

CHAIN MIXING AND CHAIN TRANSITIVE ITERATED FUNCTION SYSTEMS

M. FATEHI NIA

DEPARTMENT OF MATHEMATICS, YAZD UNIVERSITY, 89195-741 YAZD, IRAN

E-MAIL: FATEHINIAM@YAZD.AC.IR

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ABSTRACT. This paper considers some properties in topological dynamical systems in iterated function systems. First, we will introduce chain mixing and chain transitive iterated function systems then some results and an example is presented to compare with these notions in discrete dynamical systems. As our main result, using adding machine maps and topological conjugacy we show that chain mixing, chain transitive and chain recurrence properties in iterated function systems are equivalent.

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1. INTRODUCTION

Assume that (X, d) is a compact metric space. In the first place it is necessary to recall some definitions and symbols which we need in the main body of the paper ([10, 11]). An *Iterated Function System (IFS)* $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is any family of continuous functions $f_\lambda : X \rightarrow X$, $\lambda \in \Lambda$, such that Λ is a finite nonempty set (see[11]).

Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 1\}$ and Λ^T denote the set of all infinite

sequences $\{\lambda_i\}_{i \in T}$ of symbols belonging to Λ . Each element of $\Lambda^{\mathbb{Z}_+}$ can be denoted as $\sigma = \{\lambda_1, \lambda_2, \dots\}$ and we use the shorted notation

$$\mathcal{F}_{\sigma_n} = f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ \dots \circ f_{\lambda_1}.$$

A sequence $\{x_n\}_{n \in T}$ in X is said to be an orbit of the IFS $\mathcal{F}$ if there exists $\sigma \in \Lambda^T$ such that $x_{n+1} = f_{\lambda_n}(x_n)$, for every $\lambda_n \in \sigma$ [10].

Two IFSs $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\lambda | \lambda \in \Lambda\}$ with continuous maps on compact metric spaces X and Y are said to be topological conjugate, if there is a homeomorphism $h : X \rightarrow Y$ such that $g_\lambda = h \circ f_\lambda \circ h^{-1}$, for all $\lambda \in \Lambda$ [10]. Many notions in dynamics like attractors, minimality, transitivity, and shadowing can be extended to IFS (see [2, 3, 7, 8, 9]).

Topological mixing and topological transitivity are two important notions in discrete dynamical systems. In [1], Akin E. prove that topological mixing implied topological transitivity. Many authors study different properties of topological dynamical systems and their applications in other areas [12, 13, 14]. For example chain recurrence and chain transitivity have applications in the study of epidemiology, game theory, economics, and mathematical biology (see [12] for more details). In [13], Richeson and Wiseman prove that for every continuous map on a connected space, chain mixing, chain transitivity and chain recurrent properties are equivalent. Up to this point, they prove that the growth of the chain mixing times give some information about topological entropy. The main goal of this paper is to extend the following result to iterated function systems:

Theorem 1.1. [13] *Let X be connected and $f : X \rightarrow X$ be continuous. Then the following are equivalent:*

- (1) *f is chain mixing.*
- (2) *f is totally chain transitive.*
- (3) *f is chain transitive.*
- (4) *f is chain recurrent.*

For this purpose, firstly, we define the notions chain mixing and chain transitive iterated function systems. Then by some lemmas, we present our results about these notions on iterated function systems. Lemma 3.5 is one of the main results of this paper, this lemma presents some conditions for an IFS to be topologically conjugated with an adding machine map. In Theorem 3.6 we prove that if $\mathcal{F}$ is

chain transitive then for some $k > 1$, $\mathcal{F}^k$ restricted to some invariant subsets is chain mixing; or $\mathcal{F}$ factors to an adding machine map. As a corollary we have Theorem 3.7 which is a generalization of Theorem 1.1 for iterated function systems. At the end section, we consider some results that are related to this theorem.

2. MIXING, RECURRENCE AND TRANSITIVITY IN ITERATED FUNCTION SYSTEMS

At first we present some notations which shall be used later.

Given $\delta > 0$, a sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in X is called a δ -chain of $\mathcal{F}$ if there exists $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that for every $\lambda_n \in \sigma$, we have $d(x_{n+1}, f_{\lambda_n}(x_n)) < \delta$.

A point $x \in X$ is chain recurrent if for every $\epsilon > 0$, there is an ϵ -chain from x to itself [4].

