

## REPRESENTATION UP TO HOMOTOPY AND HOM-LIE ALGEBROID MODULES

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**ABSTRACT.** In this paper we introduce the concept of hom-Lie algebroid modules and  $\mathcal{VB}$  hom-Lie algebroids. Then we show the correspondence between hom-Lie algebroid modules and representation up to homotopy of hom-Lie algebroids. Because of the effective role of representation theory and Lie algebraic structures in particle physics, we show the correspondence between bi-graded hom-Lie algebraic modules and  $\mathcal{VB}$  hom-Lie algebraist. At the end, we study some properties of representation up to homotopy, using the language of hom-Lie algebroid modules.

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### 1. INTRODUCTION

Wigner in 1930, noted a natural connection between particle physics and representation theory, which links the properties of elementary particles to the structure of Lie groups and Lie algebras.

We introduced The notion of hom-Lie algebroid representation as a generalization of Lie algebroid representation in [8, 9]. One of notable problem with the typical notion of Lie algebroid representation is the absence of a well-defined adjoint representation and this problem is infeasible for hom-Lie algebroid representations. The efforts to solve this problem have led to the creation of a number of new generalizations of the notion of Lie algebroid representation, which the most popular of them is the representation up to homotopy, which is generalized for hom-Lie algebroids too. We study deformation theory and constructing characteristic classes for hom-Lie algebroids by the useful context that representation up to homotopy provides [1].

We introduced and studied the concepts of hom-Lie algebroids representation and representation up to homotopy in [8–10] and Lie-algebroid modules are studied in [7]. For the first time, the Lie algebroid modules were used to generalize the concept of Lie algebroid representations [2, 5–7, 11]

At first let us recall the definition of hom-Lie algebroids and some of hom-Lie algebroid examples.

**Definition 1.1.** [4] *Let  $A$  be vector bundle over smooth manifold  $M$ ,  $\theta : M \rightarrow M$  is a smooth map,  $[\cdot, \cdot]_A : \Gamma(A) \otimes \Gamma(A) \rightarrow \Gamma(A)$  is a bilinear map,  $\rho : A \rightarrow TM$  is a vector bundle morphism and  $\Theta : \Gamma(A) \rightarrow \Gamma(A)$  is a linear endomorphism of  $\Gamma(A)$ , then  $(A \rightarrow M, \theta, [\cdot, \cdot]_A, \rho, \Theta)$ , is a hom-Lie algebroid, if*

- (1)  $\Theta(fX) = \theta^*(f)\Theta(X)$ , for all  $X \in \Gamma(A)$ ,  $f \in C^\infty(M)$ ;
- (2) the triple  $(\Gamma(A), [\cdot, \cdot]_A, \Theta)$  is a hom-Lie algebra;
- (3) the following hom-Leibniz identity holds:

$$[X, fY]_A = \theta^*(f)[X, Y]_A + \mathcal{L}_{\rho(X)}(f)\Theta(Y), \quad \text{for all } X, Y \in \Gamma(A), f \in C^\infty(M).$$

- (4)  $(\Theta, \theta^*)$  is a representation of  $(\Gamma(A), [\cdot, \cdot]_A, \Theta)$  on  $C^\infty(M)$ .

The bilinear map  $[\cdot, \cdot]$  is called bracket and the vector bundle morphism  $\rho$  is called anchor.

**Example 1.2.** [8] *For a manifold  $M$  and a hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  There exists a trivial hom-Lie algebroid structure on  $TM \oplus (M \times \mathfrak{g})$  which is defined by an anchor  $\rho = \pi_1 : TM \oplus (M \times \mathfrak{g}) \rightarrow TM$ , a bracket*

$$[X \oplus v, Y \oplus w] = [X, Y]_{TM} \oplus (X(w) - Y(v) + [v, w]_{\mathfrak{g}}),$$

and vector bundle map

$$\begin{aligned}\Theta : \Gamma(TM \oplus (M \times \mathfrak{g})) &\rightarrow \Gamma(TM \oplus (M \times \mathfrak{g})) \\ f \oplus g &\mapsto f \oplus \alpha \circ g\end{aligned}$$

over  $M$ .

