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POINTWISE WEAKLY MIXING PROPERTY AND LI-YORKE SENSITIVITY IN NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. In this paper we present novel concept of pointwise weakly mixing property (PWMP) for autonomous (ADS) and nonautonomous (NDS) dynamical systems. We show that if NDS $(X, f_{1,\infty})$ has PWMP, then proximal cells are dense in X and in addition NDS is sensitive. Furthermore we conclude that NDS $(X, f_{1,\infty})$ is Li-Yorke sensitive and also densely Li-Yorke chaotic with pointwise weakly mixing property.

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1. Introduction

Recently, chaos theory in dynamical systems is regarded as an important subject that is studied by many researchers. There are many definitions about chaos, for instance Devaney chaos, topological chaos, omega chaos, Li-Yorke chaos and distributional chaos, for more facts refer to [5],[10],[2]. Weakly mixing property is as one of the tools to study chaos theory. Iwanik proved that weak mixing property implies chaos in the sense of Li and Yorke[9]. Shi and Chen defined the concept of Li-Yorke chaos for the nonautonomous dynamical systems (NDS)[12]. Many authors studied behavior of Li-Yorke chaos on NDS, for instance refer to [13],[12],[4],[6]. Balibrea and Oprocha showed that weak mixing of NDS is sufficient to induce chaos in the sense of Li and Yorke [3].

Akin and Kolyda introduced concept of Li-Yorke sensitivity and showed that in weakly mixing dynamical system (X, f), proximal cells is dense in $X \times X$ and (X, f) is Li-Yorke sensitive [1]. They proposed a question with this regard that"are all Li-Yorke sensitive systems Li-Yorke chaotic?". By an example, Hanakova demonstrated that the answer of the mentioned question is negative[7]. Hung and Ye proved that if proximal pairs of (X, f) is dense in $X \times X$ and f is either transitive with X infinite or sensitive, then f is densely Li-Yorke chaotic [8]. Also they showed that a topological transitive map with a periodic point is chaotic in the sense of Li-Yorke.

Rybak showed that if (X, G) is a weakly mixing system and G is an abelian semigroup, then (X, G) is Li-Yorke sensitive [11]. If NDS $(X, f_{1,\infty})$ with abelian condition, $f_i o f_j = f_j o f_i$, for any $i, j \in \mathbb{N}$, has weakly mixing property, then by a similar strategy of [11], we can show that $(X, f_{1,\infty})$ is Li-Yorke sensitive. Here we try to evaluate this matter without abelian condition. Therefore, in this paper, we attempt to introduce a new concept for autonomous and nonautonomous dynamical systems that is called pointwise weakly mixing property (PWMP). Moreover, we evaluate the behavior of this property in these systems. We also give an example that contains PWMP autonomous dynamical system and an example on NDS that is not abelian but it has PWMP.

In section 2, we express the priliminary concepts in autonomous and nonautonomous dynamical systems. In section 3, the regarded examples and the main theorems are expressed. Indeed we demonstrate that if NDS $(X, f_{1,\infty})$ has PWMP, proximal cells are dense in X and also this system has sensitive property. We show that $f_{1,\infty}$ with PWMP is Li-Yorke sensitive. Finally, we show that NDS $(X, f_{1,\infty})$ with this property, could be densely Li-Yorke chaotic.

2. PRELIMINARIES

A topological dyamical system is a pair (X, f) where X is a nonempty compact metric space with a metric d and f is a continuous map from X to itself. The *diameter* of a set $A \subset X$ is defined to be $diam(A) = \sup\{d(x, y) : x, y \in A\}$. For any $x \in X$, *orbit* of x under f denoted by $\{f^n(x); n \in \mathbb{Z}_+\}$.
The ω-limit set of $x \in X$ denoted by

$$\omega(x, f) = \{y \in X;\ \exists\{t_k\}_{k=0}^{\infty},\ t_k \to \infty \, such\ that\, f^{t_k}(x) \to y\}.$$

