n$ and $\forall \sigma \in \Lambda^{\mathbb{Z}+}$.

Conversely, let $\mathcal{F}_{\sigma_n}(A) \subset A \forall n$ and $\forall \sigma \in \Lambda^{\mathbb{Z}+}$ where $A \subset X$.

Then, $(f_{\lambda_{n-1}} \circ \cdots \circ f_{\lambda_0})(A) \subset A, \forall \sigma = \{\lambda_0, \lambda_1, \lambda_2, \dots\} \in \Lambda^{\mathbb{Z}+}$.

When $\sigma = \{\lambda_i, \lambda_i, \lambda_i, \dots\}$, we have $(f_{\lambda_i})^n(A) \subset A \forall n$ which follows that $f_{\lambda_i}(A) \subset A \forall \lambda_i \in \Lambda$. Thus, $f_\lambda(A) \subset A \forall \lambda \in \Lambda$. Hence, $A \subset X$ is invariant. $\square$

Corollary 2.2. *Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an iterated function system. A set $A \subset X$ is said to be inversely invariant if and only if $\mathcal{F}_{\sigma_n}^{-1}(A) \subset A$ for every n and for every sequence $\sigma \in \Lambda^{\mathbb{Z}+}$.*

By following [16], we define a set of equicontinuous points of an IFS, an equicontinuous IFS, an almost equicontinuous IFS and proposed the following theorems [2.4-2.8].

Definition 2.3. *Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an iterated function system. The set $\mathcal{X} \subset X$ is said to be the set of equicontinuous points in $\mathcal{F}$ if for each $x \in \mathcal{X}$ and for each $\epsilon > 0$, $\exists \delta > 0$ such that $\forall y \in B_\delta(x), \forall n \geq 0, d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(x)) < \epsilon$ for any sequence $\sigma \in \Lambda^{\mathbb{Z}+}$. If $\mathcal{X} = X$, then $\mathcal{F}$ is said to be an equicontinuous IFS.*

From [16], [pp. 61, 62], we know that the set $\mathcal{E}_\epsilon = \{x \in X : \exists \delta > 0, \forall y, z \in B_\delta(x), \forall n \geq 0, d(f^n(y), f^n(z)) < \epsilon\}$ is inversely invariant, open and $\mathcal{E} = \bigcap_{m>0} \mathcal{E}_{\frac{1}{m}}$. The followings [Theorems 2.4-2.6] are similar results for iterated function systems.

Theorem 2.4. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an iterated function system, $\epsilon > 0$ and

$$\mathcal{X}_\epsilon = \{x \in X : \exists \delta > 0, \forall n \geq 0, \forall y, z \in B_\delta(x), d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) < \epsilon \text{ for any sequence } \sigma \in \Lambda^{\mathbb{Z}^+}\}.$$

Then $\mathcal{X}_\epsilon$ is inversely invariant.

Proof. Let $\lambda \in \Lambda$ be arbitrary. Let $x \in f_\lambda^{-1}(\mathcal{X}_\epsilon)$ be a point. Then, $f_\lambda(x) \in \mathcal{X}_\epsilon$. Let $\delta_\lambda < \frac{\epsilon}{2}$ satisfies the definition of $\mathcal{X}_\epsilon$. Then,

$$(1) \quad d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) < \epsilon, \forall y, z \in B_{\delta_\lambda}(f_\lambda(x)), \forall n \geq 0$$

and for any sequence $\sigma \in \Lambda^{\mathbb{Z}^+}$. Since f_λ is continuous, $\exists 0 < \eta_\lambda < \delta_\lambda$ such that for any $z, y \in X$, $d(f_\lambda(y), f_\lambda(z)) < 2\delta_\lambda$ whenever $d(y, z) < 2\eta_\lambda$. Thus, if $y, z \in B_{\eta_\lambda}(x)$, then $f_\lambda(y), f_\lambda(z) \in B_{\delta_\lambda}(f_\lambda(x))$. Therefore, by (1) we have, $d(\mathcal{F}_{\sigma_n}(f_\lambda(y)), \mathcal{F}_{\sigma_n}(f_\lambda(z))) < \epsilon, \forall y, z \in B_{\eta_\lambda}(x), \forall n \geq 0$ and for any sequence $\sigma \in \Lambda^{\mathbb{Z}^+}$.

Let $\eta = \min\{\eta_\lambda : \lambda \in \Lambda\}$, then $d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) < \epsilon, \forall y, z \in B_\eta(x), \forall n \geq 0$ and for any sequence $\sigma \in \Lambda^{\mathbb{Z}^+}$, where $\mathcal{F}_{\sigma_n} = \mathcal{F}_{\sigma_n} \circ f_\lambda$. Therefore, $x \in \mathcal{X}_\epsilon$. Hence, $\mathcal{X}_\epsilon$ is inversely invariant. $\square$

Theorem 2.5. $\mathcal{X}_\epsilon$ is open.

