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A NOTE ON L.C.Q.K. MANIFOLDS

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Abstract. L.c.q.K. manifolds were investigated by ([4], [7], [6], [8]) and anti-invariant Riemannian submersion was studied in ([10],[1]). The present note aims to work on anti-invariant and Lagrangian Riemannian submersion of locally conformal quaternion Kaehler manifolds. Further, the geometry of foliation is discussed and some decomposition results for such submersions are obtained.

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1. Introduction

In 1976, I. Vaisman [12] initiated the work on l.c.K (locally conformal Kaehler) manifolds (also see: [4]) and l.c.q.K (locally conformal quaternion Kaehler) manifolds were taken to investigation by ([7], [6], [8]) and their submanifolds in ([11],[3]).

One can recall l.c.q.K. manifolds by means of manifolds of quaternion Hermitian type that have metric which holds conformal property with quaternion Kaehler metric in some neighborhood of any given point. In case, any l.c.q.K. manifold satisfies compact property, one may fix Lee form parallel without any restrictions [4] and this one is the reason that set it apart from l.c.K. manifold [4].

On the other side, for Riemannian manifolds $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$, Prof. B. O'Neill was the researcher who produced the concept of Riemannian submersion $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$. In 2010, B. Sahin [10] earned the credit by working on anti-invariant Riemannian submersions defined from almost Hermitian manifold onto some Riemannian manifold. Our note aims to work on anti-invariant and Lagrangian Riemannian submersion of l.c.q.K. manifolds. Further, geometry of foliation is discussed and some decomposition results for such submersions are obtained.

2. Preliminaries

We begin with l.c.q.K. manifold.

Definition 2.1. *Let us identify by J_1, J_2 and J_3 almost complex structures and by $(\widetilde{\mathcal{M}}, J_{\overline{\alpha}}, g), \overline{\alpha} \in \{1,2,3\}$, any quaternion Hermitian manifold for sub-bundle $\mathring{S}$ of End $(T\widetilde{\mathcal{M}})$ that has rank equals to 3 and spanned by $J_{\overline{\alpha}}, \overline{\alpha} \in \{1,2,3\}$. A quaternion Hermitian metric g for which the Levi-Civita connection $\widetilde{\nabla}$ holds the condition $\widetilde{\nabla}\mathring{S} \subset \mathring{S}$, is called as quaternion Kaehler metric.*

Any l.c.q.K. manifold might be constructed by quaternion Hermitian manifold equipped with a metric g such that over neighborhoods $\mathcal{U}_i$ covering $\widetilde{\mathcal{M}}$, $g\,|_{\mathcal{U}_i} = e^{f_i} g_i$, for any quaternion Kaehler metric g_i on $\mathcal{U}_i$. Here, one can locally define the Lee form ω' as follows $\omega'\,|_{\mathcal{U}_i} = df_i$, so that the following hold [7]

$$d\theta' = \omega' \wedge \theta' \ \ and \ \ d\omega' = 0.$$

When one identifies associated Levi Civita connection with $\widetilde{\nabla}$, we see [7]

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{X}_1} J_{\overline{\alpha}})\mathcal{X}_2 &= \tfrac{1}{2}\{\theta'(\mathcal{X}_2)\mathcal{X}_1 - \omega'(\mathcal{X}_2)J_{\overline{\alpha}}\mathcal{X}_1 - g(\mathcal{X}_1,\mathcal{X}_2)A - \Omega(\mathcal{X}_1,\mathcal{X}_2)\mathring{B}\} \\ &\quad + \overline{Q_{\alpha b}}(\mathcal{X}_1)J_b\mathcal{X}_2 + \overline{Q_{\alpha c}}(\mathcal{X}_1)J_c\mathcal{X}_2, \ \ \forall \mathcal{X}_1,\mathcal{X}_2 \in T\widetilde{\mathcal{M}}, \end{aligned} \tag{1}$$

in this case, $\overline{Q_{\alpha b}}$ is used for a skew symmetric matrix of local forms and Lee vector field $\mathring{B}$ assures the relation $g(\mathcal{X}_1, \mathring{B}) = \omega'(\mathcal{X}_1)$. Also, $\theta' = \omega' o J_{\overline{\alpha}}$ and $A = -J_{\overline{\alpha}}\mathring{B}$.

Assume $(\widetilde{\mathcal{M}}^m, g)$ and $(\widetilde{\mathcal{B}}^b, g_{\widetilde{\mathcal{B}}})$ to be Riemannian manifolds with $m > b$. Then, $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$ of maximal rank defines a Riemannian submersion from $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{B}}$ if lengths of horizontal vectors are preserved by the differential $\overline{\pi}_*$.

Also, $\overline{\pi}^{-1}(p*), p* \in \widetilde{\mathcal{B}}$, represents a submanifold of $\widetilde{\mathcal{M}}$ famously known as fiber with dimension equals to $(m-b)$. Also, when vector field $\mathcal{X}$ is always tangent to fibers, it is named as vertical and if $\mathcal{X}$ is always orthogonal to fibers, it is known as horizontal.

Further, horizontal vector field becomes basic when $\overline{\pi}_*\mathcal{X}_p = \mathcal{X}_{*\overline{\pi}(p)}, \forall p \in \widetilde{\mathcal{M}}$, wherein $\mathcal{X}_*$ identifies a vector field on $\widetilde{\mathcal{B}}$.

Next, suppose $\mathcal{V}$ and $\mathcal{H}$ represents projection morphisms on $ker\overline{\pi}_*$ and $(ker\overline{\pi}_*)^{\perp}$, respectively. Then we have [5]

Lemma 2.2. *When we identify Riemannian submersion between Riemannian manifolds $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$ by $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$. On Riemannian manifold $\widetilde{\mathcal{B}}$, we also identify by ∇^* a Levi-Civita connection. One notices the following*

- **(a):** $g(\mathcal{Y}_1, \mathcal{Y}_2) = g_{\widetilde{\mathcal{B}}}(\mathcal{Y}_{1*}, \mathcal{Y}_{2*})o\overline{\pi}$,
- **(b):** $([\mathcal{Y}_1, \mathcal{Y}_2]^{\mathcal{H}}) = (\mathcal{Y}_{1*}, \mathcal{Y}_{2*})$,
- **(c):** *when $\overline{\mathbb{V}}$ represents vertical field of $ker\overline{\pi}_*$, then $[\overline{\mathbb{V}}, \mathcal{Y}_1]$ is also vertical,*
- **(d):** *in corresponds with $\nabla^*_{\mathcal{Y}_{1*}}\mathcal{Y}_{2*}$, $\mathcal{H}(\nabla_{\mathcal{Y}_1}\mathcal{Y}_2)$ represents a basic vector field.*

In this case, $\mathcal{Y}_1, \mathcal{Y}_2$ stand for basic vector fields on $\widetilde{\mathcal{M}}$.

