

***- η -RICCI SOLITON WITHIN THE FRAMEWORK OF SASAKIAN MANIFOLD**

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ABSTRACT. In this paper we study *- η -Ricci soliton on Sasakian manifolds. Here, we have discussed some curvature properties on Sasakian manifold admitting *- η -Ricci soliton. We have obtained some significant results on *- η -Ricci soliton in Sasakian manifolds satisfying $R(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $\bar{P}(\xi, X) \cdot S = 0$, where $\bar{P}$ is Pseudo-projective curvature tensor. The conditions for *- η -Ricci soliton on ϕ -conharmonically flat and ϕ -projectively flat Sasakian manifolds have been obtained in this article. Lastly, we have given an example of 5-dimensional Sasakian manifolds satisfying *- η -Ricci soliton.

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1. INTRODUCTION

The notion of Ricci flow was first introduced by Hamilton [6] in 1982. The Ricci flow is an evolution equation for metrics on a Riemannian manifold. The Ricci flow equation is given by [3], [13]

$$(1) \quad \frac{\partial g}{\partial t} = -2S$$

on a compact Riemannian manifold M with Riemannian metric g .

A self-similar solution to the Ricci flow [6], [17] is called a Ricci soliton [7] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by [3], [13]

$$(2) \quad \mathcal{L}_V g + 2S + 2\lambda g = 0,$$

where $\mathcal{L}_V$ is the Lie derivative in the direction of V , S is Ricci tensor, g is Riemannian metric, V is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively.

In 2009, Jong Taek Cho and Makoto Kimura introduced the notion of η -Ricci soliton [2], given by the equation [1]:

$$(3) \quad \mathcal{L}_\xi g + 2S = 2\lambda g + 2\mu\eta \otimes \eta,$$

for constants λ and μ .

The notion of $*$ -Ricci tensor on almost Hermitian manifolds and $*$ -Ricci tensor of real hypersurfaces in non-flat complex space were introduced by Tachibana [16] and Hamada [5] respectively where the $*$ -Ricci tensor is defined by:

$$(4) \quad S^*(X, Y) = \frac{1}{2}(\text{trace}\{\phi \circ R(X, \phi Y)\}),$$

for all vector fields X, Y on M^n and ϕ is a $(1,1)$ -tensor field.

If $S^*(X, Y) = \lambda g(X, Y) + \mu\eta(X)\eta(Y)$ for all vector fields X, Y and λ, μ are smooth

functions, then the manifold is called $*$ - η -Einstein manifold.

Further if $\mu = 0$ i.e $S^*(X, Y) = \lambda g(X, Y)$ for all vector fields X, Y then the manifold becomes $*$ -Einstein.

In 2014, Kaimakamis and Panagiotidou [9] introduced the notion of $*$ -Ricci soliton which can be defined as:

$$(5) \quad \mathcal{L}_V g + 2S^* + 2\lambda g = 0,$$

for all vector fields X, Y on M^n and λ being a constant.

In 2019, the notion of $*$ -Conformal η -Ricci soliton was introduced by S. Roy, S. Dey, A. Bhattacharyya and S. K. Hui [12], where it is defined as:

$$(6) \quad \mathcal{L}_\xi g + 2S^* + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0,$$

where p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.

Now motivating from the above notions, we introduce the $*$ - η Ricci soliton as:

Definition 1.1: A Riemannian manifold (M, g) of dimension n is said to admit $*$ - η Ricci soliton if

$$(7) \quad \mathcal{L}_\xi g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where λ and μ are constants.

In the above equation if the vector field ξ is the gradient of a smooth function f (denoted by Df , D denotes the gradient operator) then the equation (7) is called gradient $*$ - η Ricci soliton and it is defined as:

$$(8) \quad Hess f + S^* + \lambda g + \mu\eta \otimes \eta = 0,$$

where $Hess f$ is the Hessian of the smooth function f .

Moreover if λ and μ are smooth functions then equations (7) and (8) are respectively called almost $*$ - η Ricci soliton and gradient almost $*$ - η Ricci soliton.

Note that $*$ - η Ricci soliton is trivial if the vector field ξ is killing and in that case the manifold becomes $*$ - η -Einstein.

