

SUBMERSIONS OF CONTACT CR -SUBMANIFOLDS OF GENERALIZED QUASI SASAKIAN MANIFOLDS

MOHD DANISH SIDDIQI

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
JAZAN UNIVERSITY, JIZAN, KINGDOM OF SAUDI ARABIA

E-MAIL: MSIDDIQI@JAZANU.EDU.SA

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ABSTRACT. In this paper, we discuss some properties of almost contact metric submersion of contact CR -submanifolds of generalized quasi-Sasakian manifold and derive some results based on their curvatures's differential geometry. We also study de-Rham cohomology of CR -submanifold of generalized quasi-Sasakian manifold under the submersion.

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1. INTRODUCTION

An interesting topic in the differential geometry is the theory of submersion in spaces endowed with additional structures. In 1966, O'Neill [9] studied Riemannian submersion between Riemannian manifold. After that Chinea defined almost contact Riemannian submersion between almost contact metric manifolds [4]. In [16] Watson defined almost Hermitian submersion between almost Hermitian manifolds and in most cases he show that the base manifold and each fiber has the same kind of structure as the total space. He also show that the vertical and

horizontal distributions are invariant. In [6] S. Kobayashi studied submersion of CR -submanifold and obtained interesting results. In 1989, N. Paghiuc was studied the submersion of semi-invariant submanifolds of a Sasakian manifold [11]. Moreover, Semi-Riemannian submersion were introduced by O'Neil in [10]. It is well know that semi-Riemannian submersions are of interest in physics, owing to their application in the Yang-Milles theory, Kaluza-Klien theory, supergravity and super-string theory (see [3], [5], [12]). Submersion of semi-invariant submanifolds of trans-Sasakian manifold have been proposed by Shahid et al., [14]. In 2014, [8] Massamba and Matamba studied the submersions of contact CR -submanifolds with symplectic structures manifolds. Furthermore, in [15] Shahid et al., also have discussed some properties of submersions of CR -submanifolds of almost Hermitian manifolds which is closely related to this topic.

Motivated by the above studies, in this paper we study submersion of contact CR -submanifold of generalized quasi-Sasakian manifold which is a generalized form of almost contact metric manifolds. It consist, Sasakian manifolds, Kenmotsu manifolds and cosymplectic manifolds.

2. Preliminaries

Let M be an odd dimensional smooth manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) , where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatiabile Riemannian metric on M such that

$$(1) \quad \phi^2 U = -U + \eta(U)\xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \quad g(U, \xi) = \eta(V)$$

$$(2) \quad g(U, \phi V) = g(\phi U, V), \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for any vector filed U, V tangent to M , where I is the identity operator on tangent space of M . Throughout the study, all differentiable functions belongs to the class $C^\infty(M)$ and algebra of smooth function on M denoted by $\mathcal{F}(M)$ also by $\Gamma(\mathcal{E})$ the $\mathcal{F}(M)$ module of set of vector bundle $\mathcal{E}$ over M .

$\mathcal{N}_\phi$ denoted the Nijenhuis tensor with respect to the tensor field ϕ is given by

$$(3) \quad \mathcal{N}_\phi(U, V) = [\phi U, \phi V] + \phi^2[U, V] - \phi[\phi U, V] + \phi[U, \phi V]$$

for all $U, V \in \Gamma(TM)$ and the fundamental 2-form is given by $\Phi(U, V) = g(U, \phi V)$ for all $U, V \in \Gamma(TM)$.

Definition 2.1.(a) If $\mathcal{N}_\phi(U, V) + 2d\eta(U, V)\xi = 0$, then an almost contact metric manifold M (ϕ, ξ, η, g) is called normal [2] for all $U, V \in \Gamma(TM)$ or equivalently [1]

$$(\nabla_U \phi)V = \phi(\nabla_U \phi)V - g(\nabla_U \xi, V)$$

for all $U, V \in \Gamma(TM)$.

(b) If $d\Phi = d\eta = 0$, then a normal almost contact metric manifold is called cosymplectic [1].

An almost contact metric manifold M is said to be *generalized quasi-Sasakian* manifold if and only if Reeb vector field ξ is Killing, such that ([1], [15], [18])

$$(4) \quad (\bar{\nabla}_U \phi)V = g(\bar{\nabla}_{\phi U} \xi, V)\xi - \eta(U)\bar{\nabla}_{\phi U} \xi$$

for all $U, V \in \Gamma(TM)$

Moreover, F is a tensor field of type $(1, 1)$ defined as $FU = -\nabla_U \xi$, for all $U, V \in \Gamma(TM)$.