An IFS $\mathcal{F}$ is chain transitive if for every $x, y \in X$ and every $\epsilon > 0$ there is an ϵ -chain from x to y .

Definition 2.1. *The IFS $\mathcal{F}$ is topologically transitive if for every nonempty sets $U, V \subset X$, there is $\sigma \in \Lambda^{\mathbb{Z}_+}$ such that $\mathcal{F}_{\sigma_n}(U) \cap V \neq \emptyset$, for some $n \geq 1$.*

In [10], the author n -th composition of an IFS, $\mathcal{F}$ as the following:

$$\mathcal{F}^n = \{g_\mu | \mu \in \Pi\} = \{f_{\lambda_n} \circ \dots \circ f_{\lambda_1} | \lambda_1, \dots, \lambda_n \in \Lambda\}.$$

We say that $\mathcal{F}$ is totally chain transitive if $\mathcal{F}^n$ is chain transitive, for all $n \geq 1$.

Definition 2.2. [10] *The IFS $\mathcal{F}$ is said to be ϵ -chain mixing, if there exists $N > 0$ such that for every $x, y \in X$ and $n > N$ we can find an ϵ -chain from x to y of length exactly n .*

The IFS $\mathcal{F}$ is chain mixing if it is ϵ -chain mixing, for every $\epsilon > 0$.

In this section we will recall and extend some well-known general results in discrete dynamical systems for IFS. Also, we give a non trivial example of IFS that are chain mixing. The proof of the following lemma is straightforward and is therefore omitted.

Lemma 2.3. *Suppose that the IFS $\mathcal{F}$ is chain mixing then $\mathcal{F}^n$ is chain transitive, for all positive integers n .*

Given two complete metric spaces (X, d_X) and (Y, d_Y) , consider the product set $X \times Y$ endowed with the metric

$$D((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ and $\mathcal{G} = \{Y; g_\gamma | \gamma \in \Gamma\}$ be two IFS. The IFS $\mathcal{H} = \{X \times Y; \Phi_{\lambda, \gamma} | \lambda \in \Lambda, \gamma \in \Gamma\}$, defined by

$\Phi_{\lambda, \gamma}(x, y) := (f_\lambda(x), g_\gamma(y))$ is called the product of the IFSs $\mathcal{F}$ and $\mathcal{G}$.

Lemma 2.4. *Suppose that $\mathcal{F}^n$ is chain transitive, for all positive integers n . Then the IFS $\mathcal{F} \times \mathcal{F}$ is chain transitive.*

Proof. Fix $\epsilon > 0$. Let (a_1, b_1) and (a_2, b_2) be two arbitrary points in $X \times X$. This is sufficient to prove that, there exist two ϵ -chains of the same length from a_1 to b_1 and from a_2 to b_2 .

Let $P_1 = \{a_1 = x_0, x_1, \dots, x_m = b_1\}$ be an ϵ -chain from a_1 to b_1 and

$P_2 = \{b_1 = y_0, y_1, \dots, y_k = b_1\}$ be an ϵ -chain from b_1 to itself.

Consider a sequence $\{\lambda_1, \lambda_2, \dots, \lambda_{m-1}\}$ in Λ . Since $\mathcal{F}^{k+1}$ is chain transitive, there is an ϵ -chain of $\mathcal{F}^{k+1}$,

$$P_3^{k+1} = \{f_{\lambda_{m-1}} \circ \dots \circ f_{\lambda_1}(a_2) = z_0, z_1, \dots, z_l = b_2\}.$$

For each $0 \leq i < l$ there exists a sequence $\{\lambda_1^i, \dots, \lambda_k^i\}$ in Λ such that

$d(f_{\lambda_k^i} \circ \dots \circ f_{\lambda_1^i}(z_i), z_{i+1}) < \epsilon$. Then,

$\{a_2, f_{\lambda_1}(a_2), \dots, f_{\lambda_{m-1}} \circ \dots \circ f_{\lambda_1}(a_2),$

$z_1, f_{\lambda_1^1}(z_1), \dots, f_{\lambda_{k-1}^1} \circ f_{\lambda_{k-2}^1} \circ \dots \circ f_{\lambda_1^1}(z_1),$

$z_2, f_{\lambda_1^2}(z_2), \dots, f_{\lambda_{k-1}^2} \circ f_{\lambda_{k-2}^2} \circ \dots \circ f_{\lambda_1^2}(z_2),$

$\vdots$

$z_{l-1}, f_{\lambda_1^{l-1}}(z_{l-1}), \dots, f_{\lambda_{k-1}^{l-1}} \circ f_{\lambda_{k-2}^{l-1}} \circ \dots \circ f_{\lambda_1^{l-1}}(z_{l-1}), z_l = b_2\}$

is an ϵ -chain for $\mathcal{F}$ of length $m + kl + 1$ from a_2 to b_2 .