The Riemannian-Christoffel curvature tensor R [10], the conharmonic curvature

tensor H [8], the projective curvature tensor P [18] and the Pseudo-projective curvature tensor $\bar{P}$ [11] in a Riemannian manifold (M, g) are defined by:

$$(9) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z,$$

$$(10) \quad \begin{aligned} H(X, Y)Z &= R(X, Y)Z - \frac{1}{(n-2)}[g(Y, Z)QX - g(X, Z)QY \\ &+ S(Y, Z)X - S(X, Z)Y], \end{aligned}$$

$$(11) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(Y, Z)QX - g(X, Z)QY],$$

and

$$(12) \quad \begin{aligned} \bar{P}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\ &- \frac{r}{n}(\frac{a}{n-1} + b)[g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where Q is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$, S is the Ricci tensor, $r = \text{tr}(S)$ is the scalar curvature, where $\text{tr}(S)$ is the trace of S , $a, b \neq 0$ are constants and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M .

The paper is organized as follows: After introduction, section 2 consists of basic definitions of Sasakian manifolds. Section 3 devotes $*$ - η -Ricci soliton on Sasakian manifolds. In section 4, we establish some significant results on $*$ - η -Ricci soliton in Sasakian manifolds satisfying $R(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $\bar{P}(\xi, X) \cdot S = 0$, where $\bar{P}$ is Pseudo-projective curvature tensor. Moreover it is shown that the Sasakian manifold, admitting $*$ - η -Ricci soliton becomes η -Einstein manifold when it satisfies $S(\xi, X) \cdot R = 0$ and Einstein manifold when it satisfies $R(\xi, X) \cdot S = 0$ and $\bar{P}(\xi, X) \cdot S = 0$. We also obtain the conditions for $*$ - η -Ricci soliton on ϕ -conharmonically flat and ϕ -projectively flat Sasakian manifolds. It is also shown that the Sasakian manifold, admitting $*$ - η -Ricci soliton becomes η -Einstein manifold when it is ϕ -conharmonically flat and Einstein manifold when it is ϕ -projectively flat. In last section, we give an example of 5-dimensional Sasakian manifolds satisfying $*$ - η -Ricci soliton.

2. PRELIMINARIES

Let M be a $(2n+1)$ dimensional connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that [12],[3]

$$(13) \quad \phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0,$$

$$(14) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(15) \quad g(X, \phi Y) = -g(\phi X, Y),$$

$$(16) \quad g(X, \xi) = \eta(X),$$

for all vector fields $X, Y \in \chi(M)$.

A $(2n+1)$ dimensional almost contact manifold M with (ϕ, ξ, η, g) structure is said to be a contact metric manifold [12] iff $d\eta(X, Y) = g(X, \phi Y)$.

Also an almost contact metric structure of M is said to be normal [12] if

$$(17) \quad 2d\eta(X, Y)\xi + [\phi, \phi](X, Y) = 0,$$

where $[\phi, \phi]$ is Nijenhuis tensor.

A normal contact metric manifold is said to be a Sasakian manifold.

In a $(2n+1)$ dimensional Sasakian manifold the following conditions hold [12]:

$$(18) \quad \nabla_X \xi = -\phi X,$$

$$(19) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$

$$(20) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(21) \quad R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(22) \quad \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$$

$$(23) \quad (\nabla_X \eta)Y = -g(\phi X, Y),$$

where R is the Riemannian curvature tensor and ∇ is the Levi-Civita connection associated with g .

[12] As every Sasakian manifold is K-contact then we have in a Sasakian manifold,

$$(24) \quad (\mathcal{L}_\xi g)(X, Y) = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ which is a Killing vector field.

3. $\ast$ - η -RICCI SOLITON ON SASAKIAN MANIFOLD

Let (M, ϕ, ξ, η, g) be a $(2n+1)$ dimensional Sasakian manifold. Consider the $\ast$ - η -Ricci soliton on M as:

$$(25) \quad (\mathcal{L}_\xi g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0,$$

for any X, Y in M .

Then from (24), we get,

$$(26) \quad S^*(X, Y) = -\lambda g(X, Y) - \mu\eta(X)\eta(Y).$$

In paper [4], Lemma 3.1, authors have proved that the $\ast$ -Ricci tensor on a $(2n+1)$ -dimensional Sasakian manifold (M, ϕ, ξ, η, g) is given by,

$$(27) \quad S^*(X, Y) = S(X, Y) - (2n - 1)g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y in M .