For Riemannian submersion identified by $\overline{\pi}$ between Riemannian manifolds $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$. A characterization for geometry of $\overline{\pi}$ with the help of O'Neills tensors $\mathcal{T}$ and $\mathcal{A}$ can be done as

$$\mathcal{A}_{\overline{F_1}}\overline{F}_2 = \mathcal{H}\nabla_{\mathcal{H}\overline{F_1}}\mathcal{V}\overline{F_2} + \mathcal{V}\nabla_{\mathcal{H}\overline{F_1}}\mathcal{H}\overline{F_2} \tag{2}$$

$$\mathcal{T}_{\overline{F_1}}\overline{F_2} = \mathcal{H}\nabla_{\mathcal{V}\overline{F_1}}\mathcal{V}\overline{F_2} + \mathcal{V}\nabla_{\mathcal{V}\overline{F_1}}\mathcal{H}\overline{F_2}, \tag{3}$$

$\overline{F_1}$ and $\overline{F_2}$ are written for vector fields on $\widetilde{\mathcal{M}}$. One can also note that if $\mathcal{T}$ vanishes identically then and only then fibres of $\overline{\pi}$ are totally geodesic. Further, consider that $\forall\overline{F_1} \in \Gamma(T\widetilde{\mathcal{M}})$, then on this set of all sections on the tangent bundle, identify skew-symmetric operators by $\mathcal{T}_{\overline{F_1}}$ and $\mathcal{A}_{\overline{F_1}}$ so that horizontal distribution and vertical distribution are reversed. Additionally, observe $\mathcal{T}$ to be vertical and $\mathcal{A}$ to be horizontal, $\mathcal{T}_{\overline{F_1}} = \mathcal{T}_{\mathcal{V}\overline{F_1}}$, $\mathcal{A} = \mathcal{A}_{\mathcal{H}\overline{F_1}}$. We also have

$$\mathcal{T}_U\overline{\mathbb{W}} = \mathcal{T}_{\overline{\mathbb{W}}}U, \forall U, \overline{\mathbb{W}} \in \Gamma(ker\overline{\pi}_*) \tag{4}$$

$$\mathcal{A}_{\mathcal{X}}\mathcal{Y} = -\mathcal{A}_{\mathcal{Y}}\mathcal{X} = \frac{1}{2}\mathcal{V}[\mathcal{X}, \mathcal{Y}], \forall \mathcal{X}, \mathcal{Y} \in (\Gamma(ker\overline{\pi}_*)^{\perp}). \tag{5}$$

Next, we have the following relations.

Lemma 2.3. *Let us suppose that $\mathcal{Y}_1, \mathcal{Y}_2 \in (\Gamma(ker\overline{\pi}_*)^{\perp})$, then*

- **(a):** $\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{W}} = \mathcal{T}_{\overline{\mathbb{V}}}\overline{\mathbb{W}} + \hat{\nabla}_{\overline{\mathbb{V}}}\overline{\mathbb{W}}$
- **(b):** $\nabla_{\overline{\mathbb{V}}}\mathcal{Y}_1 = \mathcal{H}\nabla_{\overline{\mathbb{V}}}\mathcal{Y}_1 + \mathcal{T}_{\overline{\mathbb{V}}}\mathcal{Y}_1$
- **(c):** $\nabla_{\mathcal{Y}_1}\overline{\mathbb{V}} = \mathcal{A}_{\mathcal{Y}_1}\overline{\mathbb{V}} + \mathcal{V}\nabla_{\mathcal{Y}_1}\overline{\mathbb{V}}$
- **(d):** $\nabla_{\mathcal{Y}_1}\mathcal{Y}_2 = \mathcal{H}\nabla_{\mathcal{Y}_1}\mathcal{Y}_2 + \mathcal{A}_{\mathcal{Y}_1}\mathcal{Y}_2, \quad \forall\overline{\mathbb{V}}, \overline{\mathbb{W}} \in \Gamma(ker\overline{\pi}_*)$.

In this case, $\hat{\nabla}_{\overline{\mathbb{V}}}\overline{\mathbb{W}} = \mathcal{V}\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{W}}$ and when we identify a basic vector field by $\mathcal{Y}_1$, then $\mathcal{H}\nabla_{\overline{\mathbb{V}}}\mathcal{Y}_1 = \mathcal{A}_{\mathcal{Y}_1}\overline{\mathbb{V}}$.

3. Anti-invariant Riemannian submersions

Definition 3.1. [10] *Consider complex almost Hermitian manifold $(\widetilde{\mathcal{M}}^m, g, J)$ and suppose $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ identifies any Riemannian manifold. The Riemannian submersion produced by $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$ for which $J(ker\overline{\pi}_*) \subseteq (ker\overline{\pi}_*)^{\perp}$ holds, is known as anti-invariant Riemannian submersion.*

Now, consider anti-invariant Riemannian submersion $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$ and in $(ker\overline{\pi}_*)^{\perp}$, one might identify the orthogonal complementary distribution of $J(ker\overline{\pi}_*)$ with μ. Then above definition results $J(ker\overline{\pi}_*)^{\perp} \cap (ker\overline{\pi}_*) \neq 0$, producing

$$(ker\overline{\pi}_*)^{\perp} = J(ker\overline{\pi}_*) \oplus \mu. \tag{6}$$

One might also write

$$J\mathcal{X} = B\mathcal{X} + G\mathcal{X}, \quad \forall\mathcal{X} \in \Gamma((ker\overline{\pi}_*)^{\perp}), B\mathcal{X} \in \Gamma(ker\overline{\pi}_*), G\mathcal{X} \in \Gamma(\mu), \tag{7}$$

and using the fact that $\overline{\pi}_*((ker\overline{\pi}_*)^{\perp}) = T\widetilde{\mathcal{B}}$ and in light of (7), we can show $g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*J\overline{\mathbb{V}}, \overline{\pi}_*G\mathcal{X}) = 0, \forall\overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$ proposing

$$T\widetilde{\mathcal{B}} = \overline{\pi}_*(J(ker\overline{\pi}_*)) \oplus \overline{\pi}_*(\mu). \tag{8}$$

Next, harmonic maps are considered among manifolds $(\widetilde{\mathcal{M}}, g)$ and $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ equipped with Riemannian structures [2]. Take smooth map ϕ between $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$ and give the pullback bundle by $\phi^{-1}(T\widetilde{\mathcal{B}})$ having fibres $(\phi^{-1}(T\widetilde{\mathcal{B}}))_{p'} = T_{\phi(p')}\widetilde{\mathcal{B}}, p' \in \widetilde{\mathcal{M}}$, then one might view the differential of ϕ as a section of $\mathrm{Hom}(T\widetilde{\mathcal{M}}, \phi^{-1}(T\widetilde{\mathcal{B}})) \to \widetilde{\mathcal{M}}$. Also, $\mathrm{Hom}(T\widetilde{\mathcal{M}}, \phi^{-1}(T\widetilde{\mathcal{B}}))$ induces ∇ from the Levi-Civita connection written by $\nabla^{\widetilde{\mathcal{M}}}$ and pullback connection. One also has the second fundamental form for ϕ

$$(\nabla\phi_*)(\mathcal{Y}_1, \mathcal{Y}_2) = \nabla^{\phi}_{\mathcal{Y}_1}\phi_*(\mathcal{Y}_2) - \phi_*(\nabla^{\widetilde{\mathcal{M}}}_{\mathcal{Y}_1}\mathcal{Y}_2). \tag{9}$$

Now, we write.

Lemma 3.2. *Let us suppose that* $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ *represents any l.c.q.K. manifold and* $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ *any Riemannian manifold. Then, for anti-invariant Riemannian submersion* $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$*, we have*

(i): $g(G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = 0$

(ii): $g(\nabla_{\mathcal{Z}} G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = -g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, G\mathcal{Z})$

(iii): $g(\nabla_{\overline{\mathbb{V}}} B\mathcal{W}, G\mathcal{Z}) = g(G\mathcal{Z}, \mathcal{T}_{\overline{\mathbb{V}}} B\mathcal{W}) = -g(B\mathcal{W}, \mathcal{T}_{\overline{\mathbb{V}}} G\mathcal{Z})$,

$\forall \mathcal{Z}, \mathcal{W} \in \Gamma((ker\overline{\pi}_*)^{\perp}), \overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$. *In this case,* ω' *means globally defined closed* 1*-form of manifold* $\widetilde{\mathcal{M}}$*.*

Proof. Let $\mathcal{Z}, \mathcal{W} \in \Gamma((ker\overline{\pi}_*)^{\perp})$.

(i) Equation (7) presents

$$\begin{aligned} g(G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) &= g(J_{\overline{\alpha}}\mathcal{W} - B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g(J_{\overline{\alpha}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}), \forall \overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*), \end{aligned}$$

because $B\mathcal{W} \in \Gamma(ker\overline{\pi}_*), J_{\overline{\alpha}}\overline{\mathbb{V}} \in \Gamma((ker\overline{\pi}_*)^{\perp})$. Also, $g(J_{\overline{\alpha}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = g(\mathcal{W}, \overline{\mathbb{V}}) = 0$, completing the proof.