Proof. Let $x \in \mathcal{X}_\epsilon$ and $\delta > 0$ satisfies the definition of $\mathcal{X}_\epsilon$ for x . Now, for any $y \in B_\delta(x)$ and $z, w \in B_{\frac{\delta}{2}}(y)$, we have $z, w \in B_\delta(x)$. So, $\forall n \geq 0$ and for any sequence $\sigma \in \Lambda^{\mathbb{Z}^+}$, we have $d(\mathcal{F}_{\sigma_n}(z), \mathcal{F}_{\sigma_n}(w)) < \epsilon$. Then, for $y \in X$ and $z, w \in B_{\frac{\delta}{2}}(y)$, we have $d(\mathcal{F}_{\sigma_n}(z), \mathcal{F}_{\sigma_n}(w)) < \epsilon, \forall n \geq 0$. Thus, $y \in \mathcal{X}_\epsilon$. Hence, $\mathcal{X}_\epsilon$ is open. $\square$

Theorem 2.6. $\mathcal{X} = \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}}$.

Proof. Let $x \in \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}}$ be arbitrary. Let $\epsilon > 0$ be a positive number. Then, there exists a positive integer m such that $\frac{1}{m} < \epsilon$. Since, $x \in \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}}, \exists \delta > 0$, such that $\forall y, z \in B_\delta(x), \forall n \geq 0$ and $\forall \sigma \in \Lambda^{\mathbb{Z}^+}, d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) < \frac{1}{m}$. Then $\forall y \in B_\delta(x), \forall n \geq 0, d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(x)) < \frac{1}{m} < \epsilon$. Since ϵ was chosen arbitrarily, we have $x \in \mathcal{X}$. Thus,

$$(2) \quad \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}} \subset \mathcal{X}.$$

Conversely, let $x \in \mathcal{X}$ and $m > 0$. Then, $\exists \delta > 0$ such that $\forall y \in B_\delta(x), \forall \sigma \in \Lambda^{\mathbb{Z}^+}, d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(x)) < \frac{1}{2m}$. Thus, it follows that, if $y, z \in B_\delta(x)$, then

$d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) < \frac{1}{m}$. So, $\exists \delta > 0$ such that $\forall y, z \in B_\delta(x)$, $\sigma \in \Lambda^{\mathbb{Z}_+}$ and $\forall n \geq 0$, $d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) < \frac{1}{m}$. Thus, $x \in \mathcal{X}_{\frac{1}{m}}$. Since $m > 0$ was chosen arbitrarily, we have $x \in \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}}$. Therefore,

$$(3) \quad \mathcal{X} \subset \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}}.$$

Combining equations (2) and (3), we get $\mathcal{X} = \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}}$. □

Definition 2.7. *The iterated function system $\mathcal{F}$ is said to be almost equicontinuous if $\mathcal{X}$ is a residual set.*

Following [16], [pp. 62, theorem 2.31], we have a topologically transitive dynamical system is either sensitive or almost equicontinuous and the next theorem is a similar result for iterated function systems.

Theorem 2.8. *A topologically transitive iterated function system $\mathcal{F}$ is either sensitive or almost equicontinuous.*

Proof. Let $\mathcal{F}$ be a topologically transitive iterated function system and $\epsilon > 0$. Let us assume that $\mathcal{X}_\epsilon$ is non-empty and non-dense. Then, $U = X - \overline{\mathcal{X}_\epsilon}$ is open and non-empty set. Therefore, $U \cap \mathcal{F}_{\sigma_n}^{-1}(\mathcal{X}_\epsilon) \neq \emptyset$. But, $U \cap \mathcal{F}_{\sigma_n}^{-1}(\mathcal{X}_\epsilon) \subset U \cap \mathcal{X}_\epsilon = \emptyset$. This is a contradiction. Thus, $\mathcal{X}_\epsilon$ is either empty or dense.