Using this relation (27), we can write (26) as,

$$(28) \quad S(X, Y) = [2n - 1 - \lambda]g(X, Y) - (\mu - 1)\eta(X)\eta(Y).$$

From this equation (28), we get,

$$(29) \quad S(X, \xi) = [2n - \lambda - \mu]\eta(X).$$

Again we also know in a $(2n+1)$ dimensional Sasakian manifold the Ricci tensor field satisfies:

$$(30) \quad S(X, \xi) = 2n\eta(X).$$

Then from (29) and (30), we get,

$$(31) \quad \lambda + \mu = 0.$$

Now we know,

$$(32) \quad (\nabla_X S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),$$

for any $X, Y, Z \in \chi(M)$.

Then replacing the expression of S from (28), we get,

$$(33) \quad (\nabla_X S)(Y, Z) = -(\mu - 1)[\eta(Y)(\nabla_X \eta)Z + \eta(Z)(\nabla_X \eta)Y].$$

Using (23), we get,

$$(34) \quad (\nabla_X S)(Y, Z) = (\mu - 1)[\eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y)].$$

Similarly we can obtain,

$$(35) \quad (\nabla_Y S)(Z, X) = (\mu - 1)[\eta(Z)g(\phi Y, X) + \eta(X)g(\phi Y, Z)],$$

and

$$(36) \quad (\nabla_Z S)(X, Y) = (\mu - 1)[\eta(X)g(\phi Z, Y) + \eta(Y)g(\phi Z, X)].$$

Then adding (34), (35), (36) and using (15) we get,

$$(37) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0,$$

for any $X, Y, Z \in \chi(M)$.

Thus we can state the following theorem:

Theorem 3.1. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) admits *- η -Ricci Soliton then the manifold has cyclic Ricci tensor i.e. $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$, for any $X, Y, Z \in \chi(M)$.*

Now if $\nabla S = 0$, then taking $Z = \xi$ in the expression of ∇S from (34), we obtain,

$$(38) \quad (\mu - 1)g(\phi X, Y) = 0.$$

$$\forall X, Y \in \chi(M).$$

Hence $\mu = 1$.

Then from (31), we obtain, $\lambda = -1$.

This leads to the following:

Theorem 3.2. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) is Ricci symmetric i.e $\nabla S = 0$ and admits $*$ - η -Ricci Soliton then $\mu = 1$ and $\lambda = -1$.*

4. CURVATURE PROPERTIES ON SASAKIAN MANIFOLD SATISFYING $*$ - η -RICCI SOLITON

The condition $R(\xi, X) \cdot S = 0$ implies that,

$$(39) \quad S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0,$$

for any $X, Y, Z \in \chi(M)$.

Replacing the expression of S from (28) and from the symmetries of R we get,

$$(40) \quad (\mu - 1)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0,$$

for any $X, Y, Z \in \chi(M)$.

For $Z = \xi$ we have,

$$(41) \quad (\mu - 1)[g(X, Y) - \eta(X)\eta(Y)] = 0,$$

for any $X, Y \in \chi(M)$.

Then using (14), we have,

$$(42) \quad (\mu - 1)g(\phi X, \phi Y) = 0,$$

for any $X, Y \in \chi(M)$.

It follows that,

$$(43) \quad \mu = 1.$$

Then from (31), we get,

$$(44) \quad \lambda = -1.$$

So we can state the following theorem:

Theorem 4.1. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) satisfies $R(\xi, X) \cdot S = 0$ and admits $*\text{-}\eta$ -Ricci Soliton then $\mu = 1$ and $\lambda = -1$.*

Now from (43), (44) and (28), we obtain,

$$(45) \quad S(X, Y) = 2ng(X, Y).$$

for any $X, Y \in \chi(M)$.

Then we have,

Corollary 4.2. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) satisfies $R(\xi, X) \cdot S = 0$ and admits $*\text{-}\eta$ -Ricci Soliton then the manifold becomes Einstein manifold.*

Again the condition $S(\xi, X) \cdot R = 0$ implies that,

$$(46) \quad \begin{aligned} &S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W \\ &+ S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W + S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X \\ &= 0, \end{aligned}$$

for any $X, Y, Z, W \in \chi(M)$.