(ii) Consider $\mathring{B} \in \Gamma(ker\overline{\pi}_*)$ and use (1) to produce

$$\begin{aligned} g(\nabla_{\mathcal{Z}} G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) &= -g(G\mathcal{W}, \nabla_{\mathcal{Z}} J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= -g(G\mathcal{W}, J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{Z}), \end{aligned}$$

here part (i) has been used. Now, thanks to (7) we get

$$\begin{aligned} g(\nabla_{\mathcal{Z}} G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) &= -g(G\mathcal{W}, J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, B\mathcal{Z} + G\mathcal{Z}) \\ &= -g(G\mathcal{W}, J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, G\mathcal{Z}), \end{aligned}$$

since $G\mathcal{W} \in \Gamma(\mu)$ and $B\mathcal{Z} \in \Gamma(ker\overline{\pi}_*)$. So, Lemma 2.3 presents

$$g(\nabla_{\mathcal{Z}} G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = -g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) - (G\mathcal{W}, J_{\overline{\alpha}}\mathcal{V}\nabla_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, G\mathcal{Z}).$$

As $J_{\overline{\alpha}}\mathcal{V}\nabla_{\mathcal{Z}}\overline{\mathbb{V}} \in \Gamma(J_{\overline{\alpha}}ker\overline{\pi}_*)$, we arrive at

$$g(\nabla_{\mathcal{Z}} G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = -g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, G\mathcal{Z})$$

that proves the result. □

From now, we suppose $\mathring{B} \in (ker\overline{\pi}_*)$ and during the proofs of the results, horizontal vector fields are considered as basic whenever required. Also, $ker\overline{\pi}_*$ is integrable.

Theorem 3.3. *Let us suppose that* $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ *represents any l.c.q.K. manifold and* $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ *any Riemannian manifold. Then, for anti-invariant Riemannian submersion* $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$, *we have the following equivalent assertions for* $\overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$:

(a): $(ker\overline{\pi}_*)^\perp$ *is integrable,*

(b): $g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{W}, B\mathcal{Z}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) = g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, B\mathcal{W}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) + g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) - g(G\mathcal{Z}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{W}}\overline{\mathbb{V}}) - \frac{1}{2}g(\widetilde{\mathcal{B}}\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2}g(B\mathcal{Z}, \mathring{B})g(\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}})$,

(c): $g(\mathcal{A}_{\mathcal{W}}B\mathcal{Z} - \mathcal{A}_{\mathcal{Z}}B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = -g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) + g(G\mathcal{Z}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{W}}\overline{\mathbb{V}}) + \frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(B\mathcal{Z}, \mathring{B})g(\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}})$,

$\forall \mathcal{Z}, \mathcal{W} \in \Gamma((ker\overline{\pi}_*)^\perp)$.

Proof. Definition 3.1 produces $J_{\overline{\alpha}}\mathcal{W} \in \Gamma(ker\overline{\pi}_* \oplus \mu)$ and $J_{\overline{\alpha}}\overline{\mathbb{V}} \in \Gamma((ker\overline{\pi}_*)^\perp)$. So, use of (1) gives

$$\begin{aligned} g([\mathcal{Z}, \mathcal{W}], \overline{\mathbb{V}}) &= g(J_{\overline{\alpha}}[\mathcal{Z}, \mathcal{W}], J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g(J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(J_{\overline{\alpha}}\nabla_{\mathcal{W}}\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}\theta'(\mathcal{W})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -g(\nabla_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2}\theta'(\mathcal{Z})g(\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \end{aligned}$$

in the present case, $g(\mathcal{Z}, \mathring{B}) = \omega'(\mathcal{Z})$, $\theta' = \omega' o J_{\overline{\alpha}}$ and $\Omega(\mathcal{Z}, \mathcal{W}) = g(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{W})$, so $\mathring{B} \in \Gamma(ker\overline{\pi}_*)$ and (7) implies

$$\begin{aligned} g([\mathcal{Z}, \mathcal{W}], \overline{\mathbb{V}}) &= g(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(\nabla_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad +\frac{1}{2}g(B\mathcal{Z}, \mathring{B})g(\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g(\nabla_{\mathcal{Z}}B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) + g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(\nabla_{\mathcal{W}}B\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -g(\nabla_{\mathcal{W}}G\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad +\frac{1}{2}g(B\mathcal{Z}, \mathring{B})g(\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}). \end{aligned}$$

Because $\overline{\pi}$ represents a Riemannian submersion, one achieves

$$\begin{aligned} g([\mathcal{Z}, \mathcal{W}], \overline{\mathbb{V}}) &= g(\overline{\pi}_*\nabla_{\mathcal{Z}}B\mathcal{W}, \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) + g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\mathcal{W}}B\mathcal{Z}, \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -g(\nabla_{\mathcal{W}}G\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad +\frac{1}{2}g(B\mathcal{Z}, \mathring{B})g(\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}). \end{aligned}$$

Finally, Lemma 3.2 helps us to find

$$\begin{aligned} g([\mathcal{Z},\mathcal{W}],\overline{\mathbb{V}}) &= g_{\widetilde{\mathcal{B}}}(-(\nabla\overline{\pi}_*)(\mathcal{Z},B\mathcal{W})+(\nabla\overline{\pi}_*)(\mathcal{W},B\mathcal{Z}),\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -g(G\mathcal{W},J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}})+g(G\mathcal{Z},J_{\overline{\alpha}}\mathcal{A}_{\mathcal{W}}\overline{\mathbb{V}}) \\ &\quad -\frac{1}{2}g(B\mathcal{W},\mathring{B})g(\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}})+\frac{1}{2}g(B\mathcal{Z},\mathring{B})g(\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}) \end{aligned}$$

showing that (a)⇔(b).

Next, taking account of Lemma (2.3), we observe

$$\begin{aligned} (\nabla\overline{\pi}_*)(\mathcal{Z},B\mathcal{W}) &- (\nabla\overline{\pi}_*)(\mathcal{W},B\mathcal{Z}) \\ &= -\overline{\pi}_*(\nabla_{\mathcal{Z}}B\mathcal{W})+\overline{\pi}_*(\nabla_{\mathcal{W}}B\mathcal{Z}) \\ &= -\overline{\pi}_*(\nabla_{\mathcal{Z}}B\mathcal{W}-\nabla_{\mathcal{W}}B\mathcal{Z}) \\ &= \overline{\pi}_*(\mathcal{A}_{\mathcal{W}}B\mathcal{Z}-\mathcal{A}_{\mathcal{Z}}B\mathcal{W}). \end{aligned}$$

Therefore, it is written as

$$\begin{aligned} g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z},B\mathcal{W}) &- (\nabla\overline{\pi}_*)(\mathcal{W},B\mathcal{Z}),\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*(\mathcal{A}_{\mathcal{W}}B\mathcal{Z}-\mathcal{A}_{\mathcal{Z}}B\mathcal{W}),\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g(\mathcal{A}_{\mathcal{W}}B\mathcal{Z}-\mathcal{A}_{\mathcal{Z}}B\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}). \end{aligned}$$

But, $\mathcal{A}_{\mathcal{Z}}B\mathcal{W}-\mathcal{A}_{\mathcal{W}}B\mathcal{Z}\in\Gamma((ker\overline{\pi}_*)^{\perp})$ establishing equivalence of (b) and (c). □

Definition 3.4. *Anti-invariant Riemannian submersion $\overline{\pi}$ becomes Lagrangian Riemannian submersion provided $J_{\overline{\alpha}}(ker\overline{\pi}_*)=(ker\overline{\pi}_*)^{\perp}$. Moreover, submersion $\overline{\pi}$ represents proper anti-invariant Riemannian submersion if $\mu\neq\{0\}$.* [10]