If $\mathcal{X}_\epsilon \neq \emptyset$ for any ϵ , then $\mathcal{X}_\epsilon$ is dense. Therefore, $\mathcal{X} = \bigcap_{m>0} \mathcal{X}_{\frac{1}{m}}$ is a residual set. Thus, $\mathcal{F}$ is almost equicontinuous. Again, if $\mathcal{X}_\epsilon = \emptyset \forall \epsilon$, then $\exists$ some $\epsilon > 0$ such that for any $x \in X$ and for any $\delta > 0$, $\exists y, z \in B_\delta(x)$, and $n \geq 0$ such that $d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) \geq \epsilon$ for some $\sigma \in \Lambda^{\mathbb{Z}_+}$. Then, for any $x \in X$, $\epsilon \leq d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(z)) \leq d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(x)) + d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(z))$. This follows that either $d(\mathcal{F}_{\sigma_n}(y), \mathcal{F}_{\sigma_n}(x)) \geq \frac{\epsilon}{2}$ or $d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(z)) \geq \frac{\epsilon}{2}$. Thus $\mathcal{F}$ is sensitive. □

We know that a topologically mixing system (X, f) with at least two points is sensitive [16], [pp. 65]. The following is a generalization of this result in iterated function systems.

Theorem 2.9. *A topologically mixing iterated function system $\mathcal{F}$ with at least two points is sensitive if one of the constituent map f_λ is topologically mixing.*

The following remark and the examples [2.11-2.14] are given by considering both kinds of transitivity of an IFS in which examples 2.11 and 2.12 are considered by following [15].

Remark 2.10. *It is clear from the definition that an IFS $\mathcal{F}$ is transitive if at least one of the constituent maps f_λ is transitive but the converse is not true. Thus, by considering several cases where X is compact metric space without isolated points; compact metric space with isolated points; compact but not connected; the maps f_λ are not continuous; the maps f_λ are continuous, we give counterexamples to support that the converse statement is not true.*

In the following example, we consider the case when X is a compact space with isolated points. We find f_1 and f_2 which are continuous but not transitive in dense orbit sense while the IFS $\mathcal{F} = \{X; f_1, f_2\}$ is transitive in dense orbit sense.

Example 2.11.

Let $X = \{0, \frac{1}{n} : n \geq 1\}$.

Let $f_1 : X \rightarrow X$ be defined as $f_1(0) = 0$ and $f_1(\frac{1}{n}) = \begin{cases} \frac{1}{n}, & \text{if } n = 1, 2 \\ \frac{1}{n-1}, & \text{if } n \geq 3. \end{cases}$

Let $f_2 : X \rightarrow X$ be defined as $f_2(0) = 0$ and $f_2(\frac{1}{n}) = \begin{cases} \frac{1}{2}, & \text{if } n = 1 \\ \frac{1}{n+2}, & \text{if } n \geq 2. \end{cases}$

Then, $O(x, f_1)$ and $O(x, f_2)$ are not dense in X for any $x \in X$. Hence, f_1 and f_2 are not transitive. Now, considering $f_1 \circ f_2$, we get

$(f_1 \circ f_2)(0) = 0$ and $(f_1 \circ f_2)(\frac{1}{n}) = \frac{1}{n+1}$ for $n \geq 1$. Then, $O(1, f_1 \circ f_2)$ is dense. So, $f_1 \circ f_2$ is transitive. Thus, for $\sigma = \{2, 1, 2, 1, \dots\}$, the orbit of 1, $O_\sigma(1)$ is dense in X . Hence, the IFS $\mathcal{F} = \{X; f_1, f_2\}$ is transitive.

The following is an example of a topologically transitive IFS $\mathcal{F}$ in which we take X as a compact metric space without isolated points. We find two non continuous functions f_1 and f_2 which are not transitive whereas the IFS $\mathcal{F} = \{X; f_1, f_2\}$ is transitive.

Example 2.12.

Consider $I = [0, 1]$ and $g : I \rightarrow I$ as the standard tent map.

Let $X = \{x : x \text{ is a rational number in } [0, 1]\}$.

Let $f_1 : X \rightarrow X$ be defined as $f_1(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ x, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$

We have, $U' = (\frac{1}{2}, \frac{3}{5})$ and $V' = (\frac{4}{5}, 1)$ are nonempty open sets in I . Let $U = U' \cap X$, $V = V' \cap X$, then U and V are nonempty open sets in X and $f_1^n(U) \cap V = \phi$, $\forall n \geq 0$. Therefore, f_1 is not topologically transitive. Also, f_1 is not continuous.

Let $f_2 : X \rightarrow X$ be defined as

$$f_2(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \text{if } \frac{1}{2} < x \leq \frac{3}{4} \\ 1 - x, & \text{if } \frac{3}{4} < x \leq 1. \end{cases}$$

We have $P' = (0, \frac{1}{4})$ and $Q' = (\frac{1}{4}, \frac{1}{2})$ are nonempty open sets in I . Let $P = P' \cap X$, $Q = Q' \cap X$, then P and Q are nonempty open sets in X and $f_2^n(P) \cap Q = \phi$, $\forall n \geq 0$. Therefore, f_2 is not topologically transitive. Also, f_2 is not continuous.

