

THE NUMBER OF ZEROS OF ABELIAN INTEGRAL FOR A QUINTIC HAMILTONIAN SYSTEMS WITH A ROSE-FIGURE LOOP

AIYONG CHEN^{1,2} AND HUIYANG ZHANG²

¹DEPARTMENT OF MATHEMATICS,

HUNAN FIRST NORMAL UNIVERSITY, CHANGSHA, 410205, P. R. CHINA

²SCHOOL OF MATHEMATICS AND COMPUTING SCIENCE,

GUILIN UNIVERSITY OF ELECTRONIC TECHNOLOGY, GUILIN, 541004, P. R. CHINA

E-MAIL: AIYONGCHEN@163.COM

(Received: 6 August 2019, Accepted: 7 January 2020)

ABSTRACT. In this article, we consider the near-Hamiltonian system

$$\begin{cases} \dot{x} = -y(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2) + \varepsilon f(x, y), \\ \dot{y} = x(ax^4 + ay^4 + 2ax^2y^2 + x^2 - 3y^2) + \varepsilon g(x, y), \end{cases}$$

where $a < 0$, $0 < |\varepsilon| \ll 1$, $f(x, y)$ and $g(x, y)$ are polynomials of degree n ($n = 2m$, $m \geq 3$, $m \in \mathbb{N}$). The number of isolated zeros of the corresponding Abelian integral for $h \in (-\frac{1}{12a^2}, 0)$ is estimated.

AMS Classification: 34C08, 34C09, 37G15.

Keywords: Hamiltonian system; Limit cycles; Abelian integrals; Picard-Fuchs equation.

1. INTRODUCTION AND THE MAIN RESULTS

The latter part of Hilbert's 16th problem can be expressed as: what is the maximum number of limit cycles that a planar n -degree polynomial system may have, and what are the relative positions of these limit cycles? The limit cycle is that

the orbits on each side of it approach infinitely close to this isolated closed orbit when $t \rightarrow \infty$. Extensive articles on this area has been carried out. For Hamiltonian system with small disturbance

$$(1) \quad \dot{x} = H_y + \varepsilon p(x, y), \dot{y} = -H_x + \varepsilon q(x, y),$$

where $0 < |\varepsilon| \ll 1$, p and q are polynomials of degree n . Arnold [1] raises the weakened Hilbert's 16th problem. The Abelian integral can be written as

$$(2) \quad I(h) = \oint_{\Gamma_h} q(x, y)dx - p(x, y)dy, \quad h \in \Sigma,$$

where Γ_h is the compact component of the Hamiltonian $H(x, y) = h$, $h \in \Sigma$. Σ is the range of the closed orbit Γ_h . The number of zeros of the Abelian integral of system (1) can be obtained by the Poincaré-Pontryagin Theorem, and the upper bound of the number of limit cycles of the undisturbed system is given.

Khovansky [4] and Varchenko [7], who proved the existence of $Z(m, n)$, give the common result of dealing with the weakened Hilbert 16th problem, but they can't give the exact expression of $Z(m, n)$. There are now a lot of researches on the special Hamiltonian which don't have cross terms of x and y . Zhao and Zhang [14] prove that when $H(x, y) = \frac{1}{2}y^2 + u(x)$, the degree $u(x) = 4$ and the number of Abelian integrals less than $7n + 5$. However, when $H(x, y)$ has the form $x^i y^j$, of those i and j are natural numbers, there is little related research. For

$$H(x, y) = x^2 + y^2 + ax^2y^2 - x^4 + y^4, \quad a > -2,$$

Wu, Zhang and Li [13] prove $Z(3, n) \leq 2[\frac{n-1}{4}] + 6[\frac{n-2}{4}] + 24$. Yang and Zhao [10] prove $Z(3, n) \leq 48n + 12$. For more conclusion about $Z(H, n)$, we can see [2, 3, 5, 6, 8, 9, 11, 12] and references therein.

Below we mainly analyze the following Hamiltonian function

$$(3) \quad H(x, y) = -\frac{a}{6}x^6 - \frac{a}{6}y^6 - \frac{a}{2}x^4y^2 - \frac{a}{2}x^2y^4 - \frac{1}{4}x^4 - \frac{1}{4}y^4 + \frac{3}{2}x^2y^2,$$

where $a < 0$. The corresponding system is

$$(4) \quad \begin{cases} \dot{x} = -y(ax^4 + ay^4 + 2ax^2y^2 + y^2 - 3x^2), \\ \dot{y} = x(ax^4 + ay^4 + 2ax^2y^2 + x^2 - 3y^2). \end{cases}$$