Taking the inner product with ξ , the above equation becomes,

$$(47) \quad \begin{aligned} &S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) \\ &- S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W) - S(\xi, Z)\eta(R(Y, X)W) \\ &+ S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0, \end{aligned}$$

for any $X, Y, Z, W \in \chi(M)$.

Replacing the expression of S from (28) and taking $Z = \xi$, $W = \xi$, we get,

$$(48) \quad [4n - 1 - 2\lambda - \mu][g(X, Y) - \eta(X)\eta(Y)] = 0,$$

for any $X, Y \in \chi(M)$.

Then using (14), we have,

$$(49) \quad [4n - 1 - 2\lambda - \mu]g(\phi X, \phi Y) = 0,$$

for any $X, Y \in \chi(M)$.

Now using (31), the above equation becomes,

$$(50) \quad [4n - 1 - \lambda]g(\phi X, \phi Y) = 0,$$

for any $X, Y \in \chi(M)$.

It follows that,

$$(51) \quad \lambda = 4n - 1.$$

and then from (31), we obtain,

$$(52) \quad \mu = 1 - 4n.$$

This leads to the following:

Theorem 4.3. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) satisfies $S(\xi, X) \cdot R = 0$ and admits $\ast$ - η -Ricci Soliton then $\mu = 1 - 4n$ and $\lambda = 4n - 1$.*

Now from (51), (52) and (28) we obtain,

$$(53) \quad S(X, Y) = -2ng(X, Y) + 4n\eta(X)\eta(Y),$$

for any $X, Y \in \chi(M)$.

Then we have,

Corollary 4.4. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) satisfies $S(\xi, X) \cdot R = 0$ and admits $\ast$ - η -Ricci Soliton then the manifold becomes η -Einstein manifold.*

The condition $\overline{P}(\xi, X) \cdot S = 0$ implies that

$$(54) \quad S(\overline{P}(\xi, X)Y, Z) + S(Y, \overline{P}(\xi, X)Z) = 0.$$

for any $X, Y, Z \in \chi(M)$, where $\overline{P}$ is the Pseudo-projective curvature tensor in M .
Now in a $(2n+1)$ dimensional Sasakian manifold,

$$(55) \quad \begin{aligned} \overline{P}(\xi, X)Y &= aR(\xi, X)Y + b[S(X, Y)\xi - S(\xi, Y)X] \\ &- \frac{r}{2n+1}(\frac{a}{2n} + b)[g(X, Y)\xi - g(\xi, Y)X], \end{aligned}$$

where $a, b \neq 0$ are constants.

Then using (21), (28) and (29), we get,

$$(56) \quad \begin{aligned} \overline{P}(\xi, X)Y &= [a - \frac{r}{2n+1}(\frac{a}{2n} + b)][g(X, Y)\xi - \eta(Y)X] \\ &+ b[2n - 1 - \lambda + \frac{1}{2}(p + \frac{2}{2n+1})][g(X, Y)\xi - \eta(Y)X] \\ &+ b(\mu - 1)[\eta(Y)X - \eta(X)\eta(Y)\xi]. \end{aligned}$$

Replacing the expression of S from (28) and using (56), we obtain,

$$(57) \quad \begin{aligned} S(\overline{P}(\xi, X)Y, Z) &= [2n - 1 - \lambda][a - \frac{r}{2n+1}(\frac{a}{2n} + b)][g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] \\ &+ b[2n - 1 - \lambda]^2[g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] \\ &- b(\mu - 1)[2n - 1 - \lambda][g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] \\ &- (\mu - 1)[a - \frac{r}{2n+1}(\frac{a}{2n} + b)][g(X, Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)]. \end{aligned}$$

for any $X, Y, Z \in \chi(M)$.

As from (28), we have $S(X, Y) = S(Y, X)$ for any $X, Y \in \chi(M)$, then we can write,

$$(58) \quad S(Y, \overline{P}(\xi, X)Z) = S(\overline{P}(\xi, X)Z, Y)$$

for any $X, Y, Z \in \chi(M)$.