On the other hand, the sequence $\{P_1, P_2, \dots, P_2\}$, which the sequence P_2 repeated l times is an ϵ -chain of length $m + kl + 1$ from a_1 to b_1 . So, the proof is complete. $\square$

Consider the IFS $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$, by definitions, if f_λ is chain mixing, for some $\lambda \in \Lambda$, then $\mathcal{F}$ is chain mixing. In the next example we introduce an IFS $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ such that f_λ isn't chain mixing, for any $\lambda \in \Lambda$, but $\mathcal{F}$ is chain mixing.

Example 2.5. *Consider the tent map*

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

By Lemma 3.4 and the last example in [6], $T(x)$ is chain mixing.

Let $g_1(x) : [0, 1] \rightarrow [0, 1]$ be a map defined by

$$g_1(x) = \begin{cases} T(x) & \text{if } 0 \leq x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

and $g_2(x) : [0, 1] \rightarrow [0, 1]$ be a map defined by

$$g_2(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2}, \\ T(x) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Consider $0 < \epsilon < \frac{1}{2}$, since $g_1(\frac{3}{4}) = 1$ then there isn't any ϵ -chain from $\frac{3}{4}$ to 1. This implies that g_1 , as an ordinary dynamical system, isn't chain mixing. Similar argument shows that g_2 isn't chain mixing.

But, we show that $\mathcal{F} = \{g_1, g_2; [0, 1]\}$ is a chain mixing IFS.

Let $\epsilon > 0$ be arbitrary. Since $T(x)$ is chain mixing, there is $N > 0$ such that for any $x, y \in X$ and $n > N$ there exists an ϵ -chain for tent map from x to y of length exactly n .

Suppose that $x = x_0, x_1, \dots, x_{n-1} = y$ is the corresponding ϵ -chain for tent map.

Consider the sequence $\sigma = \{\lambda_0, \lambda_1, \dots, \lambda_{n-1}\}$, where $\lambda_0 = 1$ if $x_i \in [0, \frac{1}{2}]$ and $\lambda_0 = 1$ if $x_i \in [\frac{1}{2}, 1]$. So, for all $0 \leq i \leq n-1$, we have

$g_{\lambda_i}(x_i) = T(x_i)$ and hence $d(g_{\lambda_i}(x_i), x_{i+1}) = d(T(x_i), x_{i+1}) < \epsilon$. This implies that $x = x_0, x_1, \dots, x_{n-1} = y$ is an ϵ -chain from x to y of length exactly n under $\mathcal{F}$.

Lemma 2.6. Let $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an IFS and X be a connected space. Chain recurrence implies chain transitivity.

Proof. Suppose that $\mathcal{F}$ has the chain recurrence property and consider arbitrary positive number ϵ . Two points x and y are said to be ϵ -chain equivalent, if we can find ϵ -chains from x to y and conversely. The chain recurrence property for the IFS $\mathcal{F}$ implies that ϵ -chain is an equivalence relation. By connectedness of X this is sufficient to show that every equivalence class is an open set.

Assume that x and y are ϵ -chain equivalent. We show that there is $\delta > 0$ such that x is ϵ -chain equivalent to all points of $B_\delta(y)$. Pick $\delta \leq \frac{\epsilon}{2}$ such that $d(a, b) < \delta$ implies that $d(f_\lambda(a), f_\lambda(b)) < \frac{\epsilon}{2}$, for all $a, b \in X$ and $\lambda \in \Lambda$. Consider the arbitrary point $z \in B_\delta(y)$, we show that x is ϵ -chains equivalent to z .

Let $(y_0 = x, y_1, \dots, y_n = y)$ be an ϵ -chain from x to y and $(y_n = y, y_{n+1}, \dots, y_{n+m} = y)$ be an ϵ -chain from y to itself. Then,

$$(y_0 = x, y_1, \dots, y_n = y, y_{n+1}, \dots, y_{n+m-1}, z)$$

is an ϵ -chain from x to z . In a similar way, suppose that $(x_0 = y, x_1, \dots, x_k = x)$ is an ϵ -chain from y to x .