The perturbed Hamiltonian system corresponding to (4) is

$$(5) \quad \begin{cases} \dot{x} = -y(ax^4 + ay^4 + 2ax^2y^2 + y^2 - 3x^2) + \varepsilon f(x, y), \\ \dot{y} = x(ax^4 + ay^4 + 2ax^2y^2 + x^2 - 3y^2) + \varepsilon g(x, y), \end{cases}$$

where $a < 0$, $0 < |\varepsilon| \ll 1$, f and g are polynomials of degree n ($n = 2m$, $m \geq 3$, $m \in \mathbb{N}$). Our next goal is to find an upper bound for the number of zeroes of the Abelian integral related to system (5). When $a < 0$ the system (4) has five singularities. $O(0, 0)$ is a higher-order singularity. There are four centers

$$C_{1,2} = (\pm(-\frac{1}{a})^{1/2}, 0), \quad C_{3,4} = (0, \pm(-\frac{1}{a})^{1/2}).$$

Curves $H(x, y) = 0$ and $H(x, y) = -\frac{1}{12a^2}$ correspond to singularities $O(0, 0)$ and C_i ($i=1, 2, 3, 4$), respectively. When $h \in (-\frac{1}{12a^2}, 0)$, they correspond to the closed orbits around the singularity the center C_i ($i = 1, 2, 3, 4$).

The main conclusions of this article are:

Theorem 1.1. *The number of isolated zeros of $I(h)$ of system (5) does not exceed $8[\frac{n-4}{6}] + 4[\frac{n-2}{6}] + 8$ (considering the multiplicity) for $h \in (-\frac{1}{12a^2}, 0)$. The degree of $f(x, y)$ and $g(x, y)$ in system (5) is n ($n = 2m$, $m \geq 3$, $m \in \mathbb{N}$).*

2. THE EXPRESSION OF $I(h)$

Denote $I_{i,j}(h) = \oint_{\Gamma_h} x^i y^j dx$, where i and j are natural numbers.

Lemma 2.1. *If $h \in \Sigma$, $a < 0$ and $i + j = n$ ($n = 2m$, $m \geq 3$, $m \in \mathbb{N}$), then*

$$(6) \quad I_{2i+1,2j+1}(h) = \tilde{\alpha}(h)I_{11}(h) + \tilde{\beta}(h)I_{13}(h), \quad \text{if } i + j \geq 1(i + j = \frac{n}{2} - 1),$$

$$(7) \quad I_{2i,2j}(h) = \tilde{\gamma}(h)I_{02}(h) + \tilde{\delta}(h)I_{22}(h), \quad \text{if } i + j \geq 2(i + j = \frac{n}{2}),$$

where $\tilde{\alpha}(h)$, $\tilde{\beta}(h)$, $\tilde{\gamma}(h)$ and $\tilde{\delta}(h)$ are real polynomials in h with

$$\deg \tilde{\alpha}(h) \leq [\frac{n-2}{6}], \quad \deg \tilde{\beta}(h) \leq [\frac{n-4}{6}],$$

$$\deg \tilde{\gamma}(h) \leq [\frac{n-2}{6}], \quad \deg \tilde{\delta}(h) \leq [\frac{n-4}{6}].$$

Proof. Here we only prove (6). We can identify (7) as well. Differentiating (3) with about x , we have

$$(8) \quad -ax^5 - ay^5 \frac{\partial y}{\partial x} - 2ax^3y^2 - ax^4y \frac{\partial y}{\partial x} - axy^4 - 2ax^2y^3 - y^3 \frac{\partial y}{\partial x} - x^3 + 3xy^2 + 3x^2y \frac{\partial y}{\partial x} = 0.$$

We can multiply (8) by $x^{i-5}y^j dx$ and integrate it by parts, we get

$$(9) \quad \begin{aligned} aI_{i,j} &= \frac{a(i-5)}{j+6}I_{i-6,j+6} + \frac{i-5}{j+4}I_{i-6,j+4} + \frac{a(2i-j-10)}{j+4}I_{i-4,j+4} \\ &+ \frac{3(j-i+5)}{j+2}I_{i-4,j+2} + \frac{a(i-2j-5)}{j+2}I_{i-2,j+2} - I_{i-2,j}. \end{aligned}$$

Then, multiply (3) by $x^i y^{j-6} dx$ and integrate on the closed orbit $H(x, y) = h$, we acquire

$$(10) \quad \begin{aligned} \frac{a}{6}I_{i,j} &= -\frac{a}{6}I_{i+6,j-6} - \frac{a}{2}I_{i+4,j-4} - \frac{1}{4}I_{i+4,j-6} + \frac{3}{2}I_{i+2,j-4} \\ &- hI_{i,j-6} - \frac{1}{4}I_{i,j-2} - \frac{a}{2}I_{i+2,j-2}. \end{aligned}$$