Now, consider $f_1 \circ f_2$,

$$(f_1 \circ f_2)(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then, $f_1 \circ f_2 = g|_X$. We have, for any non-degenerate subinterval U of I there exists $k > 0$ such that $g^k(U) = I$ [see [15], example 3.1.3]. Also, X is the set of periodic and eventually periodic points of g , therefore, X is dense in I . Hence, for any pair U_1 and U_2 of non empty open subintervals of I , there is a periodic or eventually periodic orbit of g which intersects both U_1 and U_2 . This follows that, $f_1 \circ f_2$ is topologically transitive. Thus, for any pair of nonempty open sets U and V in X , we can find a sequence $\sigma = \{2, 1, 2, 1, \dots\}$, such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$ for some $n \geq 0$, where n is a multiple of 2. Hence, the IFS $\mathcal{F} = \{X; f_1, f_2\}$ is topologically transitive.

The following is an example of a topologically transitive IFS $\mathcal{F}$, where X is a compact metric space without isolated points and the constituent maps f_λ are continuous self maps which are not topologically transitive.

Example 2.13.

Let $X_1 = [0, 1]$.

Let $T : X_1 \rightarrow X_1$ be defined by

$$T(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(1 - x), & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let $X_2 = [-1, 0]$.

Let $T^* : X_2 \rightarrow X_2$ be defined by

$$T^*(x) = \begin{cases} -2(x+1), & \text{if } -1 \leq x < -\frac{1}{2} \\ 2x, & \text{if } -\frac{1}{2} \leq x \leq 0. \end{cases}$$

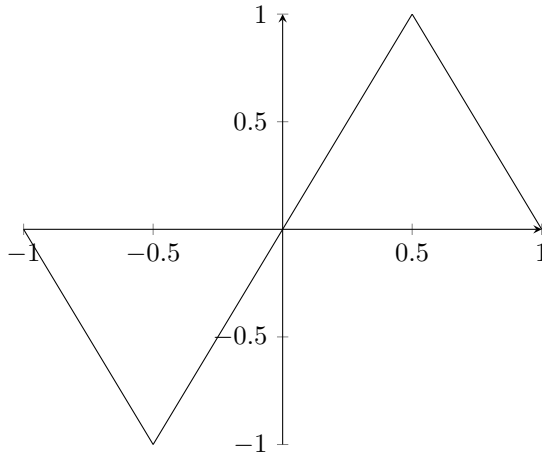


FIGURE 1. Graph of $(f_1 \circ f_2)(x)$

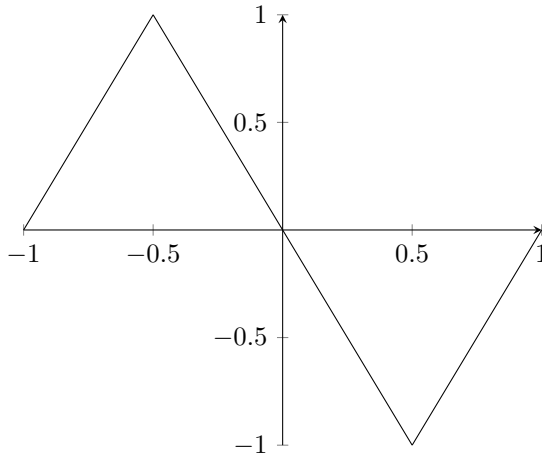


FIGURE 2. Graph of $(f_1 \circ f_2 \circ f_3)(x)$

Let $X = X_1 \cup X_2$.

Let f_1, f_2 and f_3 be continuous self maps defined on X by

$$f_1(x) = \begin{cases} T(x), & \text{if } x \in X_1 \\ x, & \text{if } x \in X_2, \end{cases}$$

$$f_2(x) = \begin{cases} x, & \text{if } x \in X_1 \\ T^*(x), & \text{if } x \in X_2, \end{cases}$$

and $f_3(x) = -x$. Then, it can be easily checked that f_1, f_2 and f_3 are not topologically transitive.