Dynamical system (X, f) is *topologically transitive* if for every pair of nonempty and open subsets $U, V \subset X$, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$. (X, f) is *weakly mixing* if $f \times f : X \times X \to X \times X$ is transitive which means that for any nonempty open sets U_1, U_2, V_1, V_2 , there is an integer $n > 0$ such that $f^n(U_1) \cap V_1 \neq \emptyset$ and $f^n(U_2) \cap V_2 \neq \emptyset$.
(X, f) is *sensitive*, if there is $\delta > 0$ such that for any $x \in X$ and neighbourhood U of x, there is a $y \in U$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \delta$. A subset E of a topological space X is said to be of *first category* in X if E can be written as the countable union of subsets which are nowhere dense in X.

Definition 2.1. *A dynamical system (X, f) has pointwise weakly mixing property (PWMP) if for any $x \in X$, there exists $z \in X$ such that for every nonempty open subsets $U, V \subset X$ and for any neighbourhood W of z, there exists $n \in \mathbb{N}$ such that*

$$f^n(x) \in W \quad , \quad f^n(U) \cap V \neq \emptyset.$$

A nonautonomous dynamical system is a pair $(X, f_{1,\infty})$ where X is a compact metric space with metric d and $f_{1,\infty} = \{f_n\}_{n\geq 1}$ is a sequence of continuous maps $f_n : X \to X$. For $i, n \in \mathbb{N}$

$$f_i^0 := id, \;\; f_i^n := f_{i+n} \, o \, f_{i+n-1} \, o \cdots o \, f_{i+1} \, o \, f_i.$$

For any $x \in X$, *orbit* x under $f_{1,\infty}$ is sequence $x, f_1(x), f_1^2(x), f_1^3(x), ...$ where $f_1^n(x) = f_n \, o \, f_{n-1} \, o \cdots o \, f_2 \, o \, f_1(x)$.
Denote nonautonomous dynamical system $(X, f_{1,\infty})$ with NDS. If in NDS $(X, f_{1,\infty})$, $f_i o f_j = f_j o f_i$ for any $i, j \in \mathbb{N}$, then $(X, f_{1,\infty})$ is said to be an *abelian* NDS.
We say that a NDS $f_{1,\infty}$ is *transitive* if for any two nonempty open subsets $U, V \subset X$, there is a positive integer n such that $f_1^n(U) \cap V \neq \emptyset$.

$(X, f_{1,\infty})$ is called *weakly mixing* if for nonempty open subsets U_1, V_1, U_2, V_2 of X, there is $n \in \mathbb{N}$ such that $f_1^n(U_1) \cap V_1 \neq \emptyset$ and $f_1^n(U_2) \cap V_2 \neq \emptyset$.
A NDS $f_{1,\infty}$ has *sensitive* property if there is $\delta > 0$ such that for any point $x \in X$ and for each $\epsilon > 0$, there exists $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f_1^n(x), f_1^n(y)) > \delta$.
A pair (x, y) in $X \times X$ is called *proximal* for NDS $(X, f_{1,\infty})$ if

$$\liminf_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0,$$

and *asymptotic* if

$$\lim_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0.$$

The sets of proximal pairs and asymptotic pairs is explained as following; if $\bar{V}_\epsilon = \{(x, y) : d(x, y) \leq \epsilon\}$, we have

$$Prox(f_{1,\infty}) = \{(x, y) : \liminf_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0\} = \bigcap_{\epsilon>0}(\bigcup_{n=1}^{\infty} (f_1^{-n} \times f_1^{-n})(\bar{V}_\epsilon),$$

$$A_{k,\epsilon} = \bigcap_{i=k}^{\infty} (f_1^{-i} \times f_1^{-i})(\bar{V}_\epsilon),$$

$$Asym_\epsilon(f_{1,\infty}) = \bigcup_k A_{k,\epsilon},$$

$Asym(f_{1,\infty}) = \{(x, y), \lim_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0\} = \bigcap_{\epsilon>0} Asym_\epsilon(f_{1,\infty})$.