From (9) and (10) we obtain

$$(11) \quad \begin{aligned} I_{i,j} &= \frac{j+6}{a(i+j+1)} \left[\left(\frac{i-5}{j+4} - \frac{3(i-5)}{2(j+6)} \right) I_{i-6,j+4} - \frac{6(i-5)}{j+6} h I_{i-6,j} \right. \\ &+ \left(\frac{a(2i-j-10)}{j+4} - \frac{3a(i-5)}{j+6} \right) I_{i-4,j+4} + \left(\frac{3(j-i+5)}{j-2} + \frac{9(i-5)}{j+6} \right) I_{i-4,j+2} \\ &+ \left(\frac{a(i-2j-5)}{j+2} - \frac{3a(i-5)}{j+6} \right) I_{i-2,j+2} \\ &\left. - \left(1 + \frac{3(i-5)}{2(j+6)} \right) I_{i-2,j} \right], \end{aligned}$$

$$(12) \quad \begin{aligned} I_{i,j} &= \frac{j}{a(i+j+1)} \left[-\frac{1}{2}I_{i+4,j-6} - \frac{a(i+j+1)}{j-4}I_{i+4,j-4} + \frac{3(i+2j-5)}{j-4}I_{i+2,j-4} \right. \\ &\left. - \frac{2a(i+j+1)}{j-2}I_{i+2,j-2} - \frac{2i+3j-4}{2(j-2)}I_{i,j-2} - 6hI_{i,j-6} \right]. \end{aligned}$$

Here we can get the result by induction on n . From (11) and (12) we can get ($n = 6, 8, 10$)

$$(13) \quad \left\{ \begin{array}{l} \frac{7a}{5}I_{1,5} = -\frac{13}{6}I_{1,3} - 7aI_{5,1} - \frac{14a}{3}I_{3,3} + 18I_{3,1}, \\ \frac{7a}{9}I_{3,3} = -\frac{14a}{15}I_{1,5} - \frac{2}{3}I_{1,3}, \\ aI_{5,1} = -\frac{a}{5}I_{1,5} + I_{1,3} - \frac{2a}{3}I_{3,3} - I_{3,1}, \\ \frac{9a}{7}I_{1,7} = -\frac{19}{10}I_{1,5} - \frac{1}{2}I_{5,1} - 3aI_{5,3} - \frac{18a}{5}I_{3,5} + 10I_{3,3} - 6hI_{1,1}, \\ \frac{9a}{5}I_{3,5} = -\frac{17}{6}I_{3,3} - 9aI_{7,1} - 6aI_{5,3} + 24I_{5,1}, \\ aI_{5,3} = -\frac{3a}{7}I_{1,7} + \frac{9}{5}I_{1,5} - \frac{6a}{5}I_{3,5} - I_{3,3}, \\ \frac{9a}{7}I_{7,1} = -\frac{1}{35}I_{1,5} - \frac{9a}{35}I_{3,5} + \frac{11}{7}I_{3,3} - \frac{6a}{7}I_{5,3} - \frac{10}{7}I_{5,1} - \frac{12}{7}hI_{1,1}, \\ \frac{11a}{9}I_{1,9} = -\frac{25}{14}I_{1,7} - \frac{1}{2}I_{5,3} - \frac{11a}{5}I_{5,5} - \frac{22a}{7}I_{3,7} + \frac{42}{5}I_{3,5} - 6hI_{1,3}, \\ \frac{11a}{7}I_{3,7} = -\frac{23}{10}I_{3,5} - \frac{1}{2}I_{7,1} - \frac{11a}{3}I_{7,3} - \frac{22a}{5}I_{5,5} + 12I_{5,3} - 6hI_{3,1}, \\ aI_{5,5} = -\frac{5a}{9}I_{1,9} + \frac{15}{7}I_{1,7} - \frac{10a}{7}I_{3,7} - I_{3,5}, \\ \frac{11a}{9}I_{7,3} = -\frac{1}{12}I_{1,7} - \frac{11a}{21}I_{3,7} + \frac{13}{5}I_{3,5} - \frac{22a}{15}I_{5,5} + \frac{4}{3}I_{5,3} - \frac{4h}{3}I_{1,3}, \\ \frac{11a}{7}I_{9,1} = -\frac{2}{35}I_{3,5} - \frac{11a}{35}I_{5,5} + \frac{15}{7}I_{5,3} - \frac{22a}{21}I_{7,3} - \frac{13}{7}I_{7,1} - \frac{24h}{7}I_{3,1}. \end{array} \right.$$

It is easy to see that when $n = 6, 8$, the result is true. If $i + j < 2l - 2$ ($l \geq 4$), the conclusion holds. When $i + j = 2l$ ($l \geq 3$), taking $(i, j) = (1, 2l - 1), (3, 2l - 3), (5, 2l - 5), \dots, (7, 2l - 7)$ in (12) and $(i, j) = (2l - 5, 5), (2l - 3, 3), (2l - 1, 1)$ in