Now, we consider $f_1 \circ f_2$ and $f_1 \circ f_2 \circ f_3$ in X .

$$\text{Thus, } f_1 \circ f_2(x) = \begin{cases} T(x), & \text{if } x \in X_1 \\ T^*(x), & \text{if } x \in X_2, \end{cases}$$

$$\text{and } f_1 \circ f_2 \circ f_3(x) = \begin{cases} 2 + 2x, & \text{if } -1 \leq x < -\frac{1}{2} \\ -2x, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 2x - 2, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

The graphs of $f_1 \circ f_2$ and $f_1 \circ f_2 \circ f_3$ are given at Figure 1 and Figure 2 respectively. Then, for any pair of nonempty open sets U, V in X_1 , $\exists \sigma = \{2, 1, 2, 1, \dots\}$ and $n \geq 0$, where n is a multiple of 2 such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$. Similarly, for any pair of nonempty open sets U, V in X_2 , $\exists \sigma = \{2, 1, 2, 1, \dots\}$ such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$, where n is a multiple of 2. Again, for any pair of nonempty open sets $U \in X_1$ and $V \in X_2$, $\exists \sigma = \{3, 2, 1, 2, 1, \dots\}$, and $n \geq 0$, where n is odd, such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$. Thus, for any non empty sets U, V in X , $\exists \sigma \in \Lambda^{\mathbb{Z}^+}$ and $n \geq 0$, such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \phi$. Hence, the IFS $\mathcal{F} = \{X; f_1, f_2, f_3\}$ is topologically transitive.

Lastly, we take X to be a compact space without isolated points and not connected. We find continuous maps f_λ which are not topologically transitive whereas the corresponding IFS $\mathcal{F}$ is topologically transitive.

Example 2.14.

Let $X_1 = [0, 1]$.

Let $T : X_1 \rightarrow X_1$ be defined by

$$T(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let $X_2 = [2, 3]$.

Let $T' : X_2 \rightarrow X_2$ be defined by

$$T'(x) = \begin{cases} 2x - 2, & \text{if } 2 \leq x \leq \frac{5}{2} \\ 2(4 - x), & \text{if } \frac{5}{2} < x \leq 3. \end{cases}$$

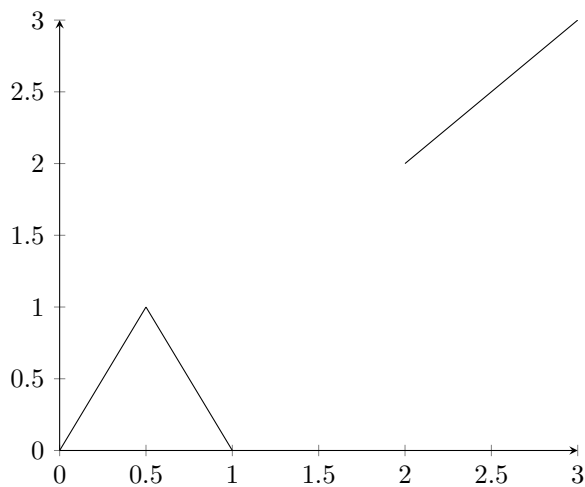


FIGURE 3. Graph of $f_1(x)$

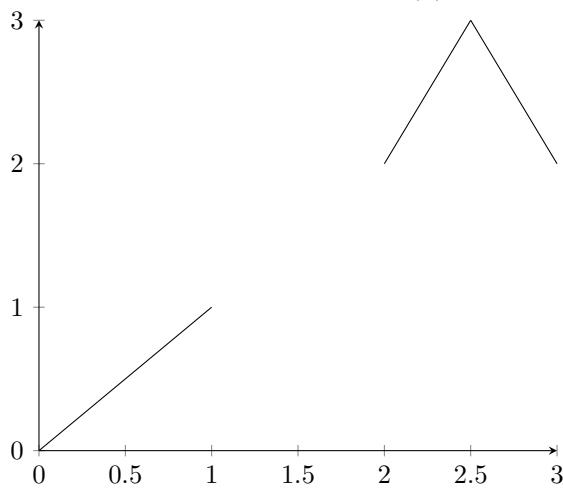


FIGURE 4. Graph of $f_2(x)$

Let $X = X_1 \cup X_2$.

Let f_1, f_2 be continuous self maps defined on X by

$$f_1(x) = \begin{cases} T(x), & \text{if } x \in X_1 \\ x, & \text{if } x \in X_2, \end{cases}$$

$$f_2(x) = \begin{cases} x, & \text{if } x \in X_1 \\ T'(x), & \text{if } x \in X_2. \end{cases}$$

Then, f_1 and f_2 are not topologically transitive as $f_\lambda^n(U) \cap V = \phi$ for any non empty open sets $U \subset X_1$ and $V \subset X_2$, where $\lambda = 1, 2$ and $\forall n \geq 0$. The graphs of f_1 and f_2 are given at Figure 3 and Figure 4 respectively.

Again, $f_3 : X \rightarrow X$ be defined as

$$f_3(x) = \begin{cases} x + 2, & \text{if } x \in X_1 \\ x - 2, & \text{if } x \in X_2. \end{cases}$$

Then, f_3 is not topologically transitive as there are non empty open sets $U = (0, \frac{1}{2})$ and $V = (\frac{5}{2}, 3)$ in X such that $f_3^n(U) \cap V = \phi$ for all $n \geq 0$.