The structure of the paper is as follows:

- In the section two we introduce the notion of representation up to homotopy of hom-Lie algebroids and we study some properties and examples.
- The third section is devoted to the hom-Lie algebroid modules and their relation by representation up to homotopy.
- In the section four we introduce the notion of  $\mathcal{VB}$  hom-Lie algebroids and their relation by hom-Lie algebroid modules.
- In the Last section we construct the notion of characteristic classes, and prove that this construction gives well define invariants of hom-Lie algebroid modules.

## 2. REPRESENTATION UP TO HOMOTOPY OF HOM-LIE ALGEBROIDS

At first let us to study the concept of representation up to homotopy of hom-Lie algeroids.

**Definition 2.1.** [9] Let  $(A \rightarrow M, \theta, [\cdot, \cdot]_A, \rho, \Theta)$  be a hom-Lie algebroid,  $\varepsilon$  be a graded vector bundle and  $\alpha$  be a degree preserving operator on  $\varepsilon$ , a representation up to homotopy of  $A$  on  $\varepsilon$  with respect to  $\alpha$ , is a degree 1 operator  $D_\alpha$  on  $\Omega_\alpha(A; \varepsilon)$  such that  $D_\alpha^2 = 0$ ,

$$(1) \quad \Theta^* D_\alpha = \alpha \circ D_\alpha$$

and

$$(2) \quad D_\alpha(\omega\eta) = D_\alpha\omega\Theta^*(\eta) + (-1)^p\Theta^*\omega D_\alpha(\eta),$$

for any  $\omega \in \Omega^p(A)$  and  $\eta \in \Omega_\alpha(A; \varepsilon)$ .

By an  $\alpha$ -representation up to homotpy we mean a representation up to homotopy with respect to  $\alpha$ .

**Proposition 2.2.** *Let  $\varepsilon$  graded vector bundles over  $M$ ,  $(\varepsilon, D_\alpha)$  is a  $\alpha$ -representations up to homotopy of  $A$  if and only if,*

- (1) *A degree 1 operator  $\partial_\alpha$  on  $\varepsilon$  making  $(\varepsilon, \partial_\alpha)$  a complex.*
- (2) *An  $A$ -connection  $\nabla^\alpha$  on  $(\varepsilon, \partial_\alpha)$  with respect to  $\alpha$ .*
- (3) *An  $\text{End}(\varepsilon)$ -valued 2-form  $\omega_2$  of total degree 1, i.e.*

$$\omega_2 \in \Omega^2(A; \underline{\text{End}}^{-1}(\varepsilon))$$

*satisfying*

$$(3) \quad \omega_2(\Theta(X_1), \Theta(X_2)) = \omega_2(X_1, X_2) \circ \alpha$$

*and*

$$(4) \quad \partial_\alpha(\omega_2) + R_{\nabla^\alpha} = 0.$$

- (4) *For each  $p > 2$ , an  $\text{End}(\varepsilon)$ -valued  $p$ -form  $\omega_p$  of total degree 1, i.e.*

$$\omega_p \in \Omega^p(A; \underline{\text{End}}^{1-p}(\varepsilon))$$

*satisfying*

$$(5) \quad \omega_p(\Theta(X_1), \Theta(X_2), \dots, \Theta(X_p)) = \omega_p(X_1, X_2, \dots, X_p) \circ \alpha$$

*and*

$$\partial_\alpha(\omega_p) + d_{\nabla^\alpha}(\omega_{p-1}) + \omega_2 \circ \omega_{p-2} + \omega_3 \circ \omega_{p-3} + \dots \omega_{p-2} \circ \omega_2 = 0.$$

*Moreover, the correspondence is characterized by*

$$(6) \quad D_\alpha(\eta) = \partial_\alpha(\eta) + d_{\nabla^\alpha}(\eta) + \omega_2 \wedge \eta + \omega_3 \wedge \eta + \dots$$

*Proof.* Since the  $\Omega_\alpha(A; \varepsilon)$  is generated as an  $\Omega(A)$ -module by  $\Gamma(\varepsilon)$  and as (2), the operator  $D_\alpha$  will be uniquely determined by what it does on  $\Gamma(\varepsilon)$  it will send each  $\Gamma(\varepsilon^k)$  into the sum

$$\Gamma(\varepsilon^{k+1}) \oplus \Gamma_\alpha^1(A; \varepsilon^k) \oplus \Gamma^2(A; \varepsilon^{k-1}) \oplus \dots$$