(11), respectively, we obtain

$$A \begin{pmatrix} I_{1,2l-1} \\ I_{3,2l-3} \\ \vdots \\ I_{2l-9,9} \\ I_{2l-7,7} \\ I_{2l-5,5} \\ I_{2l-3,3} \\ I_{2l-1,1} \end{pmatrix} = \frac{1}{a(2k+1)} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{l-4} \\ a_{l-3} \\ a_{l-2} \\ a_{l-1} \\ a_l \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & \frac{4l-2}{2l-3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{4l-6}{2l-5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{18}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \frac{14}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{10}{7} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{6}{5} & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{2}{3} & 1 \end{pmatrix},$$

$$a_1 = (2l-1) \left(\frac{5-6l}{4l-6} I_{1,2l-3} - \frac{1}{2} I_{5,2l-7} - \frac{a(2l+1)}{2l-5} I_{1,2l-3} + \frac{12l-18}{2l-5} I_{3,2l-5} \right. \\ \left. - 6h I_{1,2l-7} \right),$$

$$a_2 = (2l-3) \left(\frac{7-6k}{4k-10} I_{3,2k-5} - \frac{1}{2} I_{7,2k-9} - \frac{a(2l+1)}{2l-7} I_{7,2l-7} + \frac{12l-24}{2l-7} I_{5,2l-7} \right. \\ \left. - 6h I_{3,2l-9} \right),$$

$$a_{l-4} = 9 \left(\frac{-5-4l}{14} I_{2l-9,7} - \frac{1}{2} I_{2l-5,3} - \frac{a(2l+1)}{5} I_{2l-5,5} + \frac{12+6l}{5} I_{2l-7,5} \right. \\ \left. - 6h I_{2l-9,3} \right),$$

$$a_{k-3} = 7 \left(\frac{-3-4l}{10} I_{2l-7,5} - \frac{1}{2} I_{2l-3,1} - \frac{a(2l+1)}{3} I_{2l-3,3} + (2l+2) I_{2l-5,3} \right. \\ \left. - 6h I_{2l-7,1} \right),$$

$$a_{l-2} = 11 \left(\frac{25-5l}{99} I_{2l-11,9} - \frac{10al+5a}{99} I_{2l-9,9} + \frac{60l-135}{77} I_{2l-9,7} + \frac{4-3l}{11} I_{2l-7,5} \right. \\ \left. - \frac{12l-60}{11} h I_{2l-11,5} \right),$$

$$\begin{aligned}
 a_{l-1} &= 9\left(\frac{4-l}{21}I_{2l-9,7} - \frac{2al+a}{21}I_{2l-7,7} + \frac{4l-7}{5}I_{2l-7,5} - \frac{12l-48}{9}hI_{2l-9,3}\right. \\
 &\quad \left.+ \frac{1-l}{3}I_{2l-5,3}\right), \\
 a_l &= 7\left(\frac{3-l}{35}I_{2l-7,5} - \frac{2al+a}{35}I_{2l-5,5} + \frac{4l-5}{7}I_{2l-5,3} - \frac{12l-36}{7}hI_{2l-7,1}\right. \\
 &\quad \left.+ \frac{2-3l}{7}I_{2l-3,1}\right).
 \end{aligned}$$

A is a $l \times l$ matrix and $\det A = -3 \neq 0$. Thus

$$\begin{aligned}
 I_{i,j}(h) &= \tilde{\alpha}_{2l-4}(h)I_{11}(h) + \tilde{\beta}_{2l-4}(h)I_{13}(h) \\
 &\quad + h\tilde{\alpha}_{2l-6}(h)I_{11}(h) + h\tilde{\beta}_{2l-6}(h)I_{13}(h) \\
 &:= \tilde{\alpha}_{2l}(h)I_{11}(h) + \tilde{\beta}_{2l}(h)I_{13}(h),
 \end{aligned}$$

where $\tilde{\alpha}_{2l-s}(h), \tilde{\beta}_{2l-s}(h) (s = 4, 6)$ represent the polynomials in h satisfying that

$$\deg \tilde{\alpha}_{2l-s}(h) \leq \left\lfloor \frac{2l-s-2}{6} \right\rfloor, \quad \deg \tilde{\beta}_{2l-s}(h) \leq \left\lfloor \frac{2l-s-4}{6} \right\rfloor.$$

Then the following equation satisfies

$$\begin{aligned}
 \deg \tilde{\alpha}_{2l}(h) &= \max\left(\left\lfloor \frac{2l-2}{6} \right\rfloor, \left\lfloor \frac{2l-6}{6} \right\rfloor\right), \\
 \deg \tilde{\beta}_{2l}(h) &= \max\left(\left\lfloor \frac{2l-4}{6} \right\rfloor, \left\lfloor \frac{2l-8}{6} \right\rfloor\right).
 \end{aligned}$$

We can get

$$\deg \tilde{\alpha}_{2l}(h) \leq \left\lfloor \frac{l-1}{3} \right\rfloor, \quad \deg \tilde{\beta}_{2l}(h) \leq \left\lfloor \frac{l-2}{3} \right\rfloor.$$

By the same method we can obtain

$$\deg \tilde{\gamma}_{2l}(h) \leq \left\lfloor \frac{l-1}{3} \right\rfloor, \quad \deg \tilde{\delta}_{2l}(h) \leq \left\lfloor \frac{l-2}{3} \right\rfloor.$$

By Lemma 2.1, we can get Lemma 2.2.