Then, $(z, y_{n+1}, \dots, y_{n+m} = y = x_0, x_1, \dots, y_n = x)$ is an ϵ -chain from z to x . So, x and z are ϵ -chain equivalent. $\square$

3. MAIN RESULTS

In this section by using adding machine maps and topological conjugacy we investigate the relation between chain mixing, chain transitivity and chain recurrent properties in iterated function systems. The main idea of these results and proofs come from the article by Richeson and Wiseman [13].

Definition 3.1. (*Adding machine map.* [5]) Let $\alpha = (j_1, j_2, \dots)$ be a sequence of integers where each $j_i \geq 2$. Let Δ_α denote all sequences $(x_1, x_2, \dots)$ where $x_i \in \{0, 1, \dots, j_i - 1\}$ for each i . We put a metric d_α on Δ_α given by $d_\alpha((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^{\infty} \frac{\delta(x_i, y_i)}{2^i}$, where $\delta(x_i, y_i) = 1$ if $x_i \neq y_i$ and $\delta(x_i, y_i) = 0$ if $x_i = y_i$. Addition in Δ_α is defined as follows:

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (z_1, z_2, \dots),$$

where $z_1 = (x_1 + y_1) \bmod j_1$ and $z_2 = (x_2 + y_2 + t_1) \bmod j_2$. Here $t_1 = 0$ if $x_1 + y_1 < j_1$ and $t_1 = 1$ if $x_1 + y_1 \geq j_1$. Continue adding and carrying in this way for the whole sequence.

We define, the adding machine map, $g_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$ by

$$g_\alpha((x_1, x_2, \dots)) = (x_1, x_2, \dots) + (1, 0, 0, \dots).$$

Lemma 3.2. Assume that $\mathcal{F}$ is a chain transitive IFS. For every $\epsilon > 0$ there exists $m_\epsilon > 1$ such that for any $x \in X$, m_ϵ is the greatest common divisor of the length of all ϵ -chains from x to x .

Proof. $x, z \in X$ and consider $m_\epsilon(x)$ as the greatest common divisor of the lengths of ϵ -chains from x to x . Suppose that $(z_0 = z, z_1, \dots, z_n = z)$ is an ϵ -chain and

$(x_0 = x, x_1, \dots, z, \dots, x_{km_\epsilon(x)} = x)$ is an ϵ -chain from x to itself and containing z . Then

$$(x_0 = x, x_1, \dots, z, z_1, \dots, z_n = z, \dots, x_{km_\epsilon(x)} = x),$$

is an ϵ -chain from x to itself of the length $km_\epsilon(x) + n$. This implies that $km_\epsilon(x) + n$ is a multiple of $m_\epsilon(x)$ and consequently $m_\epsilon(x)$ divides n . $\square$

Let $\mathcal{F}$ be chain transitive. Define a relation on X by setting $a \sim_\epsilon b$, if there is an ϵ -chain from a to b of length km_ϵ , for some $k > 0$. Since $\mathcal{F}$ is chain transitive, this is clear that $\sim_\epsilon$ is an equivalence relation.

Lemma 3.3. *For each $\epsilon > 0$ there are m_ϵ equivalence class for relation $\sim_\epsilon$, such that for every $\sigma \in \Lambda^{\mathbb{Z}^+}$, each equivalence class is invariant under $\mathcal{F}_{\sigma_{m_\epsilon}}$ and $\mathcal{F}_\sigma$ cycles among the classes periodically.*

Proof. The case $m_\epsilon = 1$ is clear.

Let $m_\epsilon > 1$ and $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_k, \dots\}$ be an arbitrary sequence in Λ . Consider $x \in X$, we have an ϵ -chain from x to x of km_ϵ length. If $f_{\lambda_1}(x) \in \bar{x}$ then we have an ϵ -chain from $f_{\lambda_1}(x)$ to x of length km_ϵ , for some $k_1 > 1$. So, there exists an ϵ -chain from x to x of $k_1m_\epsilon + 1$ length, that is a contradict with $m_\epsilon > 1$. Similarly for each $i < m_\epsilon$ if $\mathcal{F}_{\sigma_i}(x) \in \bar{x}$, then we have an ϵ -chain from x to x of $k_1m_\epsilon + i$ length, that is a contradict. Then, $\bar{x}, \overline{f_{\lambda_1}(x)}, \dots, \overline{\mathcal{F}_{\sigma_i}(x)}$ are m_ϵ separated equivalence class.