Taking help of Theorem 3.3 produces:

Corollary 3.5. *Assume that $(\widetilde{\mathcal{M}},g,J_{\overline{\alpha}})$ represents l.c.q.K. manifold and $(\widetilde{\mathcal{B}},g_{\widetilde{\mathcal{B}}})$ any Riemannian manifold $(\widetilde{\mathcal{B}},g_{\widetilde{\mathcal{B}}})$. Then, for anti-invariant Riemannian submersion $\overline{\pi}:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{B}}$, we have the following equivalent assertions:*

- **(a):** $(ker\overline{\pi}_*)^{\perp}$ *is integrable*
- **(b):** $(\nabla\overline{\pi}_*)(\mathcal{Z},J_{\overline{\alpha}}\mathcal{W})=(\nabla\overline{\pi}_*)(\mathcal{W},J_{\overline{\alpha}}\mathcal{Z})-\frac{1}{2}g(B\mathcal{W},\mathring{B})\mathcal{Z}+\frac{1}{2}g(B\mathcal{Z},\mathring{B})\mathcal{W}$
- **(c):** $\overline{\pi}_*(\mathcal{A}_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}-\mathcal{A}_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z})=\frac{1}{2}g(B\mathcal{W},\mathring{B})\mathcal{Z}-\frac{1}{2}g(B\mathcal{Z},\mathring{B})\mathcal{W}$,
  $\forall\mathcal{Z},\mathcal{W}\in\Gamma((ker\overline{\pi}_*)^{\perp})$.

Proof. $\forall \overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$, one observes $J_{\overline{\alpha}}\overline{\mathbb{V}} \in \Gamma((ker\overline{\pi}_*)^{\perp})$. Therefore, (1) shows

$$\begin{aligned} g([\mathcal{Z},\mathcal{W}],\overline{\mathbb{V}}) &= g(J_{\overline{\alpha}}[\mathcal{Z},\mathcal{W}],J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g(J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(J_{\overline{\alpha}}\nabla_{\mathcal{W}}\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= g(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(\nabla_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -\frac{1}{2}g(B\mathcal{W},\mathring{B})g(\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2}g(B\mathcal{Z},\mathring{B})g(\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}). \end{aligned}$$

Equation (9) helps us to derive

$$\begin{aligned} g([\mathcal{Z},\mathcal{W}],\overline{\mathbb{V}}) &= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W},\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) - g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z},\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -\frac{1}{2}g(B\mathcal{W},\mathring{B})g(\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2}g(B\mathcal{Z},\mathring{B})g(\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &= -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z},J_{\overline{\alpha}}\mathcal{W}),\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) + g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{W},J_{\overline{\alpha}}\mathcal{Z}),\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -\frac{1}{2}g(B\mathcal{W},\mathring{B})g(\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2}g(B\mathcal{Z},\mathring{B})g(\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}). \end{aligned}$$

Thus, $(ker\overline{\pi}_*)^{\perp}$ is integrable iff

$$\begin{aligned} g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z},J_{\overline{\alpha}}\mathcal{W}),\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) &= g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{W},J_{\overline{\alpha}}\mathcal{Z}),\overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\ &\quad -\frac{1}{2}g(B\mathcal{W},\mathring{B})g(\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2}g(B\mathcal{Z},\mathring{B})g(\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}) \end{aligned}$$

proving that(a)$\Leftrightarrow$(b).

Now, for $\mathcal{Z},\mathcal{W} \in \Gamma((ker\overline{\pi}_*)^{\perp})$, taking help of (9), it is easy to see

$$\begin{aligned} (\nabla\overline{\pi}_*)(\mathcal{W},J_{\overline{\alpha}}\mathcal{Z}) &- (\nabla\overline{\pi}_*)(\mathcal{Z},J_{\overline{\alpha}}\mathcal{W}) \\ &= -\overline{\pi}_*(\nabla_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z}) + \overline{\pi}_*(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}) \\ &= \overline{\pi}_*(\mathcal{H}(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}) - \mathcal{H}(\nabla_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z})) \\ &= \overline{\pi}_*(\mathcal{A}_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W} - \mathcal{A}_{\mathcal{W}}J_{\overline{\alpha}}\mathcal{Z}) \end{aligned}$$

implying that (b)$\Leftrightarrow$(c). □

4. Totally geodesicness on $\widetilde{\mathcal{M}}$

Theorem 4.1. *Assume by $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ any l.c.q.K. manifold and by $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ any Riemannian manifold. Then, for anti-invariant Riemannian submersion $\overline{\pi}$ between $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$, one has these equivalent assertions for $\overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$:*

- **(a):** *totally geodesic foliation is defined by $(ker\overline{\pi}_*)^{\perp}$ on $\widetilde{\mathcal{M}}$,*
- **(b):** $g(\mathcal{A}_{\mathcal{Z}}B\mathcal{W},J_{\overline{\alpha}}\overline{\mathbb{V}}) = g(G\mathcal{W},J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) - manifold\tfrac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W},G\mathcal{Z})$
  $+ \tfrac{1}{2}g(B\mathcal{W},\mathring{B})g(\mathcal{Z},J_{\overline{\alpha}}\overline{\mathbb{V}}) + \tfrac{1}{2}g(\mathcal{Z},\mathcal{W})g(\mathring{B},\overline{\mathbb{V}}),$

(c): $g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{W}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) = -g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, G\mathcal{Z}) - \frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(\mathcal{Z}, \mathcal{W})g(\mathring{B}, \overline{\mathbb{V}}),$

$\forall \mathcal{Z}, \mathcal{W} \in \Gamma((ker\overline{\pi}_*)^{\perp})$.

Proof. Equation (7) leads to

$$\begin{aligned}
g(\nabla_{\mathcal{Z}}\mathcal{W}, \overline{\mathbb{V}}) &= g(J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= g(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(\mathcal{Z}, \mathcal{W})g(\mathring{B}, \overline{\mathbb{V}}) \\
&= g(\nabla_{\mathcal{Z}}B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) + g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&\quad -\frac{1}{2}g(\mathcal{Z}, \mathcal{W})g(\mathring{B}, \overline{\mathbb{V}}) \\
&= g(\mathcal{A}_{\mathcal{Z}}B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, G\mathcal{Z}) \\
&\quad -\frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(\mathcal{Z}, \mathcal{W})g(\mathring{B}, \overline{\mathbb{V}})
\end{aligned}$$

where Lemma 2.3 and Lemma 3.2 have been used. In this way, a totally geodesic foliation is defined by $(ker\overline{\pi}_*)^{\perp}$ on $\widetilde{\mathcal{M}}$ iff

$$\begin{aligned}
g(\mathcal{A}_{\mathcal{Z}}B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) &= g(G\mathcal{W}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) - \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{W}, G\mathcal{Z}) \\
&\quad +\frac{1}{2}g(B\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2}g(\mathcal{Z}, \mathcal{W})g(\mathring{B}, \overline{\mathbb{V}}).
\end{aligned}$$

So, we have (a)$\Leftrightarrow$(b). Now, considering (9), presents

$$\begin{aligned}
g(\mathcal{A}_{\mathcal{Z}}B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) &= g(\nabla_{\mathcal{Z}}B\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= g(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\mathcal{W}, \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) - g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{W}), \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) + g_{\widetilde{\mathcal{B}}}(\nabla^{\overline{\pi}}_{\mathcal{Z}}\overline{\pi}_*(J_{\overline{\alpha}}\mathcal{W}), \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&\quad -g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{W}), \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}}) + g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&\quad -g(\nabla_{\mathcal{Z}}G\mathcal{W}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{W}), \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{V}})
\end{aligned}$$

concluding that (b) is equivalent to (c). $\square$

The next result is for Lagrangian Riemannian submersion.