Lemma 2.2. *For $a < 0$, the expression of Abelian integral $I(h)$ is*

$$(14) \quad I(h) = \alpha(h)I_{11}(h) + \beta(h)I_{13}(h) + \gamma(h)I_{02}(h) + \delta(h)I_{22}(h),$$

where the $\alpha(h)$, $\beta(h)$, $\gamma(h)$ and $\delta(h)$ are real polynomials in h , $I_{i,j}(h) = \oint_{\Gamma_h} x^i y^j dx$ and for $i + j = n$ ($n = 2m$, $m \geq 3$, $m \in \mathbb{N}$),

$$\begin{aligned}
 \deg \alpha(h) &\leq \left\lfloor \frac{n-2}{6} \right\rfloor, \quad \deg \beta(h) \leq \left\lfloor \frac{n-4}{6} \right\rfloor, \\
 \deg \gamma(h) &\leq \left\lfloor \frac{n-2}{6} \right\rfloor, \quad \deg \delta(h) \leq \left\lfloor \frac{n-4}{6} \right\rfloor.
 \end{aligned}$$

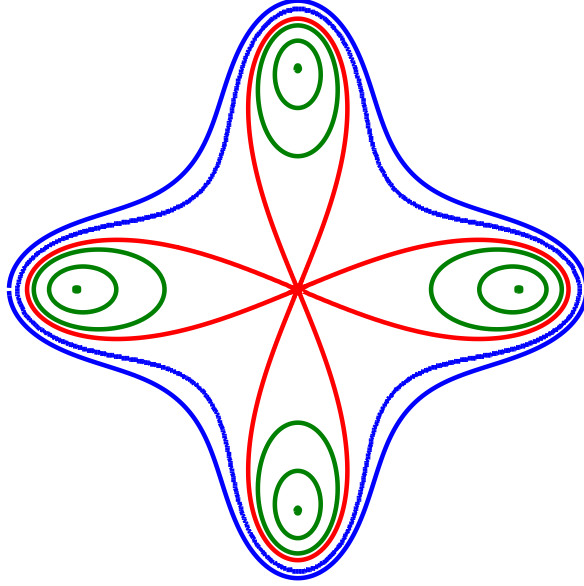


Fig. 1. Phase portraits of system (4) for $a < 0$.

3. PROOF OF THEOREM 1.1

Lemma 3.1. *If $a < 0$, the vector function $V = (I_{11}, I_{13})^T$ satisfy the following Picard-Fuchs equation*

$$(15) \quad (Bh + C)V' = V,$$

where

$$B = \begin{pmatrix} 11/18 & 0 \\ 557/(16775a) & 54/61 \end{pmatrix},$$

$$C = \begin{pmatrix} 6/(55a^2) & 51/(154a) \\ 3989/(251625a^3) & 372949/(4227300a^2) \end{pmatrix}.$$

Proof. From Hamiltonian function (3), we can get

$$\frac{\partial y}{\partial h} = \frac{1}{-y(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2)},$$

and

$$(16) \quad I'_{ij} = j \oint_{\Gamma_h} \frac{x^i y^{j-2}}{-(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2)} dx.$$

So,

$$I_{ij} = \oint_{\Gamma_h} \frac{-x^i y^{j-2}(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2)}{-(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2)} dx.$$

From (16), we can get

$$(17) \quad I_{ij} = -\frac{a}{j+2} I'_{i+4,j+2} - \frac{a}{j+6} I'_{i,j+6} - \frac{2a}{j+4} I'_{i+2,j+4} + \frac{3}{j+2} I'_{i+2,j+2} - \frac{1}{j+4} I'_{i,j+4}.$$

Multiply (16) by h , we obtain

$$(18) \quad \begin{aligned} hI'_{ij} &= j \oint_{\Gamma_h} \frac{x^i y^{j-2}(-\frac{a}{6}x^6 - \frac{a}{6}y^6 - \frac{a}{2}x^4y^2 - \frac{a}{2}x^2y^4 - \frac{1}{4}x^4 - \frac{1}{4}y^4 + \frac{3}{2}x^2y^2)}{-(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2)} dx \\ &= j(\frac{-a}{6j} I'_{i+6,j} - \frac{a}{6(j+6)} I'_{i,j+6} - \frac{a}{2(j+2)} I'_{i+4,j+2} - \frac{a}{2(j+4)} I'_{i+2,j+4} \\ &\quad - \frac{1}{4j} I'_{i+4,j} - \frac{1}{4(j+4)} I'_{i,j+4}) + \frac{3}{2(j+2)} I'_{i+2,j+2}. \end{aligned}$$