The set $Prox(f_{1,\infty})(x)$ is called the *proximal cell* of x. It consists of those points which are proximal with x.
Let $(X, f_{1,\infty})$ be a NDS. A subset $S \subset X$ is a *scrambled* set if any $x, y \in S$ witch $x \neq y$, are proximal and not asymptotic i.e.

$$\liminf_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0,$$

$$\limsup_{n\to\infty} d(f_1^n(x), f_1^n(y)) > 0.$$

[12], [4]. Denote the Li-Yorke relation on $f_{1,\infty}$ by

$$LYR = Prox(f_{1,\infty}) \setminus Asym(f_{1,\infty}).$$

$f_{1,\infty}$ is *Li-Yorke chaotic* if there exists an uncountable scrambled set. $f_{1,\infty}$ is *densely Li-Yorke chaotic* if there exsists a dense, uncountable scrambled set for

$f_{1,\infty}$.
$f_{1,\infty}$ is *Li-Yorke sensitive* if there exists $\epsilon > 0$ with the property that any neighbourhood of any $x \in X$ contains a point y proximal to x such that orbits of x and y are separated by ϵ for infinitely many times. Indeed we have

$$\liminf_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0,$$
$$\limsup_{n\to\infty} d(f_1^n(x), f_1^n(y)) > \epsilon.$$

Definition 2.2. *A nonautonomous dynamical system $(X, f_{1,\infty})$ has pointwise weakly mixing property (PWMP) if for any $x \in X$, there exists $z \in X$ such that for every nonempty open subsets $U, V \subset X$ and for any neighbourhood W of z, there exists $n \in \mathbb{N}$ such that*

$$f_1^n(x) \in W \quad , \quad f_1^n(U) \cap V \neq \emptyset.$$

Let $\Sigma = \{(s_0 s_1 s_2 \cdots) | s_k = 0 \, or \, 1\}$ be the sequence space on two symbols i.e. the set of all infinite sequences of "0" s and "1"s. Assume $s = (s_0 s_1 s_2 \cdots)$ and $t = (t_0 t_1 t_2 \cdots)$ be two elements of Σ. Define the distance between them to be

$$d(s,t) = \sum_{k \geqslant 0} \frac{|s_k - t_k|}{2^k}.$$

Define the shift map $\sigma : \sum \to \sum$ by $\sigma(s) = t$ where $\sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots)$. σ is continuous with the above metric. For any k, the iterate of shift map is $\sigma^k(s_0 s_1 s_2 \cdots) = (s_k s_{k+1} s_{k+2} \cdots)$. The space Σ with the shift map σ,(Σ, σ) is called the symbol space on 2 symbols or one sided shift space.

3. Examples and main results

Here X be a compact metric space and $f_i : X \to X$ be homeomorphism maps on X for any $i \in \mathbb{N}$.
The following example shows that there exists a dynamical system with PWMP.

Example 3.1. *Symbolic dynamical system (Σ, σ) has PWMP.*

Let $x \in \Sigma$ and consider $z \in \omega(x, \sigma)$ and W be an arbitrary neighbourhood of z. Suppose that U and V are two arbitrary nonempty open subsets of Σ. Take $s = (s_0 s_1 s_2 \cdots) \in U$ and $t = (t_0 t_1 t_2 \cdots) \in V$. There exists $\epsilon_i > 0$, $1 \leq i \leq 3$, such that $N_{\epsilon_1}(z) \subset W$, $N_{\epsilon_2}(s) \subset U$ and $N_{\epsilon_3}(t) \subset V$. Take $0 < \epsilon < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. Since

$z \in \omega(x,\sigma)$ there exists $n > 0$ with $\frac{1}{2^n} < \epsilon$, $\sigma^n(x) \in N_\epsilon(z)$. Consider $s^* \in U$ in the form of

$$s^* = (s_0 s_1 s_2 \cdots s_{n-1} t_0 t_1 t_2 \cdots),$$

Hence, we have $\sigma^n(s^*) \in V$. Then (Σ, σ) has PWMP.