Corollary 4.2. *Assume by* $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ *any l.c.q.K. manifold and by* $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ *any Riemannian manifold. Then, for Lagrangian Riemannian submersion* $\overline{\pi}$ *between* $\widetilde{\mathcal{M}}$ *and* $\widetilde{\mathcal{B}}$, *one has these equivalent assertions for* $\overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$:

- **(a):** *totally geodesic foliation by* $(ker\overline{\pi}_*)^{\perp}$ *is defined on* $\widetilde{\mathcal{M}}$,
- **(b):** $g_{\widetilde{\mathcal{B}}}(\mathcal{A}_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = \frac{1}{2} g(J_{\overline{\alpha}}\mathcal{U}, \mathring{B}) g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2} g(\mathcal{Z},\mathcal{U}) g(\mathring{B}, \overline{\mathbb{V}})$,
- **(c):** $g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{U}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) = -\frac{1}{2} g(J_{\overline{\alpha}}\mathcal{U}, \mathring{B}) g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2} g(\mathcal{Z},\mathcal{U}) g(\mathring{B}, \overline{\mathbb{V}})$,

$\forall \mathcal{Z},\mathcal{U} \in \Gamma((ker\overline{\pi}_*)^{\perp})$.

Proof. Assume $\forall \mathcal{Z},\mathcal{U} \in \Gamma((ker\overline{\pi}_*)^{\perp})$. Then, (1) produces

$$\begin{aligned}
g(\nabla_{\mathcal{Z}}\mathcal{U}, \overline{\mathbb{V}}) &= g(J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\mathcal{U}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= g(\nabla_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}\theta'(\mathcal{U}) g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2} g(\mathcal{Z},\mathcal{U}) g(\mathring{B}, \overline{\mathbb{V}}) \\
&= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}, \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}\theta'(\mathcal{U}) g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2} g(\mathcal{Z},\mathcal{U}) g(\mathring{B}, \overline{\mathbb{V}}) \\
&= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*(\mathcal{A}_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}\theta'(\mathcal{U}) g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2} g(\mathcal{Z},\mathcal{U}) g(\mathring{B}, \overline{\mathbb{V}}).
\end{aligned}$$

In this way, totally geodesic foliation by $(ker\overline{\pi}_*)^{\perp}$ is defined on $\widetilde{\mathcal{M}}$ iff

$$g_{\widetilde{\mathcal{B}}}(\mathcal{A}_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}, J_{\overline{\alpha}}\overline{\mathbb{V}}) = \frac{1}{2}\theta'(\mathcal{U}) g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) + \frac{1}{2} g(\mathcal{Z},\mathcal{U}) g(\mathring{B}, \overline{\mathbb{V}})$$

establishing equivalence between (a) and (b). Next, taking help of (9), we get

$$\begin{aligned}
g_{\widetilde{\mathcal{B}}}(\mathcal{A}_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}, J_{\overline{\alpha}}\overline{\mathbb{V}}) &= g_{\widetilde{\mathcal{B}}}(\nabla_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}, J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\mathcal{Z}} J_{\overline{\alpha}}\mathcal{U}, \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) \\
&= -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{U}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}})
\end{aligned}$$

showing

$$g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{U}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) = -\frac{1}{2}\theta'(\mathcal{U}) g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2} g(\mathcal{Z},\mathcal{U}) g(\mathring{B}, \overline{\mathbb{V}})$$

concluding (b) is equivalent to (c). □

Assume $\mathcal{Z} \in \Gamma(\mu)$. Then, (9) produces

$$(\nabla\overline{\pi}_*)(\overline{\mathbb{L}}, \mathcal{Z}) = \nabla^{\overline{\pi}}_{\overline{\mathbb{L}}}\overline{\pi}_*\mathcal{Z} - \overline{\pi}_*\nabla_{\overline{\mathbb{L}}}\mathcal{Z}, , \forall \overline{\mathbb{L}} \in \Gamma(ker\overline{\pi}_*).$$

Also,

$$(\nabla\overline{\pi}_*)(\mathcal{Z}, \overline{\mathbb{L}}) = \nabla^{\overline{\pi}}_{\mathcal{Z}}\overline{\pi}_*\overline{\mathbb{L}} - \overline{\pi}_*\nabla_{\mathcal{Z}}\overline{\mathbb{L}}.$$

Taking into consideration symmetry of second fundamental form, one gets

$$\nabla^{\overline{\pi}}_{\mathbb{L}}\pi_*\mathcal{Z} = 0. \tag{10}$$

Next, we state

Theorem 4.3. *Assume by $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ any l.c.q.K. manifold and by $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ any Riemannian manifold. Then, for anti-invariant Riemannian submersion $\overline{\pi}$ between $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$, one has these equivalent assertions for $\overline{\mathbb{V}}, \overline{\mathbb{U}} \in \Gamma(ker\overline{\pi}_*)$:*

- **(a):** *$(ker\overline{\pi}_*)$ is used to define a totally geodesic foliation on $\widetilde{\mathcal{M}}$,*
- **(b):** *$\mathcal{T}_{\overline{\mathbb{V}}}B\mathcal{Z} + \mathcal{A}_G\mathcal{Z}\overline{\mathbb{V}} = 0$ or $\mathcal{T}_{\overline{\mathbb{V}}}B\mathcal{Z} + \mathcal{A}_G\mathcal{Z}\overline{\mathbb{V}} \in \Gamma(\mu)$,*
- **(c):** *$g_{\widetilde{\mathcal{B}}}((\nabla\pi_*)(\overline{\mathbb{V}}, J_{\overline{\alpha}}\mathcal{Z}), \pi_*J_{\overline{\alpha}}\overline{\mathbb{U}}) = 0$, $\forall\mathcal{Z} \in \Gamma((ker\overline{\pi}_*)^{\perp})$.*

Proof. Equation (1) gives

$$\begin{aligned} g(\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{U}}, \mathcal{Z}) &= g(J_{\overline{\alpha}}\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{U}}, J_{\overline{\alpha}}\mathcal{Z}) \\ &= g(\nabla_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\overline{\mathbb{U}}, J_{\overline{\alpha}}\mathcal{Z}) \\ &= -g(J_{\overline{\alpha}}\overline{\mathbb{U}}, \nabla_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\mathcal{Z}) \end{aligned}$$

where we used that $(ker\overline{\pi}_*)$ and $(ker\overline{\pi}_*)^{\perp}$ are orthogonal. Next, (7) and Lemma 2.3 bring us at

$$\begin{aligned} g(\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{U}}, \mathcal{Z}) &= -g(J_{\overline{\alpha}}\overline{\mathbb{U}}, \nabla_{\overline{\mathbb{V}}}B\mathcal{Z}) - g(J_{\overline{\alpha}}\overline{\mathbb{U}}, \nabla_{\overline{\mathbb{V}}}G\mathcal{Z}) \\ &= -g(J_{\overline{\alpha}}\overline{\mathbb{U}}, \mathcal{T}_{\overline{\mathbb{V}}}B\mathcal{Z}) - g(J_{\overline{\alpha}}\overline{\mathbb{U}}, \mathcal{A}_{G\mathcal{Z}}\overline{\mathbb{V}}) \\ &= -g(J_{\overline{\alpha}}\overline{\mathbb{U}}, \mathcal{T}_{\overline{\mathbb{V}}}B\mathcal{Z} + \mathcal{A}_{G\mathcal{Z}}\overline{\mathbb{V}}) \end{aligned}$$