In addition,

$$(19) \quad \begin{aligned} I_{ij} &= \oint_{\Gamma_h} x^i y^j dx = -\frac{j}{i+1} \oint_{\Gamma_h} x^{i+1} y^{j-1} dy \\ &= -\frac{j}{i+1} \oint_{\Gamma_h} \frac{x^{i+2} y^{j+2}(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2)}{-(ay^4 + ax^4 + 2ax^2y^2 + y^2 - 3x^2)} dx \\ &= -\frac{j}{i+1} (\frac{a}{j} I'_{i+6,j} + \frac{a}{j+4} I'_{i+2,j+4} + \frac{2a}{j+2} I'_{i+4,j+2} - \frac{3}{j+2} I'_{i+2,j+2} \\ &\quad + \frac{1}{j} I'_{i+4,j}). \end{aligned}$$

By (17) – (19), we obtain

$$(20) \quad I_{ij} = \frac{1}{i+j+1} (6hI'_{ij} + \frac{1}{2} I'_{i+4,j} - \frac{3j}{j+2} I'_{i+2,j+2} + \frac{j}{2(j+4)} I'_{i,j+4}).$$

By (20), we can get

$$\begin{aligned} I_{11} &= \frac{1}{3} (6hI'_{11} + \frac{1}{2} I'_{5,1} - I'_{3,3} + \frac{1}{10} I'_{1,5}), \\ I_{13} &= \frac{1}{5} (6hI'_{13} + \frac{1}{2} I'_{5,3} - \frac{9}{5} I'_{3,5} + \frac{3}{14} I'_{1,7}). \end{aligned}$$

Noting that (11) and (12), we get the conclusion.

Lemma 3.2. *If $I_{11} \neq 0$, set $\omega(h) = \frac{I_{13}}{I_{11}}$, then $\omega(h)$ satisfies the following equation*

$$(21) \quad G(h)\omega'(h) = m_1(h)\omega(h)^2 + m_2(h)\omega(h) + m_3(h),$$

where

$$G(h) = 510300a^4h^2 + 81003a^2h + 1541, \quad m_1(h) = \frac{233325a^3}{2},$$

$$m_2(h) = 264600a^4h + \frac{88211a^2}{12}, \quad m_3(h) = -\frac{27923a}{5} - 11697a^3h.$$

Equation (21) is called the Riccati equation.

Proof. Differentiating both side of $\omega(h) = \frac{I_{13}}{I_{11}}$, we can get

$$(22) \quad \omega(h)' = \frac{I'_{13}}{I_{11}} - \frac{\omega(h)I'_{11}}{I_{11}},$$

From (15), we can get (21). This ends the proof. $\square$

Proposition 3.3.[15] *Let $\Omega = (u_1, u_2) \times (v_1, v_2)$. If $P(u, v)$ and $Q(u, v)$ continues on $\bar{\Omega}$ and smoothes in Ω . Suppose that $P'_u(u, v) \neq 0$ for $(u, v) \in \Omega$ and $Q(u, v) = 0$ is an orbit of the following equation*

$$\tilde{Q}_1(u, v)dh + \tilde{Q}_2(u, v)dy = 0.$$

Then we can get the number of solutions of system

$$\left\{ \begin{array}{l} P(u, v) = 0, \\ \left| \begin{array}{cc} \partial P / \partial u & \partial P / \partial v \\ \tilde{Q}_1(u, v) & \tilde{Q}_2(u, v) \end{array} \right| = 0. \end{array} \right.$$

equal to the number of system

$$\left\{ \begin{array}{l} P(u, v) = 0, \\ Q(u, v) = 0, \end{array} \right.$$

minus 1.

Proof of Theorem 1.1. We only consider the closed trajectory corresponds to $h \in (-\frac{1}{12a^2}, 0)$. By the phase portrait, we get $I_{02}(h) = 0$, $I_{22}(h) = 0$. For $h \in (-\frac{1}{12a^2}, 0)$, then

$$I(h) = \alpha(h)I_{11}(h) + \beta(h)I_{13}(h) = I_{11}(h)(\alpha(h) + \beta(h)\omega(h)) = I_{11}(h)\chi(h),$$

and $\#I(h) = \#(\chi(h))$. Denote $\#(r(h))$ is the number of zeros for $r(h)$ (considering the multiplicity). Let's analyze the quantity of zeros of $\chi(h)$. If the following equation is satisfied

$$\alpha(h) = \eta_k(h)\bar{\alpha}(h), \quad \beta(h) = \eta_k(h)\bar{\beta}(h),$$

then $\bar{\alpha}(h)$ and $\bar{\beta}(h)$ are polynomials of h with degree does not exceed $[\frac{n-2}{6}] - k$ and $[\frac{n-4}{6}] - k$ respectively. Denote $\#(p(h))$ is the number of zeros for $p(h)$ (considering the multiplicity) at $h \in (-\frac{1}{12a^2}, 0)$, where $p(h)$ represents the real polynomials in h . Then

$$\alpha(h) + \beta(h)\omega(h) = \eta_k(h)(\bar{\alpha}(h) + \bar{\beta}(h)\omega(h))$$

and for $h \in (-\frac{1}{12a^2}, 0)$, we can get

$$\#(\alpha(h) + \beta(h)\omega(h)) = k + \#(\bar{\alpha}(h) + \bar{\beta}(h)\omega(h)).$$