Then it will also send each  $\Omega_\alpha^p(A; \varepsilon^k)$  into the sum

$$\Omega_\alpha^p(A; \varepsilon^{k+1}) \oplus \Omega_\alpha^{p+1}(A; \varepsilon^k) \oplus \Omega_\alpha^{p+2}(A; \varepsilon^{k-1}) \oplus \dots,$$

For  $i > 1$ ,  $D_i$  is a graded  $\Omega(A)$ -linear map, where the components of  $D$  denoted by  $D_0, D_1, \dots$  and, by [1, Lemma A.1] it is the wedge product with an element in  $\Omega_\alpha(A; \text{End}(\varepsilon))$ . Moreover for each  $k$ ,  $D_1$  applies to the  $\alpha$ -derivation rule on each

the vector bundle  $\varepsilon^k$ . it comes from  $A$ -connections on these bundles. the equations (3), (4) and (5) correspond to (1) and  $D^2 = 0$ .  $\square$

A morphism  $\varphi : \varepsilon_1 \rightarrow \varepsilon_2$  between  $\alpha$ -representation up to homotopy  $(\varepsilon_1, D_\alpha)$  and  $\beta$ -representation up to homotopy  $(\varepsilon_2, D_\beta)$  of hom-Lie algebroid  $A$  is a degree zero  $\Omega(A)$ -linear map

$$\varphi : \Omega_\alpha(A; \varepsilon_1) \rightarrow \Omega_\beta(A; \varepsilon_2)$$

which commutes with  $\alpha$  and  $\beta$  and the structure differentials  $D_\alpha$  and  $D_\beta$ .

**Example 2.3.** *Let  $(A \rightarrow M, \theta, [\cdot, \cdot]_A, \rho, \Theta)$  be a hom-Lie algebroid,  $E$  be a vector bundle and  $\alpha \in \mathcal{D}(E)$ . All the representations of  $A$  with respect to  $\alpha$  is a zero degree representation up to homotopy with respect to  $\alpha$ .*

**Example 2.4.** *[9] Let  $(A \rightarrow M, \theta, [\cdot, \cdot]_A, \rho, \Theta)$  be a hom-Lie algebroid,  $\varepsilon$  be a graded vector bundle which is the trivial line bundle in degrees 0 and  $n - 1$ , and zero otherwise and  $\nabla$  is the flat connection on the trivial line bundle. Any closed  $n$ -form  $\omega \in \Omega_\alpha^n(A)$  and  $\alpha \in \mathcal{D}(M \times \mathbb{R})$  such that  $\Theta^*\omega = \alpha \circ \omega$ , induces a representation up to homotopy on the  $\varepsilon$  with the structure operator  $\nabla + \omega$ . If  $\omega$  and  $\omega'$  are cohomologous, then the induced representations up to homotopy are isomorphic with isomorphism defined by  $Id + \theta$  where  $\omega - \omega' = d\theta$ .*

Let  $(A, [\cdot, \cdot], \rho, \Theta)$  be a hom-Lie algebroid over  $M$  and  $D_\alpha$  be a representation up to homotopy of  $A$  on  $\varepsilon$  with respect to degree preserving operator  $\alpha$  on  $\varepsilon$ . There is a projection map

$$\mu : \Omega_\alpha(A; \varepsilon) \rightarrow \Gamma_\alpha(\varepsilon),$$

which

$$\ker \mu = \bigoplus_{p>0} \Omega^p(A) \otimes \Gamma_\alpha(\varepsilon).$$

The equation (2) implies that  $\ker \mu$  is  $D_\alpha$ -invariant. Therefore, there is an induced differential  $\partial_\alpha$  on  $\Gamma_\alpha(E)$ , defined by the property that the following diagram commutes:

$$\begin{array}{ccc} \Omega_\alpha(A; \varepsilon) & \xrightarrow{D_\alpha} & \Omega_\alpha(A; \varepsilon) \\ \mu \downarrow & & \downarrow \mu \\ \Gamma_\alpha(\varepsilon) & \xrightarrow{\partial_\alpha} & \Gamma_\alpha(\varepsilon) \end{array}$$