concluding that (a)⇔(b). Now, taking help of (9), we observe

$$\begin{aligned} g(\mathcal{T}_{\overline{\mathbb{V}}}B\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{U}}) \quad + &\quad g(\mathcal{A}_{G\mathcal{Z}}\overline{\mathbb{V}}, J_{\overline{\alpha}}\overline{\mathbb{U}}) \\ &= g(\mathcal{H}(\nabla_{\overline{\mathbb{V}}}B\mathcal{Z}), J_{\overline{\alpha}}\overline{\mathbb{U}}) + g(\mathcal{H}(\nabla_{\overline{\mathbb{V}}}G\mathcal{Z}), J_{\overline{\alpha}}\overline{\mathbb{U}}) \\ &= g(\nabla_{\overline{\mathbb{V}}}B\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{U}}) + g(\nabla_{\overline{\mathbb{V}}}G\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{U}}) \\ &= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\overline{\mathbb{V}}}B\mathcal{Z}, \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{U}}) + g_{\widetilde{\mathcal{B}}}(\overline{\pi}_*\nabla_{\overline{\mathbb{V}}}G\mathcal{Z}, \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{U}}) \\ &= -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, B\mathcal{Z}), \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{U}}) - g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, G\mathcal{Z}), \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{U}}) \\ &\quad + g_{\widetilde{\mathcal{B}}}(\nabla^{\overline{\pi}}_{\overline{\mathbb{V}}}\overline{\pi}_*G\mathcal{Z}, \overline{\pi}_*J_{\overline{\alpha}}\overline{\mathbb{U}}) \end{aligned}$$

which in the light of (10) gives

$$\begin{aligned} g(\mathcal{T}_{\overline{\mathbb{V}}}B\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{U}}) \quad &+ \quad g(\mathcal{A}_{G\mathcal{Z}}\overline{\mathbb{V}}, J_{\overline{\alpha}}\overline{\mathbb{U}}) \\ &= \quad -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, B\mathcal{Z}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{U}}) - g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, G\mathcal{Z}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{U}}) \\ &= \quad -g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, J_{\overline{\alpha}}\mathcal{Z}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{U}}) \end{aligned}$$

proving equivalency between (b) and (c). □

For Lagrangian Riemannian submersion $\overline{\pi}$, (8) produces $T\widetilde{\mathcal{B}} = \overline{\pi}_*(J_{\overline{\alpha}}(ker\overline{\pi}_*))$. So, one writes

Corollary 4.4. *Assume by $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ any l.c.q.K. manifold and by $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ any Riemannian manifold. Then, for Lagrangian Riemannian submersion $\overline{\pi}$ between $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$, one has these equivalent assertions for $\overline{\mathbb{U}}, \overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$:*

- **(a):** *totally geodesic foliation by $(ker\overline{\pi}_*)$ is defined on $\widetilde{\mathcal{M}}$,*
- **(b):** $\mathcal{T}_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\overline{\mathbb{U}} = 0$,
- **(c):** $(\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, J_{\overline{\alpha}}\mathcal{Z}) = 0, \quad \forall \mathcal{Z} \in \Gamma((ker\overline{\pi}_*)^\perp)$.

Proof. From Theorem 4.3 $(a) \Leftrightarrow (b)$ is obvious. We prove $(b) \Leftrightarrow (c)$. Taking into account (9) and Lemma 2.3 and using the orthogonality of $(ker\overline{\pi}_*)$ and $(ker\overline{\pi}_*)^\perp$, we obtain

$$\begin{aligned} g(\nabla_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\overline{\mathbb{U}}, J_{\overline{\alpha}}\mathcal{Z}) &= -g(J_{\overline{\alpha}}\overline{\mathbb{U}}, \nabla_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\mathcal{Z}) \\ &= -g_{\widetilde{\mathcal{B}}}(\overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{U}}, \overline{\pi}_*\nabla_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\mathcal{Z}) \\ &= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{U}}, (\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, J_{\overline{\alpha}}\mathcal{Z})) \\ g(\mathcal{T}_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\overline{\mathbb{U}}, J_{\overline{\alpha}}\mathcal{Z}) &= g_{\widetilde{\mathcal{B}}}(\overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{U}}, (\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, J_{\overline{\alpha}}\mathcal{Z})) \end{aligned}$$

since $\mathcal{T}_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\overline{\mathbb{U}} \in \Gamma(ker\overline{\pi}_*)$, it implies (b) is equivalent to (c). □

Definition 4.5. [2] *For any two Riemannian manifolds presented by $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$, respectively. The differential map $\overline{\pi}$ between them becomes totally geodesic provided $\nabla\overline{\pi}_* = 0$.*

Theorem 4.6. *Assume by $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ any l.c.q.K. manifold and by $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ any Riemannian manifold. Then, for Lagrangian Riemannian submersion $\overline{\pi}$ between $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{B}}$, one has for $\overline{\mathbb{U}}, \overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$:*

$$\mathcal{T}_{\overline{\mathbb{V}}}J_{\overline{\alpha}}\overline{\mathbb{U}} + \frac{1}{2}\omega'(\overline{\mathbb{U}})J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2}g(\overline{\mathbb{V}}, \overline{\mathbb{U}})A + \overline{Q_{\alpha b}}(\overline{\mathbb{V}})J_b\overline{\mathbb{U}} + \overline{Q_{\alpha c}}(\overline{\mathbb{V}})J_c\overline{\mathbb{U}} = 0$$

and

$$\mathcal{A}_{\mathcal{Z}} J_{\overline{\alpha}} \overline{\mathbb{U}} + \frac{1}{2}\omega'(\overline{\mathbb{U}}) J_{\overline{\alpha}} \mathcal{Z} + \frac{1}{2}\Omega(\mathcal{Z}, \overline{\mathbb{U}})\mathring{B} + \overline{Q_{\alpha b}}(\mathcal{Z}) J_b \overline{\mathbb{U}} + \overline{Q_{\alpha c}}(\mathcal{Z}) J_c \overline{\mathbb{U}} = 0,$$

iff $\overline{\pi}$ *is totally geodesic. Where* $\forall \mathcal{Z}, \mathcal{W} \in \Gamma((ker\overline{\pi}_*)^{\perp})$.

Proof. The following relation holds for $\overline{\pi}$

$$(\nabla\overline{\pi}_*)(\mathcal{Z}, \mathcal{W}) = 0. \tag{11}$$

Thanks to (1), (9) and (10), for $\overline{\mathbb{V}}, \overline{\mathbb{U}} \in (ker\overline{\pi}_*)$, we derive

$$\begin{aligned}
(\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, \overline{\mathbb{U}}) &= \nabla^{\overline{\pi}}_{\overline{\mathbb{V}}}\overline{\pi}_*(\overline{\mathbb{U}}) - \overline{\pi}_*(\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{U}}) \\
&= -\overline{\pi}_*(\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{U}}) \\
&= \overline{\pi}_*(J_{\overline{\alpha}}(J_{\overline{\alpha}}(\nabla_{\overline{\mathbb{V}}}\overline{\mathbb{U}})) \\
&= \overline{\pi}_*(J_{\overline{\alpha}}(\nabla_{\overline{\mathbb{V}}} J_{\overline{\alpha}}\overline{\mathbb{U}}) + \frac{1}{2}\omega'(\overline{\mathbb{U}}) J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2} g(\overline{\mathbb{V}}, \overline{\mathbb{U}}) A \\
&\quad + \overline{Q_{\alpha b}}(\overline{\mathbb{V}}) J_b \overline{\mathbb{U}} + \overline{Q_{\alpha c}}(\overline{\mathbb{V}}) J_c \overline{\mathbb{U}})) \\
&= \overline{\pi}_*(J_{\overline{\alpha}}(\mathcal{T}_{\overline{\mathbb{V}}} J_{\overline{\alpha}}\overline{\mathbb{U}}) + \frac{1}{2}\omega'(\overline{\mathbb{U}}) J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2} g(\overline{\mathbb{V}}, \overline{\mathbb{U}}) A \\
&\quad + \overline{Q_{\alpha b}}(\overline{\mathbb{V}}) J_b \overline{\mathbb{U}} + \overline{Q_{\alpha c}}(\overline{\mathbb{V}}) J_c \overline{\mathbb{U}})).
\end{aligned} \tag{12}$$