If $\bar{h}_l \in (-\frac{1}{12a^2}, 0)$ satisfies $\bar{\beta}(\bar{h}_l) = 0$, $0 \leq l \leq [\frac{n-4}{6}] - k$. Then the interval $(-\frac{1}{12a^2}, 0)$ is divided into $l + 1$ sub-intervals $J_1, J_2, \dots, J_{l+1}$. For each interval J_i ($i = 1, 2, \dots, l + 1$), we can obtain

$$\bar{\alpha}(h) + \bar{\beta}(h)\omega(h) = 0 \Leftrightarrow \omega(h) + \frac{\bar{\alpha}(h)}{\bar{\beta}(h)} = 0 \Leftrightarrow \begin{cases} y = \omega(h), \\ y = -\frac{\bar{\alpha}(h)}{\bar{\beta}(h)}. \end{cases}$$

Let $P(h, y) = y + \frac{\bar{\alpha}(h)}{\bar{\beta}(h)}$, $Q(h, y) = y - \omega(h)$. Above system can be written as

$$(23) \quad \begin{cases} P(h, y) = 0, \\ Q(h, y) = 0. \end{cases}$$

Differentiating $Q(h, y) = 0$ respect to h and combining (21), we get

$$\omega'(h) = y' = Q(h)^{-1}[m_1(h)\omega(h)^2 + m_2(h)\omega(h) + m_3(h)].$$

Therefore $Q(h, y) = 0$ is an orbit of the equation $\tilde{Q}_1(h, y)dh + \tilde{Q}_2(h, y)dy = 0$, where

$$\begin{cases} \tilde{Q}_1(h, y) = Q(h)^{-1}\varsigma(h), \\ \tilde{Q}_2(h, y) = -1, \end{cases}$$

and $\varsigma(h) = m_1(h)y^2 + m_2(h)y + m_3(h)$. Let's take J_1 for instance. Suppose that $J_1 = (\rho_1, \rho_2)$. For any $\forall \varepsilon > 0$, by using Proposition 3.3 on $\forall \varepsilon > 0$ to obtain

$$\left| \begin{array}{cc} \partial P / \partial h & \partial P / \partial y \\ \tilde{Q}_1(h, y) & \tilde{Q}_2(h, y) \end{array} \right| = -\frac{d}{dh} \left(\frac{\bar{\alpha}(h)}{\bar{\beta}(h)} \right) - Q(h)^{-1}\varsigma(h).$$

Since $P'_y(h, y) = 1 \neq 0$, the quantities of solutions of system (23) haven't exceed 1 plus the number of solutions of the system

$$(24) \quad \begin{cases} P(h, y) = 0, \\ -\frac{d}{dh}(\frac{\bar{\alpha}(h)}{\bar{\beta}(h)}) - Q(h)^{-1}\zeta(h) = 0. \end{cases}$$

Substituting $y = -\bar{\alpha}(h)/\bar{\beta}(h)$ into the function $-\frac{d}{dh}(\frac{\bar{\alpha}(h)}{\bar{\beta}(h)}) - Q(h)^{-1}\zeta(h) = 0$, we can reckon the quantities of zeros of the $\bar{f}(h)$, where

$$\frac{d}{dh}(\frac{\bar{\alpha}(h)}{\bar{\beta}(h)}) + Q(h)^{-1}[m_1(h)(-\frac{\bar{\alpha}(h)}{\bar{\beta}(h)})^2 + m_2(h)(-\frac{\bar{\alpha}(h)}{\bar{\beta}(h)}) + m_3(h)] := \bar{f}(h).$$

Then $\bar{f}(h) = (\bar{\beta}^2 Q(h))^{-1} f_1(h)$, where

$$f_1(h) = m_1(h)\bar{\alpha}^2 - m_2(h)\bar{\alpha}\bar{\beta} + m_3(h)\bar{\beta}^2 + Q(h)(\bar{\alpha}'\bar{\beta} - \bar{\alpha}\bar{\beta}').$$

Then we admit $\deg f_1(h) \leq [\frac{n-2}{6}] + [\frac{n-4}{6}] - 2k + 1$. It is easy to see that the degree of polynomials $f_1(h)$ about h is no more than $[\frac{n-2}{6}] + [\frac{n-4}{6}] - 2k + 1$.