Definition 2.2.[17] A submanifold M is said to be contact CR -submanifold of generalized quasi-Sasakian manifold $\bar{M}$ if ξ is tangent to M and there exists a differential distribution $D : x \rightarrow D_x \subset T_x M$ such that

- (1) D_x is invariant under ϕ i.e. $\phi D_x \subset D_x$, for each $x \in M$,
- (2) the orthogonal complementary distribution $D^\perp : x \rightarrow D_x^\perp \subset T_x M$ of the distribution D on M is totally real i.e. $\phi D^\perp \subset T_x^\perp M$,
- (3) $TM = D \oplus D^\perp \oplus \{\xi\}$, where $T_x M$ and $T_x^\perp M$ are tangent space and the normal space of M at x , respectively.

We call D (resp. $D^\perp$) the horizontal (resp. vertical) distribution. We denote g the metric tensor field of M as well as the induced on M . Let $\bar{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\bar{M}$ (resp. M).

The Gauss and Weingarten formulas for M are respectively given by

$$(5) \quad \bar{\nabla}_U V = \nabla_U V + h(U, V),$$

$$(6) \quad \bar{\nabla}_U N = -A_N U + \nabla_U^\perp N$$

for $U, V \in TM$, $N \in T^\perp M$, where $h : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM^\perp)$ is normal bundle valued symmetric bilinear form on M , the linear operator A_N is fundamental form tensor of Wiengarten with respect to the normal section N , and the differential

operator $\nabla^\perp$ defines a linear connection on the normal bundle $TM^\perp$, called the normal connection on M . Moreover, we have

$$(7) \quad g(h(U, V), N) = g(A_N U, V).$$

The submanifold M is said to be totally geodesic if h vanishes identically.

The projection of TM to D and $D^\perp$ are denoted by h and v , respectively, i.e. for any $X \in \Gamma(TM)$, we have

$$(8) \quad U = hU + vU + \eta(U)\xi.$$

Applying ϕ to U , we have

$$(9) \quad \phi U = \mathcal{P}U + \mathcal{Q}U,$$

where $\mathcal{P}U = \phi hU$ and $\mathcal{Q}U = \phi vU$ are tangential and normal components of ϕU , respectively.

The normal bundle to M has the decomposition

$$(10) \quad TM^\perp = \phi D^\perp \oplus \mu,$$

where μ denotes the orthogonal complementary distribution of $\phi D^\perp$ and is an invariant normal submanifold of $TM^\perp$ under ϕ , for any $V \in TM^\perp$, we put

$$(11) \quad N = pN + qN$$

$pN \in \phi D^\perp$, $qN \in \mu$. From the above equation, we have

$$(12) \quad \phi N = fN + nN, \quad \forall N \in TM^\perp,$$

where $fN = \phi pN \in D^\perp$ and $nN = \phi qN \in \mu$.

3. Contact CR -Submersions from submanifold of a generalized quasi-Sasakian manifold

In this section, we deal with some properties of contact CR -submersions from submanifold of a generalized quasi-Sasakian manifold.

Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR -submersions between the contact CR -submanifold M of a generalized quasi-Sasakian manifold $\bar{M}(\phi, \xi, \eta, g)$ and an almost contact metric manifold M' with the almost contact

metric structure (ϕ', ξ', η', g') [10]. Now, we discuss the properties of following distributions [13].

- (1) If $\psi_* : TM \rightarrow TM'$ is the tangent mapping to ψ , then $D^\perp = \ker(\psi_*)$.
- (2) $\psi_* : D_x \oplus \{\xi\} \rightarrow T_{\psi(x)}M'$ is an isometry for each x that satisfies $\psi_* \circ \phi = \phi' \circ \psi_*$, $\eta = \eta' \circ \psi_*$, $\psi_*(\xi_x) = \xi'_{\psi(x)}$, where $T_{\psi(x)}M'$ denotes the tangent space of M' at $\psi(x)$.

A vector field U on M is said to be basis if $U \in \Gamma(D_x \oplus \{\xi\})$ and U is ψ -related to a vector field on M' , then there exists a vector field $U_* \in TM'$ such that $\psi_*(U_x) = U_*(\psi(x))$, for each $x \in M$. By the above definition 2.2, it shows that the structural vector field ξ is a basis vector field.

Lemma 3. [11] *Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR-submersions and ∇ and ∇^* are denoted by the Levi-Civita connection on M and M' , respectively. If U and V be the basic vector fields, ψ -related to U_* and V_* . Then we have*

- (i) $g(U, V) = g'(U_*, V_*) \circ \psi$
- (ii) $h([U, V])$ is a basic vector field ψ -related to $[U_*, V_*]$.
- (iii) $[W, U] \in D^\perp$, for any $W \in D^\perp$;
- (iv) $h(\nabla_U V)\xi$ is a basic vector field ψ -related to $\nabla_{U_*}^* V_*$.