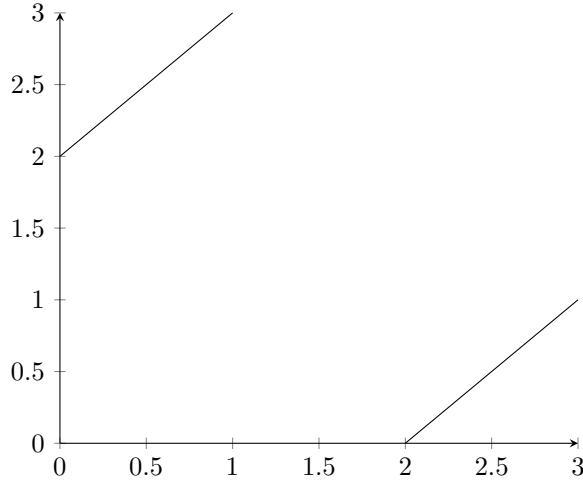
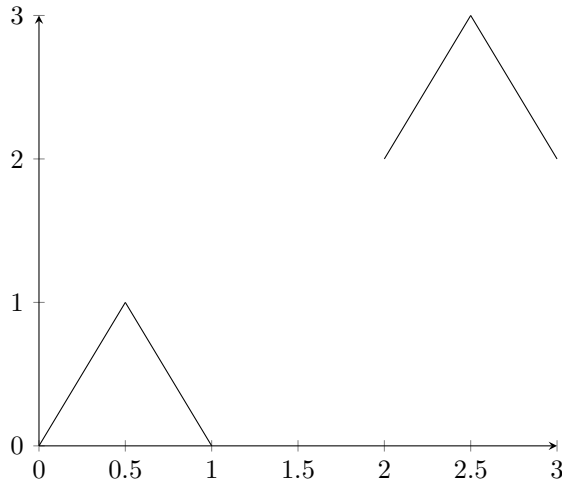
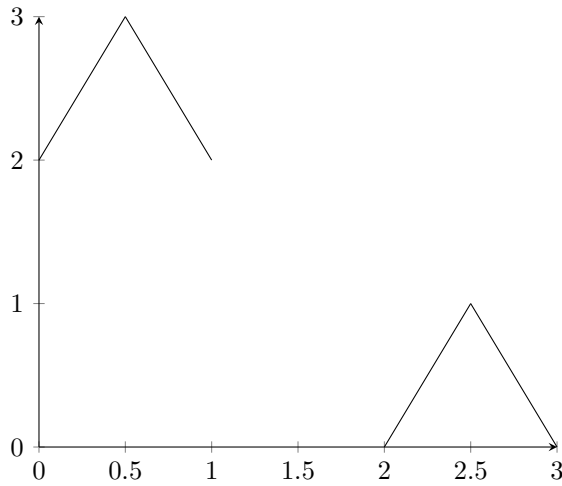


FIGURE 5. Graph of $f_3(x)$

Now, considering $f_1 \circ f_2$ and $f_1 \circ f_2 \circ f_3$, we have

$$(f_1 \circ f_2)(x) = \begin{cases} T(x), & \text{if } x \in X_1 \\ T'(x), & \text{if } x \in X_2 \end{cases}$$


 FIGURE 6. Graph of $(f_1 \circ f_2)(x)$

 FIGURE 7. Graph of $(f_1 \circ f_2 \circ f_3)(x)$

$$\text{and } (f_1 \circ f_2 \circ f_3)(x) = \begin{cases} 2 + 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(2 - x), & \text{if } \frac{1}{2} < x \leq 1 \\ 2(x - 2), & \text{if } 2 \leq x \leq \frac{5}{2} \\ 2(3 - x), & \text{if } \frac{5}{2} < x \leq 3. \end{cases}$$

Then, $f_1 \circ f_2$ and $f_1 \circ f_2 \circ f_3$ are continuous self maps on X .

Thus, by similar explanation as the above example [2.13], we have the IFS $\mathcal{F} =$

$\{X; f_1, f_2, f_3\}$ is topologically transitive. Also, the graphs of f_3 , $f_1 \circ f_2$ and $f_1 \circ f_2 \circ f_3$ are given at Figure 5, Figure 6 and Figure 7 respectively.