Then similarly from (57) by interchanging Y, Z , we get,

$$\begin{aligned}
 (59) \quad & S(\overline{P}(\xi, X)Z, Y) \\
 &= [2n - 1 - \lambda][a - \frac{r}{2n+1}(\frac{a}{2n} + b)][g(X, Z)\eta(Y) - g(X, Y)\eta(Z)] \\
 &\quad + b[2n - 1 - \lambda]^2[g(X, Z)\eta(Y) - g(X, Y)\eta(Z)] \\
 &\quad - b(\mu - 1)[2n - 1 - \lambda][g(X, Z)\eta(Y) - g(X, Y)\eta(Z)] \\
 &\quad - (\mu - 1)[a - \frac{r}{2n+1}(\frac{a}{2n} + b)][g(X, Z)\eta(Y) - \eta(X)\eta(Y)\eta(Z)].
 \end{aligned}$$

for any $X, Y, Z \in \chi(M)$.

Now using (57), (58) and (59), (54) becomes,

$$\begin{aligned}
 (60) \quad & (\mu - 1)[a - \frac{r}{2n+1}(\frac{a}{2n} + b)][g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0.
 \end{aligned}$$

for any $X, Y, Z \in \chi(M)$.

Taking $Z = \xi$ in the above equation, we get,

$$(61) \quad (\mu - 1)[a - \frac{r}{2n+1}(\frac{a}{2n} + b)][g(X, Y) - \eta(X)\eta(Y)] = 0.$$

for any $X, Y \in \chi(M)$.

Then by using (14), we obtain,

$$(62) \quad (\mu - 1)[a - \frac{r}{2n+1}(\frac{a}{2n} + b)]g(\phi X, \phi Y) = 0.$$

for any $X, Y \in \chi(M)$.

Hence we get, $(\mu - 1)[a - \frac{r}{2n+1}(\frac{a}{2n} + b)] = 0$.

Then either $\mu - 1 = 0$, or $a - \frac{r}{2n+1}(\frac{a}{2n} + b) = 0$.

Which implies that, either,

$$(63) \quad \mu = 1.$$

or,

$$(64) \quad r = \frac{2n(2n+1)a}{a+2nb}.$$

Now if $\mu = 1$ then from (31), we get,

$$(65) \quad \lambda = -1.$$

Then we can state the following theorem:

Theorem 4.5. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) satisfies $\bar{P}(\xi, X) \cdot S = 0$ and admits $*\text{-}\eta$ -Ricci Soliton then either $\mu = 1$ and $\lambda = -1$ or $r = \frac{2n(2n+1)a}{a+2nb}$, where $a, b \neq 0$ are constants.*

Now if $\mu = 1$ and $\lambda = -1$, then from (28), we get,

$$(66) \quad S(X, Y) = 2ng(X, Y).$$

So we have,

Note: *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) satisfies $\bar{P}(\xi, X) \cdot S = 0$ and admits $*\text{-}\eta$ -Ricci Soliton then the manifold becomes Einstein manifold, provided $r \neq \frac{2n(2n+1)a}{a+2nb}$, where $a, b \neq 0$ are constants.*

Definition 4.6. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition

$$\phi^2 H(\phi X, \phi Y) \phi Z = 0$$

is called ϕ -conharmonically flat [15], where H is the conharmonic curvature tensor of M^n .

Let (M, ϕ, ξ, η, g) be a $(2n+1)$ dimensional ϕ -conharmonically flat Sasakian manifold.

Now it is obvious that $\phi^2 H(\phi X, \phi Y) \phi Z = 0$ iff

$$(67) \quad g(H(\phi X, \phi Y) \phi Z, \phi W) = 0,$$

for any $X, Y, Z, W \in \chi(M)$.

As the manifold is a $(2n+1)$ dimensional ϕ -conharmonically flat, then using (1.11)

and (4.30) we get,

$$\begin{aligned}
 g(R(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{2n-1} [g(\phi Y, \phi Z)S(\phi X, \phi W) \\
 &- g(\phi X, \phi Z)S(\phi Y, \phi W) + g(\phi X, \phi W)S(\phi Y, \phi Z) \\
 (68) \quad &- g(\phi Y, \phi W)S(\phi X, \phi Z)].
 \end{aligned}$$

Let $\{e_1, e_2, \dots, e_{2n}, \xi\}$ be a local orthonormal basis of $T_p(M)$.