We define operator $\tilde{\nabla}^*$ corresponding to ∇^* for any basic vector field in M . Adopting for any $U, V \in \Gamma(D \oplus \{\xi\})$,

$$(13) \quad \tilde{\nabla}_U^* V = h(\nabla_U V) + \eta(\nabla_U V)\xi.$$

Using (iv), $\tilde{\nabla}_U^* V$ is a basic vector field, and we turn up

$$(14) \quad \pi_*(\tilde{\nabla}_U^* V) = \tilde{\nabla}_{U_*}^* V_*.$$

Define the tensor field S by, for any $U, V \in \Gamma(D \oplus \{\xi\})$,

$$(15) \quad \nabla_U V = (\tilde{\nabla}_U^* V) + S(U, V),$$

where $S(X, Y)$ is the vertical part of $\nabla_U V$. It is know that S is skew-symmetric and satisfies

$$(16) \quad S(U, V) = \frac{1}{2}v[U, V], \quad U, V \in \Gamma(D \oplus \{\xi\}).$$

Now, we want to discuss the effect of a given structure define on ambient $\bar{M}$ on the determination of the corresponding structure on the contact CR-submanifold

M and the base space M' .

The curvature tensors R, R^* of the connection ∇, ∇^* on M and M' , respectively, are related by [13], for any $U, V, Z, W \in \Gamma(D \oplus \{\xi\})$

$$(17) \quad \begin{aligned} R(U, V, Z, W) &= R^*(U_*, V_*, Z_*, W_*) - g(S(V, Z), S(U, W)) \\ &\quad + g(S(U, Z), S(V, W)) + 2g(S(U, V), S(Z, W)), \end{aligned}$$

where $\phi_*U = U_*, \phi_*V = V_*, \phi_*Z = Z_*$ and $\phi_*W = W_*$. Now, we notice that the ambient manifold, namely generalized quasi-Sasakian manifold. First of all, we have, for any $U, V \in \Gamma(D \oplus \{\xi\})$,

$$(18) \quad \begin{aligned} \bar{\nabla}_U V &= \nabla_U V + h(U, V) = \nabla_U V + ph(U, V) + qh(U, V) \\ &= \tilde{\nabla}_U^* V + S(U, V) + ph(U, V) + qh(U, V). \end{aligned}$$

Using this, we have

$$(19) \quad \phi \bar{\nabla}_U = \phi \tilde{\nabla}_U^* V + \phi S(U, V) + \phi ph(U, V) + \phi qh(U, V).$$

Setting V with ϕV into the equation (27), we turn up

$$(20) \quad \bar{\nabla}_U \phi V = \tilde{\nabla}_U^* \phi V + S(U, \phi V) + ph(U, \phi V) + qh(U, \phi V).$$

On the other hand using the definition of generalized quasi-Sasakian manifold. If $\bar{M}$ is a generalized quasi-Sasakian manifold, we find

$$(21) \quad (\bar{\nabla}_U \phi) V = \bar{\nabla}_U \phi V - \phi(\bar{\nabla}_U V) = g(\bar{\nabla}_{\phi U} \xi, V) \xi - \eta(V)(\bar{\nabla}_{\phi U} \xi).$$

Substituting (28) and (29) in (30), we turn up

$$(22) \quad \begin{aligned} &\tilde{\nabla}_U^* \phi V + S(U, \phi V) + ph(U, \phi V) + qh(U, \phi V) \\ &- \phi \tilde{\nabla}_U^* V - \phi S(U, V) - \phi ph(U, V) - \phi qh(U, V) \\ &= g(\bar{\nabla}_{\phi U} \xi, V) \xi - \eta(V)(\bar{\nabla}_{\phi U} \xi) \end{aligned}$$

Comparing the following distributions $(D \oplus \{\xi\}), D^\perp, \phi D^\perp$ and μ on both side in the above equation, we turn up the following:

Theorem 3.5. *Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR-submersion from submanifold of generalized quasi-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M' . Then we have*

$$(23) \quad \tilde{\nabla}_U^* \phi V = \eta(V) \tilde{\nabla}_{\phi U}^* \xi + \eta(V) S(\phi U, \xi) + \eta(V) h(\phi U, \xi)$$

$$+g(\tilde{\nabla}_{\phi U}^* \xi, V)\xi + g(C(\phi U, \xi), V)\xi + g(h(\phi U, \xi), V)\xi,$$

$$(24) \quad S(U, \phi V) = \phi ph(U, V),$$

$$(25) \quad \phi S(U, V) = ph(U, \phi V),$$

$$(26) \quad \phi qh(U, V) = qh(U, \phi V)$$

for any $U, V \in (D \oplus \{\xi\})$.