Remark 2.15. *If $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ are two conjugated IFS with h as conjugacy where (X, d) and (Y, d') are compact metric spaces, then $\mathcal{G}_{\sigma_n} \circ h = h \circ \mathcal{F}_{\sigma_n}$, for all $\sigma \in \Lambda^{\mathbb{Z}_+}$.*

Theorem 2.16. *Suppose (X, d) and (Y, d') are compact metric spaces and let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ be two conjugated IFS, then $\mathcal{F}$ is topologically transitive if and only if $\mathcal{G}$ does.*

Proof. Let $h : X \rightarrow Y$ be the conjugacy, then $\mathcal{G}_{\sigma_n} \circ h = h \circ \mathcal{F}_{\sigma_n}$, for all $\sigma \in \Lambda^{\mathbb{Z}_+}$ [by Remark 2.15]. First, let $\mathcal{F}$ be topologically transitive. Let U and V be a pair of non empty open sets in Y . Since h is a homeomorphism, $h^{-1}(U)$ and $h^{-1}(V)$ are a pair of non empty open sets in X . Since $\mathcal{F}$ is topologically transitive, therefore $\exists \sigma \in \Lambda^{\mathbb{Z}_+}$ such that $\mathcal{F}_{\sigma_n}(h^{-1}(U)) \cap h^{-1}(V) \neq \emptyset$, for some $n \geq 0$. Thus, $\exists$ some $x \in h^{-1}(U)$ such that $\mathcal{F}_{\sigma_n}(x) \in h^{-1}(V)$, for some $n \geq 0$, i.e., $h(x) \in U$ and $h(\mathcal{F}_{\sigma_n}(x)) \in V$, for some $n \geq 0$. This follows that $h(x) \in U$ and $\mathcal{G}_{\sigma_n}(h(x)) \in V$ for some $n \geq 0$. Thus $\mathcal{G}_{\sigma_n}(U) \cap V \neq \emptyset$, for some $\sigma \in \Lambda^{\mathbb{Z}_+}$ and for some $n \geq 0$. Hence, $\mathcal{G}$ is topologically transitive. Similarly, we can prove the converse part. Therefore, $\mathcal{F}$ is topologically transitive if and only if $\mathcal{G}$ does. $\square$

Further, from the definition of a minimal IFS, we found that an IFS $\mathcal{F}$ is minimal if at least one of the constituent maps f_λ is minimal but the converse statement is not true. The following is a counterexample.

Example 2.17.

Let $X = S^1$ be the unit circle and λ , an irrational number such that $0 < \lambda < 1$. Let f_1 and f_2 be mappings on X defined by $f_1(\theta) = 2\theta$ and $f_2(\theta) = \frac{\theta}{2} + \pi\lambda$, respectively. Then, the mappings f_1 and f_2 are not minimal. Now, $(f_1 \circ f_2)(\theta) = \theta + 2\pi\lambda, \forall \theta \in X$ which is an irrational rotation. Therefore, it is minimal. Thus, for $\sigma = \{2, 1, 2, 1, \dots\}$, the orbit $O_\sigma(\theta)$ is dense for every $\theta \in X$. Hence, the IFS $\mathcal{F} = \{X; f_1, f_2\}$ is minimal.

Similarly, by the definition of a sensitive IFS, an IFS $\mathcal{F}$ is sensitive if at least one of the constituent maps f_λ is sensitive but the converse statement is not true and

the following is a counterexample in which X is a compact space and the constituent maps f_λ are non-sensitive maps which are not continuous.

Example 2.18.

Let $X = [0, 1]$. Let f_1 and f_2 be two discontinuous self maps on X defined by

$$f_1(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ x, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

$$f_2(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{2x-1}{2}, & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x-1, & \text{if } \frac{3}{4} \leq x \leq 1. \end{cases}$$

Then we can easily check that f_1 and f_2 are not sensitive. Now,

$$f_1 \circ f_2(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x-1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Thus, we get the baker map which is sensitive [5, 16]. Hence, the IFS $\mathcal{F} = \{X; f_1, f_2\}$ is sensitive.

Lastly, we define a distal IFS and found the following results.

Definition 2.19. An IFS $\mathcal{F}$ is said to be distal if for x, y in X , $\inf\{d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) : n \in \mathbb{Z}_+, \sigma \in \Lambda^{\mathbb{Z}_+}\} = 0$, then $x = y$.

Proposition 2.20. If a function $f : X \rightarrow X$ is distal, then $f(x) \neq f(y)$ whenever $x \neq y$.

Proof. Let $f : X \rightarrow X$ be distal. Let $x, y \in X$ be such that $x \neq y$. Suppose if possible, let $f(x) = f(y)$, then $d(f(x), f(y)) = 0$, which implies that $\inf\{d(f^n(x), f^n(y)) : n \in \mathbb{Z}_+\} = 0$. Therefore, $x = y$ as f is distal, which is a contradiction. Thus, $f(x) \neq f(y)$. Since $x, y \in X$, $x \neq y$ was taken arbitrarily, therefore $f(x) \neq f(y)$ whenever $x \neq y$ if f is distal. $\square$

Corollary 2.21. If a function $f : X \rightarrow X$ is distal, then whenever $x \neq y$, $f^n(x) \neq f^n(y) \forall n \geq 0$.