Assume that $f_1(h)$ has ν_j zeros on J_j . Then $\omega(h) + \frac{\bar{\alpha}(h)}{\bar{\beta}(h)}$ has at most $\nu_1 + 1$ zeros on $(\rho_1 + \varepsilon, \rho_2 - \varepsilon)$, which means that $\bar{\alpha}(h) + \bar{\beta}(h)\omega(h)$ has at most $\nu_1 + 1$ zeros on (ρ_1, ρ_2) for any $\varepsilon > 0$. Hence for $h \in \cup J_i$, $\bar{\alpha}(h) + \bar{\beta}(h)\omega(h)$ has at most $\nu_1 + \nu_2 + \cdots + \nu_{l+1} + l + 1$ zeros. Noting that $\nu_1 + \nu_2 + \cdots + \nu_{m+1} \leq [\frac{n-2}{6}] + [\frac{n-4}{6}] - 2k + 1$ and $\bar{\alpha}(\bar{h}_l) \neq 0$, we can obtain $\bar{\alpha}(h) + \bar{\beta}(h)\omega(h) = 0$ has at most $[\frac{n-2}{6}] + 2[\frac{n-4}{6}] - 3k + 2$ zeros for $h \in (-\frac{1}{12a^2}, 0)$. According to the phase portraits, when $h \in (-\frac{1}{12a^2}, 0)$, each h corresponds to four periodic orbits. We acquire

$$\sharp(I(h), h \in \Sigma) \leq 8[\frac{n-4}{6}] + 4[\frac{n-2}{6}] + 8.$$

We can get that $I(h)$ of system (5) doesn't exceed $8[\frac{n-4}{6}] + 4[\frac{n-2}{6}] + 8$ if n is even ($n = 2m$, $m \geq 3$, $m \in \mathbb{N}$) (considering the multiplicity) for $h \in (-\frac{1}{12a^2}, 0)$.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 11671107 and No. 11971163).

REFERENCES

- [1] V.I. Arnold, Ten problems in: theory of singularities and its applications, Adv. Sov. Math., Vol. 1, 1-8 (1990).
- [2] F. Dumortier, R. Roussarie, Abelian integrals and limit cycles, J. Differential Equations, Vol. 227, Number 1, 116-165 (2018).

- [3] A. Gasull, J. Yu, X. Zhang, Vector fields with homogeneous nonlinearities and many limit cycles, *J. Differential Equations*, Vol. 258, Number 9, 3286-3303 (2015).
- [4] A.G. Khovansky, Real analytic manifolds with finiteness properties and complex Abelian integrals, *Funct. Anal. Appl.*, Vol. 18, Number 2, 119-128 (1984).
- [5] C. Liu, Estimate of the number of zeros of Abelian integrals for an elliptic Hamiltonian with figure-of-eight loop, *Nonlinearity*, Vol. 16, Number 3, 1151-1163 (2003).
- [6] N. Nyamoradi, H.R.Z. Zangeneh, Linear estimate of the number of zeros of Abelian integrals for a kind of quintic Hamiltonians, *Bull. Iranian Math. Soc.*, Vol. 37, Number 37, 101-116 (2011).
- [7] A. N. Varchenko, Estimate of the number of zeros of Abelian integrals depending on parameters and limit cycles, *Funct. Anal. Appl.*, Vol. 18, Number 2, 98-108 (1984).
- [8] J. Wu, Y. Zhang, C. Li, On the number of zeros of Abelian integrals for a kind of quartic Hamiltonians, *J. Differential Equations*, Vol. 228, Number 1, 329-335 (2014).
- [9] J. Yang, On the Algebraic Structure and the Number of Zeros of Abelian Integral for a Class of Hamiltonians with Degenerate Singularities, *Bulletin Brazilian Mathematical Society*, Vol. 49, Number 4, 1-21 (2018).
- [10] J. Yang, L. Zhao, The cyclicity of period annuli for a class of cubic Hamiltonian systems with nilpotent singular points. *J. Differential Equations*, Vol. 263, Number 9, 1-28 (2017).
- [11] J. Yang, L. Zhao, Bounding the number of limit cycles of discontinuous differential systems by using Picard-Fuchs equations, *J. Differential Equations*, Vol. 264, Number 9, 5734-5757 (2018).
- [12] J. Yang, L. Zhao, Zeros of Abelian integrals for a quartic Hamiltonian with figure-of-eight loop through a nilpotent saddle, *Nonlinear Analysis: Real World Applications*, Vol. 27, Number 9, 350-365 (2016).
- [13] X. Zhou, C. Li, On the algebraic structure of Abelian integrals for a kind of perturbed cubic Hamiltonian systems, *J. Math. Anal. Appl.* Vol. 359, Number 1, 209-215 (2009).
- [14] Y. Zhao, Z. Zhang, Linear estimate of the number of zeros of Abelian integrals for a kind of quartic Hamiltonians, *J. Differential Equations*, Vol. 155, Number 1, 73-88 (1999).
- [15] H. Zhu, C. Rousseau, Finite cyclicity of graphics with a nilpotent singularity of saddle or elliptic type, *J. Differential Equations*, Vol. 178, Number 2, 325-436 (2002).