**Definition 2.5.** A degree preserving  $\Omega(A)$ -module automorphism  $u$  is a  $\alpha$ -gauge transformation of  $\Omega_\alpha(A; \varepsilon)$ , if the following diagram commutes:

$$\begin{array}{ccc} \Omega_\alpha(A; \varepsilon) & \xrightarrow{u} & \Omega_\alpha(A; \varepsilon) \\ \mu \downarrow & & \downarrow \mu \\ \Gamma_\alpha(\varepsilon) & \xrightarrow{Id} & \Gamma_\alpha(\varepsilon) \end{array}$$

Two  $\alpha$ -representation up to homotopy  $D_\alpha$  and  $D'_\alpha$  are said gauge equivalent; if there exists a  $\alpha$ -gauge transformation  $u$  which  $D_\alpha = uD'_\alpha u^{-1}$ .

### 3. HOM-LIE ALGEBROID MODULES

Let  $(A, [\cdot, \cdot], \rho, \Theta)$  be a hom-Lie algebroid over  $M$ .

**Definition 3.1.** A hom-Lie algebroid  $\alpha$ -module over  $A$ , is a vector bundle  $\mathcal{B} \rightarrow A[1]$  together with  $\alpha \in \mathcal{D}(\mathcal{B})$  and a degree 1 operator  $\mathcal{Q}_\alpha$  on  $\Gamma_\alpha(\mathcal{B})$  such that  $\mathcal{Q}_\alpha^2 = 0$ ,  $\Theta^* \mathcal{Q}_\alpha = \alpha \circ \mathcal{Q}_\alpha$  and

$$\mathcal{Q}_\alpha(\omega\eta) = d_A(\omega)\Theta^*(\eta) + (-1)^p \Theta^* \omega \mathcal{Q}_\alpha(\eta),$$

where  $\omega \in \Omega^p(A)$  and  $\eta \in \Gamma_\alpha(\mathcal{B})$ .

Let  $(\mathcal{B}, \mathcal{Q}_\alpha)$  and  $(\mathcal{B}', \mathcal{Q}'_{\alpha'})$  be two hom-Lie algebroid  $\alpha$ -module and  $\alpha'$ -module over  $A$ , respectively. A hom-Lie algebroid module morphism from  $(\mathcal{B}, \mathcal{Q}_\alpha)$  to  $(\mathcal{B}', \mathcal{Q}'_{\alpha'})$  is a linear map  $\psi : \mathcal{B} \rightarrow \mathcal{B}'$  covering the identity map on  $A[1]$ , such that

$$\psi \mathcal{Q}_\alpha = \mathcal{Q}'_{\alpha'} \psi$$

and

$$\alpha' \circ \psi = \psi \circ \alpha.$$

**Proposition 3.2.** Representations up to homotopy of  $A$  on bounded graded vector bundle  $\varepsilon$  with respect to  $\alpha$  are in one-to-one correspondence with hom-Lie algebroid  $(\pi_A^* \alpha)$ -modules of the form  $\pi_A^* \varepsilon$ , where  $\pi_A$  is the projection map from  $A[1]$  to  $M$ .

*Proof.* Let  $\varepsilon$  be a bounded graded vector bundle over  $M$  and  $\alpha$  be a degree preserving operator on  $\varepsilon$ . There is canonical module isomorphism between  $\Gamma_{\pi_A^* \alpha}(\pi_A^* \varepsilon)$  and  $\Omega(A) \otimes_{C^\infty(M)} \Gamma_\alpha(\varepsilon)$  which is equal to  $\Omega_\alpha(A; \varepsilon)$ . So there exists a unique degree 1 operator  $\mathcal{Q}$  on  $\Gamma_{\pi_A^* \alpha}(\pi_A^* \varepsilon)$  correspond to any degree 1 operator  $D$  on  $\Omega_\alpha(A; \varepsilon)$ . It

is easy to see that  $D$  is a representation up to homotopy of  $A$  on  $\varepsilon$ , if and only if  $\mathcal{Q}^2 = 0$ ,  $\Theta^* \mathcal{Q} = \pi_A^* \alpha \circ \mathcal{Q}$  and

$$\mathcal{Q}(\omega\eta) = d_A(\omega)\Theta^*(\eta) + (-1)^p \Theta^* \omega \mathcal{Q}(\eta),$$

where  $\omega \in \Omega^p(A)$  and  $\eta \in \Gamma_{\pi_A^* \alpha}(\pi_A^* \varepsilon)$ . So it is a hom-Lie algebroid  $\pi_A^*$ -module.  $\square$