Proposition 3.2. *$f_{1,\infty}$ has PWMP if and only if for fixed $k \in \mathbb{N}$, $f_{k,\infty}$ has PWMP.*

Proof. Suppose $f_{1,\infty}$ has PWMP. Fixed $k \in \mathbb{N}$. We show that $f_{k,\infty}$ has PWMP. For this, for any $x \in X$, let z be in the definition of PWMP for $f_{1,\infty}$. Consider W be an arbitrary neighbourhood of z and U, V arbitrary nonempty open subsets of X. There are $x' \in X$ and a nonempty open subset $U' \subset X$ such that $x = f_{k-1}\, o \cdots o\, f_2\, o\, f_1(x')$ and $U = f_1^{k-1}(U')$. Since $f_{1,\infty}$ has PWMP, there is $m > k$ such that $f_1^m(x') \in W$ and $f_1^m(U') \cap V \neq \emptyset$. Then we have $f_k^{m-k}(x) \in W$ and $f_k^{m-k}(U) \cap V \neq \emptyset$. Hence $f_{k,\infty}$ has PWMP.
Now, suppose that $f_{k,\infty}$ has PWMP. We show that $f_{1,\infty}$ has PWMP. For this, for any $x \in X$ let z be in the definition of PWMP for $f_{k,\infty}$. Consider W be an arbitrary neighbourhood of z and $U, V \subset X$ be arbitrary nonempty open subsets. There are $x' \in X$ and $U' \subset X$ such that $x' = f_{k-1}\, o \cdots o\, f_2\, o\, f_1(x)$ and $U' = f_1^{k-1}(U)$. Since $f_{k,\infty}$ has PWMP, then there is $m \in \mathbb{N}$ such that $f_k^m(x') \in W$ and $f_k^m(U') \cap V \neq \emptyset$. Hence $f_1^{m+k}(x) \in W$ and $f_1^{m+k}(U) \cap V \neq \emptyset$.

□

In the following example, by using the above propsition we construct a nonautonomous dynamical system which is not abelian NDS but has PWMP.

Example 3.3. *Let Σ be the sequence space on two symbols that is mentioned above and define $h_0, h_1 : \Sigma \to \Sigma$ by*

$$h_0(s) = h_0(s_0 s_1 s_2 \cdots) = (0\, s_0 s_1 s_2 \cdots),$$

$$h_1(s) = h_1(s_0 s_1 s_2 \cdots) = (1\, s_0 s_1 s_2 \cdots),$$

for any $s = (s_0 s_1 s_2 \cdots) \in \Sigma$. Define NDS $(\Sigma, f_{1,\infty})$ as following. Fixed $k \in \mathbb{N}$. $f_1 = h_0$ and for $2 \leq i \leq k$, $f_i = h_1$ and for $i > k$, $f_i = \sigma$.
According to this construction, we have $f_{k,\infty}$ is indeed the NDS in example 3.1. So by Proposition 3.2 , $f_{1,\infty}$ has PWMP. On the other hand, we can see $h_0\, o\, h_1 \neq h_1\, o\, h_0$, so $f_{1,\infty}$ is a nonabelian NDS that has PWMP.

The following question can be a part of future researches.

Question: Is there any example on dynamical systems that has pointwise weakly mixing property but is not weakly mixing?

In [11] Theorem 4.2 , for weakly mixing system (X, G) where X is a compact space and G is an abelian semigroup, it is proved that (X, G) is Li-Yorke sensitive. The same strategy of Theorem 4.2 [11] can be used for following theorem.

Theorem 3.4. *Let $(X, f_{1,\infty})$ be an abelian NDS that has weakly mixing property. Then it is Li-Yorke sensitive.*

As a main result of this paper, we ommit abelian condition and use PWMP instesd of weakly mixing property in the previous theorem to obtain Li-Yorke sensitivity.