Putting $X = W = e_i$ and summing over $i = 1, 2, \dots, (2n+1)$, we get,

$$\begin{aligned}
 g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \frac{1}{2n-1} [g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\
 &- g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) + g(\phi e_i, \phi e_i)S(\phi Y, \phi Z) \\
 (69) \quad &- g(\phi Y, \phi e_i)S(\phi e_i, \phi Z)].
 \end{aligned}$$

Now in a $(2n+1)$ dimensional Sasakian manifold, we have,

$$(70) \quad g(\phi e_i, \phi e_i) = 2n,$$

$$(71) \quad S(\phi e_i, \phi e_i) = r - 2n,$$

$$(72) \quad g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z),$$

$$(73) \quad g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z),$$

and

$$(74) \quad g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) - g(\phi Y, \phi Z).$$

Now using (70)-(74) in (69), we get,

$$\begin{aligned}
 S(\phi Y, \phi Z) - g(\phi Y, \phi Z) &= \frac{1}{2n-1} [(r-2n)g(\phi Y, \phi Z) - S(\phi Y, \phi Z) \\
 (75) \quad &+ 2nS(\phi Y, \phi Z) - S(\phi Y, \phi Z)].
 \end{aligned}$$

which implies that,

$$(76) \quad S(\phi Y, \phi Z) = (r-1)g(\phi Y, \phi Z).$$

Now from (28), we have,

$$(77) \quad S(\phi Y, \phi Z) = [2n-1-\lambda]g(\phi Y, \phi Z).$$

Using (31), the above equation becomes,

$$(78) \quad S(\phi Y, \phi Z) = [2n+\mu-1]g(\phi Y, \phi Z).$$

Again from (28), we have,

$$(79) \quad r = \sum_{i=1}^{2n+1} S(e_i, e_i) = (2n+1)[2n-1-\lambda] - \mu + 1.$$

Using (31), the above equation becomes,

$$(80) \quad r = 2n(2n + \mu).$$

Then using (78) and (80), (76) reduces to,

$$(81) \quad [2n + \mu - 1]g(\phi Y, \phi Z) = [2n(2n + \mu) - 1]g(\phi Y, \phi Z).$$

for any $Y, Z \in \chi(M)$, and it follows that,

$$(82) \quad [(2n + \mu - 1) - (2n(2n + \mu) - 1)] = 0,$$

which implies that,

$$(83) \quad \mu = -2n.$$

Then using (31), we get,

$$(84) \quad \lambda = 2n.$$

This leads to the following:

Theorem 4.7. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) is ϕ -conharmonically flat and admits $*\text{-}\eta$ -Ricci Soliton then $\mu = -2n$ and $\lambda = 2n$.*

Now using (83), (84) and (28), we obtain,

$$(85) \quad S(X, Y) = -g(X, Y) + (2n + 1)\eta(X)\eta(Y).$$

Then we have,

Corollary 4.8. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) is ϕ -conharmonically flat and admits $*\text{-}\eta$ -Ricci Soliton then the manifold becomes η -Einstein manifold.*

Definition 4.9. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0$$

is called ϕ -projectively flat [15], where P is the Projective curvature tensor of M^n . Let (M, ϕ, ξ, η, g) be a $(2n+1)$ dimensional ϕ -projectively flat Sasakian manifold. Now it is obvious that $\phi^2 P(\phi X, \phi Y)\phi Z = 0$ iff

$$(86) \quad g(P(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for any $X, Y, Z, W \in \chi(M)$.

As the manifold is a $(2n+1)$ dimensional ϕ -projectively flat, then using (12) and (86), we get,

$$(87) \quad \begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{2n}[g(\phi Y, \phi Z)S(\phi X, \phi W) \\ &- g(\phi X, \phi Z)S(\phi Y, \phi W)]. \end{aligned}$$

Let $\{e_1, e_2, \dots, e_{2n}, \xi\}$ be a local orthonormal basis of $T_p(M)$.