On the other hand $\{x, f_{\lambda_1}(x), \dots, \mathcal{F}_{\sigma_{m_\epsilon}}(x)\}$ is an ϵ -chain from x to x . Then $\mathcal{F}_{\sigma_{m_\epsilon}}(x) \in \bar{x}$ and the proof is complete. $\square$

Remark 3.4. *Let $\epsilon_1 \leq \epsilon_2$. We have the following properties:*

- (1) *Since every ϵ_1 -chain is also an ϵ_2 -chain then every $\sim_{\epsilon_2}$ equivalence class is the union of some $\sim_{\epsilon_1}$ equivalence class. So, m_{ϵ_2} divided m_{ϵ_1} .*
- (2) *Let $C_\epsilon = \{\bar{x}_1, \dots, \bar{x}_{m_\epsilon}\}$ be the set of all $\sim_\epsilon$ equivalence class, then $\bar{x}_1, \dots, \bar{x}_{m_\epsilon}$ are closed and open, pairwise disjoint, nonempty subsets of X .*

Lemma 3.5. *Let $\{\epsilon_i\}$ be a sequence decreasing to 0 and $\{m_{\epsilon_i}\}$ be strictly increasing of positive integer numbers. Suppose that for each positive integer i there is a cover C_i of X with the following properties:*

- (1) *For each i , C_i contains $\{m_{\epsilon_i}\}$ pairwise disjoint, nonempty, clopen sets which are cyclically by $\mathcal{F}_\sigma$.*

(2) C_{i+1} partitions C_i , for all $i \geq 1$.

(3) If $V_1 \supset V_2 \supset V_3 \supset \dots$ is a nested sequence with $V_i \in C_i$, then $\cap_{i=1}^{\infty} V_i$ consists of a single point.

Then for every $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \dots\} \in \Lambda^{\mathbb{Z}_+}$, $\mathcal{F}_\sigma$ is topologically conjugate to g_α , where $\alpha = \{m_{\epsilon_1}, \frac{m_{\epsilon_2}}{m_{\epsilon_1}}, \frac{m_{\epsilon_3}}{m_{\epsilon_2}}, \dots\}$.

Proof. Take $Q_1 = \{Y_{1,0}, \dots, Y_{1,m_{\epsilon_1}}\}$ and $Q_i = \{Y_{i,0}, \dots, Y_{i, \frac{m_{\epsilon_i}}{m_{\epsilon_{i-1}}}}\}$, for every $i > 1$ as constructed in proof of Theorem 2.3. of [5] which $\cap_{i \geq 1} Y_{i, t_i}$ consists a single point, where $t_i \in \{0, \dots, \frac{m_{\epsilon_i}}{m_{\epsilon_{i-1}}} - 1\}$, for all $i \geq 1$.

We define $h : X \rightarrow \Delta_\alpha$ as follows:

Let $x \in X$. For each positive integer i , there is a unique $k_i \leq m_{\epsilon_i} - 1$, such that $x \in X_{i,k_i}$. Now $\cap_{i \geq 1} Y_{i,k_i}$ consists a single point y . Let $h(x) = y$. By proof of Theorem 2.3. of [5], h is continuous and bijective. Now, we show that $h \circ \mathcal{F}_{\sigma_n} = g_\alpha^n \circ h$ for all $n \geq 1$.

Let $x \in X$ be arbitrary. For each positive integer i there is unique $k_i \leq m_{\epsilon_i} - 1$, such that $x \in X_{i,k_i}$. There is a unique $t_i \leq m_{\epsilon_i} - 1$ defined by

$t_i = (k_i + 1) \bmod m_{\epsilon_i}$. Then each of the points $h \circ \mathcal{F}_{\sigma_n}$ and $g_\alpha^n \circ h$ is an element of $\cap_{i \geq 1} Y_{i,t_i}$. Since this intersection consists of a single point, we have

$$h \circ \mathcal{F}_{\sigma_1}(x) = g_\alpha \circ h(x).$$

Replacing $f_{\lambda_1}(x)$ by x and repeat the above technique,

$$h \circ \mathcal{F}_{\sigma_2}(x) = h(f_{\lambda_2}(f_{\lambda_1}(x))) = g_\alpha \circ h((f_{\lambda_1}(x))) = g_\alpha(g_\alpha(h(x))).$$

So, by induction on n we have

$$h(\mathcal{F}_{\sigma_n}(x)) = h(f_{\lambda_n}(\mathcal{F}_{\sigma_{n-1}}(x))) = g_\alpha(h(\mathcal{F}_{\sigma_{n-1}}(x))) = g_\alpha(g_\alpha^{n-1}(h(x))) = g_\alpha^n(h(x)).$$

□

The following theorem is one of the main results of this paper.