Proof. If $\bar{M}$ is a generalized quasi-Sasakian manifold, we have

$$(27) \quad \begin{aligned} \bar{\nabla}_{\phi U} \xi &= \nabla_{\phi U} \xi + h(\phi U, \xi) \\ &= \tilde{\nabla}_{\phi U}^* \xi + S(\phi U, \xi) + S(\phi U, \xi) + h(\phi U, \xi) \end{aligned}$$

for any $U, V \in \Gamma(D \oplus \{\xi\})$. Putting (41) into (31), we turn up

$$(28) \quad \begin{aligned} &\tilde{\nabla}_U^* \phi V + S(U, \phi V) + ph(U, \phi V) + qh(U, \phi V) \\ &- \phi \tilde{\nabla}_U^* V - \phi S(U, V) - \phi ph(U, V) - \phi qh(U, V) \\ &= \eta(Y) \tilde{\nabla}_{\phi U}^* \xi + \eta(V) S(\phi U, \xi) + \eta(V) h(\phi U, \xi) \\ &+ g(\tilde{\nabla}_{\phi U}^* \xi, V)\xi + g(S(\phi U, \xi), V)\xi + g(h(\phi U, \xi), V)\xi. \end{aligned}$$

Comparing the following distributions $(D \oplus \{\xi\})$, $D^\perp$, $\phi D^\perp$ and μ respectively, on both sides of (37), we get (32), (33), (34) and (35). $\square$

Lemma 3.6. *Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR-submersion from submanifold of generalized quasi-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M' . Then we have $S(\xi, \xi) = 0$, $h(\xi, \xi) = 0$ and $S(U, \xi) = \phi ph(\phi U, \xi) \forall X \in (D \oplus \{\xi\})$.*

Proof. Putting $V = \xi$ in (34) and using the fact $\phi \xi = 0$, we get $\phi(\phi U, \xi) = \phi S(U, \xi)$. Applying ϕ to this equation and using (1), we have for any $U \in (D \oplus \{\xi\})$,

$$\phi ph(\phi U, \xi) = \phi^2 S(U, \xi) = S(U, \xi) - \eta(S(U, \xi))\xi = S(U, \xi),$$

since $\eta(S(U, \xi)) = 0$ due to the fact that S is vertical and ξ is a basic vector, and this proves the last relation. The first two relations are obvious. $\square$

Theorem 3.7. *Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR-submersion from submanifold of generalized quasi-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M' . Furthermore, if $\bar{M}$ is generalized quasi-Sasakian manifold, then M' is also a generalized quasi-Sasakian manifold.*

Proof. Using (3.19) of Theorem 3.5, we have for any $U, V \in (D \oplus \{\xi\})$,

$$\begin{aligned} (\tilde{\nabla}_U^* \phi)V &= \eta(V)\tilde{\nabla}_{\phi U}^* \xi + \eta(V)S(\phi U, \xi) + \eta(V)h(\phi U, \xi) \\ &+ g(\tilde{\nabla}_{\phi U}^* \xi, V)\xi + g(S(\phi U, \xi), V)\xi + g(h(\phi U, \xi), V)\xi. \end{aligned}$$

Applying ψ_* to the above equation and using Lemma 3.4, and equation (23), we turn up

$$\begin{aligned} \psi_*((\tilde{\nabla}_X^* \phi)V) &= \psi_*(\eta(V)\tilde{\nabla}_{\phi U}^* \xi) + \psi_*(\eta(V)S(\phi U, \xi)) + \psi_*(\eta(V)h(\phi U, \xi)) \\ &+ \psi_*(g(\tilde{\nabla}_{\phi U}^* \xi, V)\xi) + \psi_*(g(S(\phi U, \xi), V)\xi) + \psi_*(g(h(\phi U, \xi), V)\xi). \end{aligned}$$

That is,

$$\begin{aligned} (\nabla_{U_*}^* \phi')V_* &= \eta'(V_*)\nabla_{\phi' U_*}^* \xi' + \eta'(V_*)S'(\phi' U_*, \xi') + \eta'(V_*)h'(\phi' U_*, \xi') \\ &+ g'(\nabla_{\phi' U_*}^* \xi', V_*)\xi' + g'(S'(\phi' U_*, \xi'), V_*)\xi' + g'(h'(\phi' U_*, \xi'), V_*)\xi', \end{aligned}$$

which proves that M' is a generalized quasi-Sasakian manifold

$$(\tilde{\nabla}_{U_*}^* \phi')V = g'(\nabla_{\phi' U_*}^* \xi', V_*)\xi' - \eta'(V_*)(\nabla_{\phi' U_*}^* \xi').$$

From (34) $\phi(U, V) = 0$ and $qh(U, V) = 0$, and therefore $h(U, V) = 0 \forall U, V \in (D \oplus \{\xi\})$. This proves that M is $(D \oplus \{\xi\})$ -totally geodesic. $\square$

Theorem 3.8. *Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR-submersion between submanifold of generalized quasi-Sasakian manifold $\bar{M}$ and an almost contact metric manifold M' . Then we have*

- (i) $ph(U, \phi V) + ph(\phi U, V) = 0$,
- (ii) $ph(\phi U, \phi V) = ph(U, V)$,
- (iii) $qh(\phi U, \phi V) = -qh(U, V)$,
- (iv) $S(\phi U, \phi V) = S(U, V)$

for any $U, V \in (D \oplus \{\xi\})$.