Let  $(\mathcal{B}, \mathcal{Q}_\alpha)$  and  $(\mathcal{B}', \mathcal{Q}'_{\alpha'})$  be two hom-Lie algebroid modules over  $A$ . Then  $\mathcal{Q}_\alpha \oplus \mathcal{Q}'_{\alpha'}$  is a hom-Lie algebroid  $(\alpha \oplus \alpha')$ -module structure on  $\mathcal{B} \oplus \mathcal{B}'$ , where

$$(\mathcal{Q}_\alpha \oplus \mathcal{Q}'_{\alpha'})(x, y) = \mathcal{Q}_\alpha x + \mathcal{Q}'_{\alpha'} y$$

for  $x \in \mathcal{B}$  and  $y \in \mathcal{B}'$ . Also there is a hom-Lie algebroid  $(\alpha \otimes \alpha')$ -module structure on  $\mathcal{B} \otimes \mathcal{B}'$  given by

$$\mathcal{Q}(x \otimes y) = (\mathcal{Q}_\alpha x) \otimes y + (-1)^{|x|} x \otimes (\mathcal{Q}'_{\alpha'} y),$$

for  $x \in \mathcal{B}$  and  $y \in \mathcal{B}'$ .

**Proposition 3.3.** *Let  $(\mathcal{B}, \mathcal{Q}_\alpha)$  be a hom-Lie algebroid  $\alpha$ -module over  $A$  and  $\mathcal{B}^* \rightarrow A[1]$  be the vector bundle dual to  $\mathcal{B}$ . If  $\langle \cdot, \cdot \rangle$  is the pairing taking  $\Gamma(\mathcal{B}^*) \otimes \Gamma(\mathcal{B})$  to  $\Omega(A)$ , then there exists a unique hom-Lie algebroid  $\alpha^*$ -module structure on  $\mathcal{B}^*$ , which the pairing  $\langle \cdot, \cdot \rangle$  is a hom-Lie algebroid module morphism from  $\mathcal{B}^* \otimes \mathcal{B}$  to trivial rank 1 hom-Lie algebroid id-module  $(A[1] \times \mathbb{R}, d_A)$ .*

*Proof.* We denote by  $\mathcal{Q}_{\alpha^*}^*$  the induced hom-Lie algebroid  $\alpha^*$ -module structure on  $\mathcal{B}^*$ , which is uniquely determined by the equation

$$(7) \quad d_A \langle \xi, x \rangle = \langle \mathcal{Q}_{\alpha^*}^* \xi, x \rangle + (-1)^{|\xi|} \langle \xi, \mathcal{Q}_\alpha x \rangle,$$

for  $\xi \in \mathcal{B}^*$  and  $x \in \mathcal{B}$ . Requiring (7) to hold is equivalent to asking that the pairing  $\langle \cdot, \cdot \rangle$  be a hom-Lie algebroid module morphism from  $\mathcal{B}^* \otimes \mathcal{B}$  to  $(A[1] \times \mathbb{R}, d_A)$ .  $\square$

#### 4. $\mathcal{VB}$ HOM-LIE ALGEBROIDS

Let all sides of the commutative square

$$\begin{array}{ccc} D & \xrightarrow{q_B^D} & B \\ q_A^D \downarrow & & \downarrow q_B \\ A & \xrightarrow{q_A} & M \end{array}$$

are vector bundles, This is called a double vector bundle, if

- (1)  $q_B^D$  is a vector bundle morphism over  $q_A$ ;
- (2)  $+ : D \times_B D \rightarrow D$  is a vector bundle morphism over  $+ : A \times_M A \rightarrow A$ , where  $+_B$  is the addition map for the vector bundle  $D \rightarrow B$ .

The core  $C$  of a double vector bundle is the intersection of the kernels of  $q_A^D$  and  $q_B^D$ . It has a natural vector bundle structure over  $M$ .

**Definition 4.1.** [3]

- (1) Let  $c : M \rightarrow C$  be a section and  ${}^B 0 : B \rightarrow B$  is the zero section. the corresponding core section  $\hat{c} : B \rightarrow D$  is defined as

$$(8) \quad \hat{c}(b_m) = {}^B 0_{b_m} + \overline{c(m)},$$

where  $m \in M$ ,  $b_m \in B_m$ . The space of core sections is denoted by  $\Gamma_c(B, D)$ .