Also, (9) produces

$$\begin{aligned}
(\nabla\overline{\pi}_*)(\mathcal{Z}, \overline{\mathbb{U}}) &= \nabla^{\overline{\pi}}_{\mathcal{Z}}\overline{\pi}_*(\overline{\mathbb{U}}) - \overline{\pi}_*(\nabla_{\mathcal{Z}}\overline{\mathbb{U}}) \\
&= -\overline{\pi}_*(\nabla_{\mathcal{Z}}\overline{\mathbb{U}}) \\
&= \overline{\pi}_*(J_{\overline{\alpha}}(J_{\overline{\alpha}}(\nabla_{\mathcal{Z}}\overline{\mathbb{U}})) \\
&= \overline{\pi}_*(J_{\overline{\alpha}}(\nabla_{\mathcal{Z}} J_{\overline{\alpha}}\overline{\mathbb{U}}) + \frac{1}{2}\omega'(\overline{\mathbb{U}}) J_{\overline{\alpha}}\mathcal{Z} + \frac{1}{2}\Omega(\mathcal{Z}, \overline{\mathbb{U}})\mathring{B} \\
&\quad + \overline{Q_{\alpha b}}(\mathcal{Z}) J_b \overline{\mathbb{U}} + \overline{Q_{\alpha c}}(\mathcal{Z}) J_c \overline{\mathbb{U}})) \\
&= \overline{\pi}_*(J_{\overline{\alpha}}(\mathcal{A}_{\mathcal{Z}} J_{\overline{\alpha}}\overline{\mathbb{U}}) + \frac{1}{2}\omega'(\overline{\mathbb{U}}) J_{\overline{\alpha}}\mathcal{Z} + \frac{1}{2}\Omega(\mathcal{Z}, \overline{\mathbb{U}})\mathring{B} \\
&\quad + \overline{Q_{\alpha b}}(\mathcal{Z}) J_b \overline{\mathbb{U}} + \overline{Q_{\alpha c}}(\mathcal{Z}) J_c \overline{\mathbb{U}})),
\end{aligned} \tag{13}$$

and so $J_{\overline{\alpha}}$ is non-singular and required result holds by (11),(12) and (13). □

5. Some decomposition results

Definition 5.1. [9] *Let* g *represents Riemannian metric tensor associated with* $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}} \times \widetilde{\mathcal{B}}$ *and identify by* $\mathcal{D}_{\widetilde{\mathcal{M}}}$ *and* $\mathcal{D}_{\widetilde{\mathcal{B}}}$ *the canonical foliations that perpendicularly intersect everywhere. Then, one can identify by* g *the metric tensor of*

(i): *twisted product* $\widetilde{\mathcal{M}} \times_f \widetilde{\mathcal{B}}$ *iff* $\mathcal{D}_{\widetilde{\mathcal{B}}}$ *and* $\mathcal{D}_{\widetilde{\mathcal{M}}}$ *are totally umbilical and totally geodesic foliations, respectively;*

(ii): *warped product* $\widetilde{\mathcal{M}} \times_f \widetilde{\mathcal{B}}$ *iff* $\mathcal{D}_{\widetilde{\mathcal{B}}}$ *and* $\mathcal{D}_{\widetilde{\mathcal{M}}}$ *are is a spherical foliation and totally geodesic foliations, respectively;*

(iii): *usual product of Riemannian manifolds iff* $\mathcal{D}_{\widetilde{\mathcal{B}}}$ *and* $\mathcal{D}_{\widetilde{\mathcal{M}}}$ *are totally geodesic foliations.*

Theorems 4.1 and 4.3 lead to the following result for locally conformal quaternion Kaehler manifold.

Theorem 5.2. *Assume by* $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ *any l.c.q.K. manifold and by* $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ *any Riemannian manifold. Then, for anti-invariant Riemannian submersion* $\overline{\pi}$ *between* $\widetilde{\mathcal{M}}$ *and* $\widetilde{\mathcal{B}}$*,* $\widetilde{\mathcal{M}}$ *represents a locally product manifold iff*

$$\begin{aligned} g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{U}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) &= -g(G\mathcal{U}, J_{\overline{\alpha}}\mathcal{A}_{\mathcal{Z}}\overline{\mathbb{V}}) + \frac{1}{2}\omega'(\overline{\mathbb{V}})g(G\mathcal{U}, G\mathcal{Z}) \\ &\quad -\frac{1}{2}g(B\mathcal{U}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(\mathcal{Z}, \mathcal{U})g(\mathring{B}, \overline{\mathbb{V}}) \end{aligned}$$

and

$$g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, J_{\overline{\alpha}}\mathcal{Z}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{U}}) = 0, \quad \forall \mathcal{Z}, \mathcal{U} \in \Gamma((ker\overline{\pi}_*)^\perp), \forall \overline{\mathbb{V}}, \overline{\mathbb{U}} \in \Gamma(ker\overline{\pi}_*).$$

The following result also follows from Corollary 4.2 and Corollary 4.4.

Theorem 5.3. *Assume* $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ *represents l.c.q.K. manifold and* $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ *identifies any Riemannian manifold. Then, for Lagrangian Riemannian submersion* $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$*,* $\widetilde{\mathcal{M}}$ *is locally product manifold iff*

$$g_{\widetilde{\mathcal{B}}}((\nabla\overline{\pi}_*)(\mathcal{Z}, J_{\overline{\alpha}}\mathcal{W}), \overline{\pi}_* J_{\overline{\alpha}}\overline{\mathbb{V}}) = -\frac{1}{2}g(J_{\overline{\alpha}}\mathcal{W}, \mathring{B})g(\mathcal{Z}, J_{\overline{\alpha}}\overline{\mathbb{V}}) - \frac{1}{2}g(\mathcal{Z}, \mathcal{W})g(\mathring{B}, \overline{\mathbb{V}})$$

and

$$(\nabla\overline{\pi}_*)(\overline{\mathbb{V}}, J_{\overline{\alpha}}\mathcal{Z}) = 0, \quad \forall \mathcal{Z}, \mathcal{W} \in \Gamma((ker\overline{\pi}_*)^\perp), \forall \overline{\mathbb{V}}, \overline{\mathbb{U}} \in \Gamma(ker\overline{\pi}_*).$$

Next, one has

Theorem 5.4. *Assume* $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ *represents l.c.q.K. manifold and* $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ *means any Riemannian manifold. Then, for Lagrangian Riemannian submersion* $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$*,* $\widetilde{\mathcal{M}}$ *identifies a locally twisted product manifold* $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^\perp} \times_f \widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)}$ *iff*

$$\mathcal{T}_{\overline{\mathbb{V}}} J_{\overline{\alpha}}\mathcal{Z} = -g(\mathcal{Z}, \mathcal{T}_{\overline{\mathbb{V}}}\overline{\mathbb{V}})\|\overline{\mathbb{V}}\|^{-2} J_{\overline{\alpha}}\overline{\mathbb{V}}, \quad \forall \overline{\mathbb{U}}, \overline{\mathbb{V}} \in \Gamma(ker\overline{\pi}_*)$$

and

$$g_{\widetilde{\mathcal{B}}}(\mathcal{A}_{\mathcal{Z}} J_{\overline{\alpha}} \mathcal{W}, J_{\overline{\alpha}} \overline{\mathbb{V}}) = \frac{1}{2} g(J_{\overline{\alpha}} \mathcal{W}, \mathring{B}) g(\mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{V}}) + \frac{1}{2} g(\mathcal{Z}, \mathcal{W}) g(\mathring{B}, \overline{\mathbb{V}}),$$

$\forall \mathcal{Z}, \mathcal{W} \in \Gamma((ker\overline{\pi}_*)^{\perp})$. *Wherein* $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^{\perp}} \times_f \widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)}$ *is used to denote integral manifold of* $(ker\overline{\pi}_*)^{\perp}$ *and* $(ker\overline{\pi}_*)$.