Proof. (i) Interchanging U and V in (34) gives

$$\phi S(V, U) = ph(V, \phi U) = ph(\phi U, V).$$

Then

$$ph(U, \phi V) + ph(\phi U, V) = \phi S(U, V) + \phi S(U, V) = \phi S(U, V) - \phi S(U, V) = 0.$$

(ii) Putting $U = \phi U$ in (35), we turn up

$$ph(\phi U, \phi V) = \phi S(\phi U, V) = -\phi S(V, \phi U)$$

using (33) in the above equation, we deduce

$$\begin{aligned} ph(\phi U, \phi V) &= -\phi S(V, \phi U) = -\phi(\phi ph(V, U)) = -\phi^2 ph(V, U) \\ &= ph(V, U) - \eta(h(U, V))\xi = ph(V, U). \end{aligned}$$

(iii) Putting $U = \phi U$ in (35) and using again same equation, we find

$$qh(\phi U, \phi V) = \phi qh(\phi U, V) = \phi qh(V, \phi U) = \phi^2 qh(V, U) = -qh(U, V).$$

(iv) Putting $U = \phi U$ in (33) and using (34) yields

$$\begin{aligned} S(\phi U, \phi V) &= \phi ph(\phi U, V) = \phi phh(V, \phi U) = \phi^2 S(V, U) \\ &= -S(V, U) + \eta(S(V, U)\xi) = S(U, V). \end{aligned}$$

□

B.Y. Chen [3] proposed, that the anti-invariant distribution $D^\perp$ on a CR -submanifold of a Kahler manifold is always integrable. This is still valid for a CR -submanifold of a locally conformal kahler manifold [12]. Now, we have following result

Theorem 3.9. *Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR -submersion from the submanifold of generalized quasi-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M' . If the vertical distribution $D^\perp$ and the horizontal distribution $D \oplus \{\xi\}$ are parallel and integrable, respectively, then M is CR -product manifolds.*

Proof. Since the horizontal distribution $D \oplus \{\xi\}$ is integrable, then for any $U, V \in D \oplus \{\xi\}$, we have $[U, V] \in D \oplus \{\xi\}$. Therefore $v[U, V] = 0$. Now using (26), we have $S(X, Y) = 0, \forall U, V \in D \oplus \{\xi\}$. Using the value of $S(U, V)$ in (25), we have $\nabla_U V = \tilde{\nabla}_U^* V \in D \oplus \{\xi\}$, which shows that $D \oplus \{\xi\}$ is parallel. Since the

horizontal distribution $D \oplus \{\xi\}$ and vertical $D^\perp$ are both parallel. Thus, using Rham's Theorem, it follows that M is product $M_1 \times M_2$, where M_1 is invariant submanifold of $\bar{M}$ and M_2 is the totally real submanifold of $\bar{M}$. Hence M is CR -product. $\square$

4. Curvature characteristics of generalized quasi-Sasakian manifold

The outline of this section, is the study of ϕ -sectional curvature of generalized quasi-Sasakian manifold $\bar{M}$.

Theorem 4.1. *Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR -submersion from the submanifold of generalized quasi-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M' . Then the ϕ -bisectional curvature of $\bar{M}$ and M' are related by*

$$\begin{aligned} \frac{1}{2}\bar{B}_\phi(U, V) &= \frac{1}{2}B'_\phi(U_*, V_*) - \|ph(U, V)\|^2 - \|qh(U, \phi V)\|^2 \\ &\quad - g(ph(U, U), ph(V, V)) + \|qh(U, V)\|^2, \end{aligned}$$

where $U, V \in D \oplus \{\xi\}$.

Proof. We know that

$$\bar{B}_\phi(U, V) = \bar{R}(U, \phi U, \phi V, V).$$

Put $V = \phi U, Z = \phi V, W = V$ in Gauss equation

$$\bar{R}(U, V, Z, W) = R(U, V, Z, W) - g(h(U, W), h(V, Z)) + g(h(U, Z), h(V, W)),$$

we turn up

$$\begin{aligned} \bar{R}(U, \phi U, \phi V, V) &= R(X, \phi U, \phi V, V) - g(h(U, V), h(\phi U, \phi V)) \\ &\quad + g(h(U, \phi V), h(\phi U, V)). \end{aligned}$$