- (2) A linear section  $\mathcal{X} \in \Gamma(B, D)$  is a bundle morphism  $\mathcal{X} : B \rightarrow D$  from  $B \rightarrow M$  to  $D \rightarrow A$ . The space of linear sections is denoted by  $\Gamma_\ell(B, D)$ .

**Definition 4.2.** [3] For a double vector bundle  $(D; A, B; M)$ , we say that

$$((D, B, \rho_D, \Theta_D, \Theta_B), (A, M, \rho_A, \theta_A, \theta_M))$$

is a  $\mathcal{VB}$ -hom-algebroid if  $(D, B, \rho_D, \Theta_D, \Theta_B)$  is a hom-Lie algebroid, the anchor  $\rho_D : D \rightarrow TB$  is a bundle morphism over  $\rho_A : A \rightarrow TM$ ,  $\Theta_D : D \rightarrow D$  is a bundle morphism over  $\theta_A : A \rightarrow A$ ,  $\Theta_B : B \rightarrow B$  is a bundle morphism over  $\theta_M : M \rightarrow M$  and the three Lie bracket conditions below are satisfied:

- (1)  $[\Gamma_\ell(B, D), \Gamma_\ell(B, D)]_D \subseteq \Gamma_\ell(B, D)$ ,
- (2)  $[\Gamma_\ell(B, D), \Gamma_c(B, D)]_D \subseteq \Gamma_c(B, D)$ ,
- (3)  $[\Gamma_c(B, D), \Gamma_c(B, D)]_D = 0$ .

**Proposition 4.3.** There is a one-to-one correspondence between  $\mathcal{VB}$ -hom-algebroid structures  $D$  over  $A$  and hom-Lie algebroid module structures of  $A$  on bi-graded vector bundle  $D[1]$ .

*Proof.* A hom-Lie algebroid structure on  $A \rightarrow M$  is equivalent to a degree 1 homological vector field on the graded manifold  $A[1]$ . Here,  $A[1]$  is the graded manifold whose algebra of functions is  $\bigwedge \Gamma(A^*)$ , and the operator  $d_A$ , as a derivation of this algebra, is viewed as a vector field on  $A[1]$ . The modifier homological indicates that  $d_A^2 = 0$ . Let  $\mathcal{B}$  be a bi graded vector bundle, then  $\mathcal{B} = D[1]$ , for some vector

bundle  $D \rightarrow B$ . The fact that  $\mathcal{B}$  also has a vector bundle structure over  $A[1]$  implies that  $D$  is a double vector bundle and a hom-Lie algebroid module structure on  $\mathcal{B}$  is equivalent to a  $\mathcal{VB}$ -hom-algebroid structure on  $D$  over  $A$ . In the case of a  $\mathcal{VB}$ -hom-algebroid, we may form the graded manifold  $D[1]_B$ , whose algebra of functions  $C^\infty(D[1]_B)$  is  $\Omega(D)$ . The operator  $d_D$  is viewed as a homological vector field on  $D[1]_B$ .

The algebra  $C^\infty(D[1]_B)$  has a natural double-grading arising from the double vector bundle structure

$$\begin{array}{ccc} D[1]_B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A[1] & \longrightarrow & M \end{array}$$

and this double-grading coincides with the double-grading of  $\Omega(D)$ .  $\square$

## 5. CHARACTERISTIC CLASS ASSOCIATED TO A REPRESENTATION

Let  $A \rightarrow M$  be a hom-Lie algebroid, and let  $\varepsilon \rightarrow M$  be a graded vector bundle that is bounded in degree. We introduce the notion of  $A$ -superconnection with respect to  $\alpha$ , where  $\alpha$  is a degree preserving operator on  $\varepsilon$ .

**Definition 5.1.** *A degree 1 operator  $D_\alpha$  on  $\Omega_\alpha(A; \varepsilon)$  is an  $A$ -superconnection with respect to  $\alpha$  on  $\varepsilon$ , if satisfying the hom-Leibniz rule and  $\Theta^* D_\alpha = \alpha \circ D_\alpha$ .*

A flat  $A$ -superconnection with respect to  $\alpha$  is an  $A$ -superconnection, where  $D_\alpha^2$  vanished.