Proof. Assume $\forall \mathcal{Z} \in \Gamma((ker\overline{\pi}_*)^{\perp})$,$\overline{\mathbb{V}}, \overline{\mathbb{U}} \in \Gamma(ker\overline{\pi}_*)$ and consider that $(ker\overline{\pi}_*)^{\perp}$ and $(ker\overline{\pi}_*)$ are orthogonal. Then, (1) and Lemma 2.4 propose

$$\begin{aligned} g(\nabla_{\overline{\mathbb{V}}} \overline{\mathbb{U}}, \mathcal{Z}) &= -g(\nabla_{\overline{\mathbb{V}}} \mathcal{Z}, \overline{\mathbb{U}}) \\ &= -g(J_{\overline{\alpha}} \nabla_{\overline{\mathbb{V}}} \mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{U}}) \\ &= -g(\nabla_{\overline{\mathbb{V}}} J_{\overline{\alpha}} \mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{U}}) \\ &= -g(\mathcal{T}_{\overline{\mathbb{V}}} J_{\overline{\alpha}} \mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{U}}). \end{aligned}$$

Hence, totally umbilical condition hold on $(ker\overline{\pi}_*)$ iff

$$\mathcal{T}_{\overline{\mathbb{V}}} J_{\overline{\alpha}} \mathcal{Z} = -\mathcal{Z}(\lambda) J_{\overline{\alpha}} \overline{\mathbb{V}}, \tag{14}$$

λ being a function on $\widetilde{\mathcal{M}}$. This way, (14) gives

$$\begin{aligned} g(-\mathcal{Z}(\lambda) J_{\overline{\alpha}} \overline{\mathbb{V}}, J_{\overline{\alpha}} \overline{\mathbb{V}}) &= g(\mathcal{T}_{\overline{\mathbb{V}}} J_{\overline{\alpha}} \mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{V}}) \\ -\mathcal{Z}(\lambda) \|\overline{\mathbb{V}}\|^2 &= g(\mathcal{T}_{\overline{\mathbb{V}}} J_{\overline{\alpha}} \mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{V}}) \\ &= g(\nabla_{\overline{\mathbb{V}}} J_{\overline{\alpha}} \mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{V}}) \\ &= g(J_{\overline{\alpha}} \nabla_{\overline{\mathbb{V}}} \mathcal{Z}, J_{\overline{\alpha}} \overline{\mathbb{V}}) \\ &= -g(\mathcal{Z}, \mathcal{T}_{\overline{\mathbb{V}}} \overline{\mathbb{V}}) \\ \mathcal{Z}(\lambda) &= g(\mathcal{Z}, \mathcal{T}_{\overline{\mathbb{V}}} \overline{\mathbb{V}}) \|\overline{\mathbb{V}}\|^{-2} \qquad (15) \end{aligned}$$

wherein (1) has been taken into use. Therefore, (14) and (15) produce

$$\mathcal{T}_{\overline{\mathbb{V}}} J_{\overline{\alpha}} \mathcal{Z} = -g(\mathcal{Z}, \mathcal{T}_{\overline{\mathbb{V}}} \overline{\mathbb{V}}) \|\overline{\mathbb{V}}\|^{-2} J_{\overline{\alpha}} \overline{\mathbb{V}}$$

and so result follows with the help of Corollary 4.2. □

Next, for Lagrangian Riemannian submersion, we study:

Theorem 5.5. *Assume* $(\widetilde{\mathcal{M}}, g, J_{\overline{\alpha}})$ *represents l.c.q.K. manifold and* $(\widetilde{\mathcal{B}}, g_{\widetilde{\mathcal{B}}})$ *means any Riemannian manifold. Then, Lagrangian Riemannian submersion* $\overline{\pi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{B}}$ *does not exist such that* $\widetilde{\mathcal{M}}$ *be representing a locally proper twisted product manifold* $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^{\perp}} \times_f \widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)}$.

Proof. Set $\forall \mathcal{Z} \in \Gamma((ker\overline{\pi}_*)^{\perp}),\overline{\mathbb{V}},\overline{\mathbb{U}} \in \Gamma(ker\overline{\pi}_*)$. Further, identify by $\widetilde{\mathcal{M}}$ a locally twisted product $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^{\perp}} \times_f \widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)}$. So, with def. 5.1, one finds $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^{\perp}}$ and $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)}$ to be totally umbilical and totally geodesic foliations, respectively. Also, one can see $g(\nabla_{\mathcal{Z}}\mathcal{W},\overline{\mathbb{V}}) = g(h(\mathcal{Z},\mathcal{W}),\overline{\mathbb{V}})$, h represents the second fundamental form of $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^{\perp}}$. Taking view of totally umbilical foliation $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^{\perp}}$, one writes

$$g(\nabla_{\mathcal{Z}}\mathcal{W},\overline{\mathbb{V}}) = g(H,\overline{\mathbb{V}})g(\mathcal{Z},\mathcal{W}), \tag{16}$$

in present case, H has been used for the mean curvature vector field of $\widetilde{\mathcal{M}}_{(ker\overline{\pi}_*)^{\perp}}$.

Also, taking help from (1) and Lemma 2.3, one obtains

$$\begin{aligned} g(\nabla_{\mathcal{Z}}\mathcal{W},\overline{\mathbb{V}}) &= -g(\mathcal{W},\nabla_{\mathcal{Z}}\overline{\mathbb{V}}) \\ &= -g(J_{\overline{\alpha}}\mathcal{W},J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\overline{\mathbb{V}}) \\ &= -g(J_{\overline{\alpha}}\mathcal{W},\mathcal{A}_{\mathcal{Z}}J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2}\omega'(\overline{\mathbb{V}})J_{\overline{\alpha}}\mathcal{Z}). \end{aligned} \tag{17}$$

Therefore, (16) and (17) produce

$$\begin{aligned} g(H,\overline{\mathbb{V}})g(\mathcal{Z},\mathcal{W}) &= -g(J_{\overline{\alpha}}\mathcal{W},\mathcal{A}_{\mathcal{Z}}J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2}\omega'(\overline{\mathbb{V}})J_{\overline{\alpha}}\mathcal{Z}) \\ g(H,\overline{\mathbb{V}})g(J_{\overline{\alpha}}\mathcal{W},J_{\overline{\alpha}}\mathcal{Z}) &= -g(J_{\overline{\alpha}}\mathcal{W},\mathcal{A}_{\mathcal{Z}}J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2}\omega'(\overline{\mathbb{V}})J_{\overline{\alpha}}\mathcal{Z}) \\ -g(H,\overline{\mathbb{V}})\|\mathcal{Z}\|^2 &= g(\mathcal{A}_{\mathcal{Z}}J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2}\omega'(\overline{\mathbb{V}})J_{\overline{\alpha}}\mathcal{Z},J_{\overline{\alpha}}\mathcal{Z}) \\ &= g(\nabla_{\mathcal{Z}}J_{\overline{\alpha}}\overline{\mathbb{V}} + \frac{1}{2}\omega'(\overline{\mathbb{V}})J_{\overline{\alpha}}\mathcal{Z},J_{\overline{\alpha}}\mathcal{Z}) \\ &= g(J_{\overline{\alpha}}\nabla_{\mathcal{Z}}\overline{\mathbb{V}},J_{\overline{\alpha}}\mathcal{Z}) \\ &= -g(\overline{\mathbb{V}},\nabla_{\mathcal{Z}}\mathcal{Z}). \end{aligned}$$

Hence, one concludes

$$g(H,\overline{\mathbb{V}})\|\mathcal{Z}\|^2 = g(\overline{\mathbb{V}},\mathcal{A}_{\mathcal{Z}}\mathcal{Z}).$$

In addition, (5) implies $\mathcal{A}_{\mathcal{Z}}\mathcal{Z} = 0$, resulting $g(H,\overline{\mathbb{V}})\|\mathcal{Z}\|^2 = 0$. As $H \in \Gamma(ker\overline{\pi}_*)$, using Riemannian metric g, one obtains $H = 0$ that establishes $(ker\overline{\pi}_*)^{\perp}$ to be totally geodesic and $\widetilde{\mathcal{M}}$ becomes usual product of Riemannian manifolds. □

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