Theorem 3.6. *Let $\mathcal{F}$ be a chain transitive IFS, then either:*

(1) *there exists a $k \geq 1$ such that $\mathcal{F}$ cyclically shifts k clopen equivalence class of X and $\mathcal{F}^k$ is chain mixing on every equivalence class; or*

(2) *For every $\sigma \in \Lambda^{\mathbb{Z}_+}$ there exists an adding machine map g such that $\mathcal{F}_\sigma$ is topologically conjugated to g .*

Proof. Fix $\epsilon > 0$. By Lemma 3.3, $\mathcal{F}$ cycles among the class periodically and each class is invariant under $\mathcal{F}_{\sigma_k}$, for every $\sigma \in \Lambda^{\mathbb{Z}+}$. The quantity m_ϵ is nondecreasing as $\epsilon \rightarrow 0$ and $\epsilon_1 \leq \epsilon_2$ implies that m_{ϵ_2} divides m_{ϵ_1} .

Now we have two cases: $\{m_\epsilon\}_{\epsilon>0}$ is bounded or m_ϵ increase without bound.

(1) Let m_ϵ has a maximum value k .

If $k = 1$ then there exists an unique equivalence class and similar arguments to proof of Theorem 6 in [13] shows that $\mathcal{F}$ is chain mixing.

Assume that $k = m_{\epsilon_0} > 1$ for some $\epsilon_0 > 0$, then the two equivalence relations $\sim$ and $\sim_{\epsilon_0}$ are the same. Hence we have k equivalence class such that each class is invariant under $\mathcal{F}_{\sigma_k}$ and $\mathcal{F}$ cycles among the equivalence classes periodically, for every $\sigma \in \Lambda^{\mathbb{Z}+}$. By Lemma 3.3, each equivalence class is invariant under $\mathcal{F}_{\sigma_k}$ and similar to the case $k = 1$, $\mathcal{F}_{\sigma_k}$ on each equivalence class is chain mixing.

(2) Let m_ϵ is unbounded and $\sigma \in \Lambda^{\mathbb{Z}+}$ be arbitrary. So, we have a decreasing sequence $\{\epsilon_i\}$ of positive numbers convergence to 0 and the related strictly increasing sequences $\{m_{\epsilon_i}\}$. Then for every $\{\epsilon_i\}$ we have $\{m_{\epsilon_i}\}$ equivalence class. Thus, by Remark 3.4 and Lemma 3.5 $\mathcal{F}_\sigma$ is topologically conjugate to g_α , where $\alpha = \{m_{\epsilon_1}, \frac{m_{\epsilon_2}}{m_{\epsilon_1}}, \frac{m_{\epsilon_3}}{m_{\epsilon_2}}, \dots\}$. $\square$

Now we obtain the main result of this paper. This is an extension of a similar result in [13] for iterated function systems.

Theorem 3.7. *Suppose that $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ is an IFS, consists of continuous functions on connected space X be connected. Therefore the following statements are equivalent:*

- (1) $\mathcal{F}$ is a chain mixing IFS.
- (2) $\mathcal{F}$ is a totally chain transitive IFS.
- (3) $\mathcal{F}$ is a chain transitive IFS.
- (4) $\mathcal{F}$ has the chain recurrence property.

Proof. By definitions this is clear that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. Since X is connected, by Theorem 3.6 we have $(3) \Rightarrow (1)$ and by Lemma 2.6 $(4) \Rightarrow (3)$, so the proof is complete. $\square$

Lemma 2.4 and Theorem 3.7 imply the following result.

Corollary 3.8. *Let X be connected and $\mathcal{F} = \{X; f_\lambda | \lambda \in \Lambda\}$ be an IFS, consists of continuous functions and that $\mathcal{F}^n$ is chain transitive, for all positive integers n . Then we have the following properties:*

- (1) $\mathcal{F} \times \mathcal{F}$ is chain mixing.
- (2) $\mathcal{F} \times \mathcal{F}$ is totally chain transitive.
- (3) $\mathcal{F} \times \mathcal{F}$ is chain recurrent.

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