Theorem 2.22. The IFS $\mathcal{F}$ is distal if and only if f_λ is distal $\forall \lambda \in \Lambda$.

Proof. Let f_λ be distal $\forall \lambda \in \Lambda$. Suppose $\mathcal{F}$ is not distal. Then, $\exists$ a distinct pair x, y in X such that $\inf\{d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) : n \in \mathbb{Z}_+, \sigma \in \Lambda^{\mathbb{Z}_+}\} = 0$. Then, for

every $\epsilon > 0$, $\exists \sigma \in \Lambda^{\mathbb{Z}_+}, n \geq 0$ such that $d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) < \epsilon$, which implies that $0 \leq d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) < \epsilon$. Then, $d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) = 0$. Thus, it follows that $x = y$ as f_λ is distal $\forall \lambda \in \Lambda$, which is a contradiction. Therefore, $\mathcal{F}$ is distal.

Conversely, let $\mathcal{F}$ be distal. Then, for if x, y in X , $\inf\{d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) : n \in \mathbb{Z}_+, \sigma \in \Lambda^{\mathbb{Z}_+}\} = 0$, then $x = y$. Suppose if possible, f_λ is not distal for some $\lambda \in \Lambda$. Then, $\exists$ a distinct pair x, y in X such that $\inf\{d(f_\lambda^n(x), f_\lambda^n(y)) : n \in \mathbb{Z}_+\} = 0$, which implies that $\inf\{d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) : n \in \mathbb{Z}_+, \sigma \in \Lambda^{\mathbb{Z}_+}\} = 0$. Thus, $\mathcal{F}$ is not distal, which is a contradiction. Therefore, f_λ is distal $\forall \lambda \in \Lambda$. Hence, the IFS $\mathcal{F}$ is distal if and only if f_λ is distal $\forall \lambda \in \Lambda$. $\square$

Theorem 2.23. *Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an equicontinuous IFS where f_λ is a homeomorphism $\forall \lambda \in \Lambda$. Then, $\mathcal{F}$ is distal.*

Proof. From definition 2.3, it is clear that if $\mathcal{F}$ is equicontinuous, then f_λ are equicontinuous $\forall \lambda \in \Lambda$. Thus, f_λ are equicontinuous homeomorphisms $\forall \lambda \in \Lambda$. Since equicontinuous homeomorphisms are distal [4], f_λ are distal $\forall \lambda \in \Lambda$. Hence, by theorem 2.22, $\mathcal{F}$ is distal. $\square$

Theorem 2.24. *Suppose (X, d) and (Y, d') are compact metric spaces and let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ be two conjugated IFS, then $\mathcal{F}$ is distal if and only if $\mathcal{G}$ does.*

Proof. Let $h : X \rightarrow Y$ be the conjugacy. Then, $\mathcal{G}_{\sigma_n} \circ h = h \circ \mathcal{F}_{\sigma_n}$ for all $\sigma \in \Lambda^{\mathbb{Z}_+}$ [by Remark 2.15]. First, let $\mathcal{G}$ be distal. Suppose, if possible $\mathcal{F}$ is not distal. Then, there exists a pair of distinct points $x, y \in X$ such that $\inf\{d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) : n \in \mathbb{Z}_+, \sigma \in \Lambda^{\mathbb{Z}_+}\} = 0$. Thus, for every $\epsilon > 0$, $\exists \sigma \in \Lambda^{\mathbb{Z}_+}, n \geq 0$ such that $d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) < \epsilon$, i.e. $0 \leq d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) < \epsilon$. This implies that $d(\mathcal{F}_{\sigma_n}(x), \mathcal{F}_{\sigma_n}(y)) = 0$, which follows that $\mathcal{F}_{\sigma_n}(x) = \mathcal{F}_{\sigma_n}(y)$. Thus, $(h \circ \mathcal{F}_{\sigma_n})(x) = (h \circ \mathcal{F}_{\sigma_n})(y)$. Therefore, $(\mathcal{G}_{\sigma_n} \circ h)(x) = (\mathcal{G}_{\sigma_n} \circ h)(y)$. Since $\mathcal{G}$ is distal, it follows that $h(x) = h(y)$, which implies that $x = y$. This is a contradiction. Hence, $\mathcal{F}$ is distal. Similarly, we can prove for $\mathcal{G}$ when $\mathcal{F}$ is distal. Hence, $\mathcal{F}$ is distal if and only if $\mathcal{G}$ does. $\square$

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