Substituting $h = ph + qh$ in the above equation and using (26), we derive

$$\begin{aligned} \bar{R}(U, \phi U, \phi V, V) &= \\ &R(U, \phi U, \phi V, V) - g(ph(U, V) + qh(U, V), ph(\phi U, \phi V) + qh(\phi U, \phi V)) \\ &\quad + g(ph(U, \phi V) + qh(U, \phi V), ph(\phi U, V) + qh(\phi U, V)) \\ &= R(U, \phi U, \phi V, V) - g(ph(U, V), ph(\phi U, \phi V)) - g(qh(U, V), ph(\phi U, \phi V)) \\ &\quad - g(qh(U, V), ph(\phi U, \phi V)) - g(qh(U, V), qh(\phi U, \phi V)) \\ &\quad + g(ph(U, \phi V), ph(\phi U, V)) + g(ph(U, \phi V), qh(\phi U, V)) \end{aligned}$$

$$\begin{aligned}
& +g(qh(U, \phi V), ph(\phi U, V)) + g(qh(U, \phi V), qh(\phi U, V)) \\
& = R(U, \phi U, \phi V, V) - g(ph(U, V) + qh(U, V), ph(\phi U, \phi V) + qh(\phi U, \phi U)) \\
& \quad + g(ph(U, \phi V), ph(\phi U, V)) + g(qh(U, \phi V), qh(\phi U, V)) \\
& = R(U, \phi U, \phi V, V) - g(ph(U, V) + qh(U, V), ph(\phi U, \phi V) + qh(\phi U, \phi V)) \\
& \quad + g(ph(U, \phi V), ph(\phi U, V)) + g(qh(U, \phi V), qh(\phi U, V)) \\
(29) \quad & = R(U, \phi U, \phi V, V) - \|ph(U, V)\|^2 + 2\|qh(U, V)\|^2 - \|ph(U, \phi V)\|^2.
\end{aligned}$$

Now by putting $V = \phi U$, $Z = \phi V$, $W = V$ in (26) it follows

$$\begin{aligned}
R(U, \phi U, \phi V, V) & = R^*(U_*, \phi' U_*, \phi' V_*, V_*) - g(S(\phi U, \phi V), S(U, V)) \\
& \quad + g(S(U, \phi V), S(\phi U, V)) + 2g(S(U, \phi U), S(\phi V, V)) \\
& = R^*(U_*, \phi' U_*, \phi' V_*, V_*) - g(S(\phi U, \phi V), S(U, V)) \\
(30) \quad & - g(S(U, \phi V), S(V, \phi U)) - 2g(S(U, \phi U), S(V, \phi V)).
\end{aligned}$$

Applying ϕ to equation $\phi S(U, V) = ph(U, \phi V)$, we get

$$\phi^2 S(U, V) = \phi ph(U, \phi V).$$

This gives

$$-S(U, V) + \eta(S(U, V))\xi = \phi ph(U, \phi V)$$

or

$$S(U, V) = -\phi ph(U, \phi V).$$

Using above relation in (39), we conclude

$$\begin{aligned}
(31) \quad R(U, \phi U, \phi V, V) & = R^*(U_*, \phi' U_*, \phi' V_*, V_*) - \|ph(U, V)\|^2 \\
& \quad - \|ph(U, \phi V)\|^2 - 2g(ph(U, U), ph(V, V)).
\end{aligned}$$

Put this value of $R(U, \phi U, \phi V, V)$ in (38), we obtain

$$\begin{aligned}
R(U, \phi U, \phi V, V) & = R^*(U_*, \phi' U_*, \phi' U_*, V_*) - \|ph(U, V)\|^2 - \|ph(U, \phi V)\|^2 \\
& \quad - 2g(ph(U, U), ph(V, V)) - \|ph(U, V)\|^2 + 2\|qh(U, V)\|^2 - \|ph(U, \phi V)\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
(32) \quad \bar{B}_\phi(U, V) & = B'_\phi(U, V) - 2\|ph(U, V)\|^2 - 2\|ph(U, \phi V)\|^2 \\
& \quad - 2g(ph(U, U), ph(V, V)) + 2\|qh(U, V)\|^2.
\end{aligned}$$

□

Corollary 4.2. [15] Let $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR-submersion from submanifold of generalized quasi-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M' . Then the ϕ -bisectional curvature of $\bar{M}$ and M' are related by

$$\frac{1}{2}\bar{K}(X) = \frac{1}{2}K'(X_*) - 2\|ph(U, U)\|^2 + \|qh(U, U)\|^2,$$

where $U \in (D \oplus \{\xi\})$.