For any  $A$ -superconnection  $D$  with respect to  $\alpha$  we can obtain the Chern-Weil forms

$$ch_k(D_\alpha) := str(D_\alpha^{2k}),$$

where  $str$  denotes the supertrace. Given a pair of  $A$ -superconnections  $D_\alpha$  and  $D'_\alpha$  with respect to  $\alpha$  on  $\varepsilon$ . Let  $I$  be the unit interval and consider the product hom-Lie algebroid  $A \times TI \rightarrow M \times I$ . Let  $p$  be the projection map from  $M \times I$  to  $M$ . We can form an  $A \times TI$ -superconnection with respect to  $\alpha \times id$  on  $p^*\varepsilon$  given by

$$(9) \quad \mathcal{T}_{D_\alpha, D'_\alpha} = tD_\alpha + (1-t)D'_\alpha.$$

The transgression forms  $cs_k(D_\alpha, D'_\alpha) \in \Omega^{2k-1}(A)$  are defined as

$$(10) \quad cs_k(D_\alpha, D'_\alpha) := \int ch_k(\mathcal{T}_{D_\alpha, D'_\alpha}).$$

Now we can define characteristic classes associated to a representation up to homotopy  $D_\alpha$  with respect to  $\alpha$ , as follows. Choose a metric on  $E_i$  for each  $i$ . We may use the metric to obtain an adjoint operator  $D^\dagger$  on  $\Omega(A; \varepsilon)$ , given by the equation

$$d_A \langle \omega_1, \omega_2 \rangle = \langle D_\alpha \omega_1, \omega_2 \rangle + (-1)^{|\omega_1|} \langle \omega_1, D_\alpha^\dagger \omega_2 \rangle.$$

The operator  $D_\alpha^\dagger$  satisfies the hom-Leibniz rule and squares to zero, but it is generally not homogeneous of degree 1.

The Chern-Simons forms associated to a representation up to homotopy  $D_\alpha$  with respect to  $\alpha$  is defined as

$$cs_k(D_\alpha) := cs_k(D_\alpha, D_\alpha^\dagger).$$

The following theorem implies that the cohomology classes of  $cs_k(D_\alpha)$  are well-defined invariants of hom-Lie algebroid modules.

**Theorem 5.2.** *Let  $D_\alpha$  be a representation up to homotopy with respect to  $\alpha$ . The cohomology classes  $[cs_k(D_\alpha)]$  are independent of the choice of metric and invariant with respect to gauge transformations.*

*Proof.* Let  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$  be two metrics on  $\varepsilon$ . The space of metrics is convex, therefore there exists a path  $\vartheta$  of metrics such that  $\vartheta(0)(x, y) = \langle x, y \rangle_1$ ,  $\vartheta(1)(x, y) = \langle x, y \rangle_2$  and  $u_t \in \text{End}(\varepsilon)$ , such that  $\vartheta(t)(x, y) = \vartheta(0)(u_t(x), y)$ . The adjoint operators  $\mathcal{D}_\alpha^{\dagger t}$  corresponds to  $\vartheta(t)$  satisfy the equation

$$\mathcal{D}_\alpha^{\dagger t} = (u_t)^{-1} \mathcal{D}_\alpha^{\dagger 0} u_t.$$

By [7, Proposition 7.5] we have

$$[cs_k(\mathcal{D}_\alpha, \mathcal{D}_\alpha^{\dagger 0})] = [cs_k(\mathcal{D}_\alpha, \mathcal{D}_\alpha^{\dagger 1})] + [cs_k(\mathcal{D}_\alpha^{\dagger 0}, \mathcal{D}_\alpha^{\dagger 1})],$$

and  $[cs_k(\mathcal{D}_\alpha^{\dagger 0}, \mathcal{D}_\alpha^{\dagger 1})] = 0$ , [7, Proposition 7.6]. Therefore the cohomology classes  $[cs_k(\mathcal{D}_\alpha)]$  are metric invariance.

The space of gauge transformations is

$$\bigoplus_{p,q} \Omega^p \otimes (\text{Hom}(E_k, E_{k-p})),$$

which is path connected. Hence the gauge-invariance follows as above.  $\square$

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