Putting $X = W = e_i$ and summing over $i = 1, 2, \dots, (2n+1)$, we get,

$$(88) \quad \begin{aligned} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \frac{1}{2n}[g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)]. \end{aligned}$$

Now using (70)-(74), the above equation becomes,

$$(89) \quad S(\phi Y, \phi Z) - g(\phi Y, \phi Z) = \frac{1}{2n}[(r - 2n)g(\phi Y, \phi Z) - S(\phi Y, \phi Z)],$$

which implies that,

$$(90) \quad S(\phi Y, \phi Z) = \frac{r}{2n+1}g(\phi Y, \phi Z).$$

Using (78) and (80) in the above equation, we get,

$$(91) \quad \mu = 1.$$

Then from (31), we obtain,

$$(92) \quad \lambda = -1.$$

Then we can state the following theorem:

Theorem 4.10. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) is ϕ -projectively flat and admits $\ast$ - η -Ricci Soliton then $\mu = 1$ and $\lambda = -1$.*

Now using (91), (92) and (28), we obtain,

$$(93) \quad S(X, Y) = 2ng(X, Y).$$

Then we have,

Corollary 4.11. *If a $(2n+1)$ dimensional Sasakian manifold (M, ϕ, ξ, η, g) is ϕ -projectively flat and admits $*\text{-}\eta$ -Ricci Soliton then the manifold becomes Einstein manifold.*

5. EXAMPLE OF A 5-DIMENSIONAL SASAKIAN MANIFOLD ADMITTING $*\text{-}\eta$ -RICCI SOLITON

We consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are standard coordinates in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4, e_5 be a linearly independent frame field on M given by [12],

$$e_1 = 2(y \frac{\partial}{\partial z} - \frac{\partial}{\partial x}), \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = -2 \frac{\partial}{\partial z}, \quad e_4 = 2(v \frac{\partial}{\partial z} - \frac{\partial}{\partial u}), \quad e_5 = -2 \frac{\partial}{\partial v}.$$

Let g be the Riemannian metric defined by,

$$g(e_i, e_j) = 0, i \neq j, i, j = 1, 2, 3, 4, 5,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M and ϕ be the $(1, 1)$ -tensor field defined by,

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0, \quad \phi e_4 = e_5, \quad \phi e_5 = -e_4.$$

Then, using the linearity of ϕ and g , we have $\eta(e_3) = 1, \phi^2(Z) = -Z + \eta(Z)e_3$ and $g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$, for any $Z, U \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines a Sasakian structure on M .

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g . Then we have,

$$[e_1, e_2] = 2e_3, [e_4, e_5] = 2e_3, \text{ and } [e_i, e_j] = 0 \text{ for others } i, j.$$

The connection ∇ of the metric g is given by,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszuls formula.

Using Koszuls formula, we can easily calculate,

$$\begin{aligned}\nabla_{e_1}e_1 &= 0, \nabla_{e_1}e_2 = e_3, \nabla_{e_1}e_3 = -e_2, \nabla_{e_1}e_4 = 0, \nabla_{e_1}e_5 = 0, \\ \nabla_{e_2}e_1 &= -e_3, \nabla_{e_2}e_2 = 0, \nabla_{e_2}e_3 = e_1, \nabla_{e_2}e_4 = 0, \nabla_{e_2}e_5 = 0, \\ \nabla_{e_3}e_1 &= -e_2, \nabla_{e_3}e_2 = e_1, \nabla_{e_3}e_3 = 0, \nabla_{e_3}e_4 = 0, \nabla_{e_3}e_5 = e_4, \\ \nabla_{e_4}e_1 &= 0, \nabla_{e_4}e_2 = 0, \nabla_{e_4}e_3 = -e_5, \nabla_{e_4}e_4 = 0, \nabla_{e_4}e_5 = e_3, \\ \nabla_{e_5}e_1 &= \nabla_{e_5}e_2 = \nabla_{e_5}e_3 = \nabla_{e_5}e_4 = \nabla_{e_5}e_5 = 0.\end{aligned}$$

It can be easily seen that for $e_3 = \xi$, (ϕ, ξ, η, g) is a Sasakian structure on M . Consequently, (M, ϕ, ξ, η, g) is a Sasakian manifold.

On this manifold (M, ϕ, ξ, η, g) , we can easily verify our results.

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