Proof. Putting $U = V$ in the equation (41), we turn up

$$\begin{aligned}\bar{B}_\phi(U, V) &= \bar{K}(U) = K'(X_*) - 2\|ph(U, U)\|^2 - 2\|ph(U, \phi U)\|^2 \\ &\quad - 2g(ph(U, U), ph(V, V)) + 2\|qh(U, V)\|^2. \\ &= K'(U_*) - 4\|ph(U, U)\|^2 - 2\|ph(U, \phi U)\|^2 + 2\|qh(U, V)\|^2.\end{aligned}$$

Putting $V = U$ in (34) of Theorem 3.5,

$$ph(U, \phi U) = \phi S(U, U) = 0.$$

Thus we get

$$\bar{K}(X) = K'(U_*) - 4\|ph(U, U)\|^2 + 2\|qh(U, U)\|^2,$$

□

Theorem 4.3. $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a contact CR-submersion between submanifold of generalized quasi-Sasakian manifold $\bar{M}$ and an almost contact metric manifold M' with integrable distribution $(D \oplus \{\xi\})$. Then the ϕ -sectional curvature $\bar{K}$ and K' of $\bar{M}$ and M' , respectively, satisfying

$$(33) \quad \bar{K}(U) \geq K'(U_*), \quad \forall U \in (D \oplus \{\xi\}), \quad \|U\| = 1, \quad \phi_* U = U_*,$$

and the equality holds if and only if M is $(D \oplus \{\xi\})$ -totally geodesic.

Proof. The first assertion holds from (40). The equality holds if and only if $h(U, U) = 0$ and $h(U, \phi U) = 0$, for any $U \in (D \oplus \{\xi\})$, $\|U\| = 1$. From $h(U, U) = 0$, $U \in (D \oplus \{\xi\})$, $\|U\| = 1$, and linearity of h it follows immediately that $h(U, V) = 0$, for any $U, V \in (D \oplus \{\xi\})$, and proves that M is $(D \oplus \{\xi\})$ -totally geodesic. □

5. de-Rham Cohomology of contact CR submersion of generalized quasi-Sasakian manifold

This section, discuss the submersion $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ of contact CR -submanifold M with minimal horizontal distribution $(D \oplus \{\xi\})$ effects the topology of M [14].

Assume that M be a contact CR -submanifold of a generalized quasi-Sasakian manifold $\bar{M}$ with almost contact metric structure (ϕ, ξ, η, g) with $\dim(D \oplus \{\xi\}) = 2p + 1$ and $\dim M = m$. Let us choose a local orthonormal frame $\{(e_1, e_2, \dots, e_p, \phi e_1, \phi e_2, \dots, \phi e_p, e_{2p+1} = \xi, e_{2p+2}, \dots, e_m)\}$ on M such that $\{(e_1, e_2, \dots, e_p, \phi e_1, \phi e_2, \dots, \phi e_p, e_{2p+1} = \xi)\}$ is a local orthonormal frame of $(D \oplus \{\xi\})$ and $\{e_{2p+2}, e_{2p+3}, \dots, e_m\}$ is that of $D^\perp$. Let $\{\omega^1, \omega^2, \dots, \omega^{2p+1}, \omega^{2p+2}, \dots, \omega^m\}$ be the dual frame of 1-form to the above local orthonormal frame. Define $2p + 1$ form Ω on M by [14]

$$(34) \quad \Omega = \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^{2p+1},$$

which is globally defined on M .

Definition [15] Let $\mathbb{D}$ be a distribution on a Riemannian manifold M of dimension q . If $\sum_{i=1}^q \nabla_{e_i} e_i \in \mathbb{D}$, then the distribution $\mathbb{D}$ is said to be minimal, where ∇ is the Riemannian connection on M and $\{e_1, e_2, \dots, e_q\}$ is a local orthonormal frame of $\mathbb{D}$.

Theorem 5.2. Let $\bar{M}$ be a generalized quasi-Sasakian manifold and M be a closed contact CR -submanifold of $\bar{M}$ with minimal $(D \oplus \{\xi\})$. Let M' be a almost contact metric manifold and $\psi : (M, \phi, \xi, \eta, g) \rightarrow (M', \phi', \xi', \eta', g')$ be a submersion. Then the $2p + 1$ -form Ω is closed which define a canonical de Rham cohomology class $[\Omega] \in H^{2p+1}(M, R)$, where $2p + 1 = \dim(D \oplus \{\xi\})$. Moreover the cohomology group $H^{2p+1}(M, R)$ is non-trivial if $D^\perp$ is minimal.

Proof. From definition (43) of Ω , we have

$$(35) \quad d\Omega = \sum_{i=1}^{2p+1} (-1)^{i-1} \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^{2p+1}.$$

From the above equation it follow that $d\Omega = 0$ if and only if [12]

$$(36) \quad d\Omega(Z, W, E_1, \dots, E_{2p}) = 0, \text{ and } d\Omega(Z, E_1, \dots, E_{2p+1}) = 0$$

for $Z, W \in D^\perp$ and $E_1, \dots, E_{2p+1} \in (D \oplus \{\xi\})$. Choosing the vectors $E_1, \dots, E_{2p+1} \in (D \oplus \{\xi\})$ as a local orthonormal frame

$\{e_1, e_2, \dots, e_p, \phi e_1, \phi e_2, \dots, \phi e_p, e_{2p+1} = \xi\}$ of $(D \oplus \{\xi\})$ to which $\{\omega^1, \omega^2, \dots, \omega^{2p+1}\}$ works as dual frame of 1-forms, we get by a straightforward computation that the first equation (44) holds if and only if $(D \oplus \{\xi\})$ is minimal. However, from the definition of submersion it follows that $D^\perp$ is integrable. The hypothesis of theorem gives that $(D \oplus \{\xi\})$ is minimal. Hence the form Ω is closed, and it defines a de Rham cohomology class $[\Omega] \in H^{2p+1}(M, R)$ [14].

Now suppose that $D^\perp$ is minimal and we proceed to show that in this case

$$H^{2p+1}(M, R) \neq 0.$$

To accomplish this we show that the form Ω is harmonic which would then make the cohomology class $[\Omega]$ non-trivial. Define a $(m-2p-1)$ -form $\Omega^\perp$ on M by setting

$$\Omega^\perp = \omega^{2p+2} \wedge \dots \wedge \omega^m,$$

where $\{\omega^{2p+2}, \dots, \omega^m\}$ is the dual frame to the local orthonormal frame $\{e_{2p+1}, \dots, e_m\}$ of $D^\perp$. Then with the similar argument for Ω , it follows that $d\Omega^\perp = 0$ if $(D \oplus \{\xi\})$ is integrable and $D^\perp$ is minimal. It should be noted that minimality of $(D \oplus \{\xi\})$ implies its integrability. Since both conditions are met, we have $d\Omega^\perp = 0$. this proves that the $2p+1$ -form Ω is co-closed, that is $\delta\Omega = 0$. Since $d\Omega = \delta\Omega = 0$ and M is closed submanifold, we get that Ω is harmonic $2p+1$ -form; and this complete the proof. □

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REFERENCES

- [1] A. Bejancu, Geometry of CR -submanifold, D. Reidel publishing company, Holland, 1986.
- [2] D. E. Blair, Contact manifold in Riemannian geometry, Lectur Notes in Math, 59, Springer Verlag, 1976.
- [3] B. Chen, Cohomology of CR -submanifolds, Ann. Fac. Sci. Toulouse Math. 3 (5), no. 2, 167-172, 1981.
- [4] C. Chineia, Almost contact metric submersions, Rend. Circ. Mat. Palermo, 43(1), 89-104, 1985.
- [5] M. Falcitelli, S. Ianus, A. M. Pastore and M. Visinescu, Some applications of Riemannian submersion in physics, Rev. Roum. Phys. 48 , 627-639, 2003.

- [6] S. Kobayashi, Submersion of CR - submanifolds, Tohoku Math. J. 39(2), no. 1, 95-100, 1987.
- [7] K. Kenmotsu, A class of almost contact Riemannian manifolds Tohoku Math. J. 24 (2), 93-103, 1972.
- [8] F. Massamba, T. T. Matamba., Horizontally submersions of contact CR -submanifolds. Turkish Journal of Mathematics, 38, 436–453, 2014.
- [9] B. O'Neill. The fundamental equations of a submersions, Mich. Math. J., 13, 458-469, 1996.
- [10] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, 1983..
- [11] N. Papaghiuc, Submersion of semi-invariant submanifolds of a Sasakian manifold, An. Stiint. Univ. Al. I. Cuza Iasi Sect. I a Mat. 35, no. 3 21-288, 1989.
- [12] M. Visinescu, Space-time compactification induced by nonlinear sigma models, gauge fields and submersions , Czechoslovak J. Phys. B 37, no.4, 525-528, 1987.
- [13] M. D. Siddiqi, A. Haseeb and M. Ahmad, Skew Semi-invariant Submanifolds of Generalized Quasi-Sasakian Manifolds, Carpathian Math. Publ., 9 (2), 188-197, 2017.
- [14] M. H. Shahid, F. R. Al-Solamy, J. B. Jun and M. Ahamd, Submersion of Semi-Invariant Submanifolds of Trans-Sasakian Manifold, Bull. Malays. Math. Sci. Soc. (2) 36(1) 6371, 2013.
- [15] M. H. Shahid, F. R. Al-Solamy and M. Jamali, Submersions of CR Submanifolds. In: Dragomir S., Shahid M., Al-Solamy F. (eds) Geometry of Cauchy-Riemann Submanifolds. Springer, Singapore (2016).
- [16] B. Watson, B., Riemannian submersions and nonlinear gauge field equation of general relativity, in Global Analysis- Analysis on Manifolds, 324-349, Teubner-Texte Math., 57 Teubner, Leizig, 1983.
- [17] K. Yano and M. Kon, M. Contact CR -submanifolds, Kodai Math. J., 5(2), 238-252, 1982.
- [18] R. Shamsur and Ahmad. A., On the geometry of hypersurfaces of a certain connection in a Quasi-Sasakian manifold. International J.Math. Combin., 3 (2011), 23-33.