Theorem 3.5. *If $(X, f_{1,\infty})$ has PWMP, then it is Li-Yorke sensitive.*

In order to prove of the Theorem 3.5, first we express and prove the following lemmas.

Lemma 3.6. *If $(X, f_{1,\infty})$ has PWMP, then for every $x \in X$, the proximal cell $Prox(f_{1,\infty})(x)$ is dense in X.*

Proof. To show that proximal cell $Prox(f_{1,\infty})(x)$ is dense in X, for nonempty open subset $U \subset X$, it is enough to find a point like that $y \in U$ proximal with x.

For any $x \in X$ there exists $z \in X$ such that for every arbitrary nonempty open subset $U \subset X$ and every arbitrary neighbourhood of z, the description of PWMP is satisfied. For any $i \in \mathbb{N}$, we consider an open ball O_i of z with radius $\frac{1}{i}$.

For open set U and open ball O_1, there is $n_1 \in \mathbb{N}$ such that

$$f_1^{n_1}(x) \in O_1 \ \ and \ \ f_1^{n_1}(U) \cap O_1 \neq \emptyset.$$

Let $U_0 = U$ and $U_1 = U_0 \cap f_1^{-n_1}(O_1)$. Also because of definition of PWMP, for open set U_1 and open ball O_2, there is $n_2 \in \mathbb{N}$ such that

$$f_1^{n_2}(x) \in O_2 \ \ and \ \ f_1^{n_2}(U_1) \cap O_2 \neq \emptyset.$$

Let $U_2 = U_1 \cap f_1^{-n_2}(O_2)$. By the following procedure, for open set U_{k-1} and open ball O_k, there is $n_k \in \mathbb{N}$ such that

$$f_1^{n_k}(x) \in O_k \ \ and \ \ f_1^{n_k}(U_{k-1}) \cap O_k \neq \emptyset,$$

let $U_k = U_{k-1} \cap f_1^{-n_k}(O_k)$. If y is a point of the nonempty intersection $\bigcap_k \bar{U_k}$, then $f_1^{n_k}(y) \in O_k$. Therefore $d(f_1^{n_k}(x), f_1^{n_k}(y)) \leq \frac{1}{k}$. Thus, y is a point of U which is proximal to x. $\square$

Lemma 3.7. *If $(X, f_{1,\infty})$ has PWMP, then it is sensitive.*

Proof. For every $x \in X$ consider $d(x) = \sup\{d(x,y) \mid y \in X\}$ and $\alpha = \min\{d(x) \mid x \in X\}$. We have $\alpha > 0$. Take $e = \frac{\alpha}{4}$.
Let $x \in X$ and open subset $U \subset X$ be a neighbourhood of x. There exists $z \in X$ such that for any arbitrary nonempty open subsets $U, V \subset X$ and any arbitrary neighbourhood of z, PWMP is satisfied. Let $W \subset X$ be arbitrary neighbourhood of z. There exists $z' \in X$ such that $d(z,z') > d(z) - \frac{\alpha}{8} > \alpha - \frac{\alpha}{8} = \frac{7}{8}\alpha$. Consider $W' \subset X$ be an arbitrary neighbourhood of z' small enough such that $d(W', W) > \frac{\alpha}{4}$. By PWMP, there exists $n \in \mathbb{N}$ such that

$$f_1^n(x) \in W,$$

$$f_1^n(U) \cap W' \neq \emptyset.$$

Then $U \cap f_1^{-n}(W') \neq \emptyset$. Consider $y \in U \cap f_1^{-n}(W')$. Thus $f_1^n(y) \in W'$ and $d(f_1^n(x), f_1^n(y)) > e = \frac{\alpha}{4}$.

$\square$

The following theorem is a part of Asymptotic Theorem that stated in [8]. We express and prove this theorem for NDS based on proof of Asymptotic Theorem.

Theorem 3.8. *If NDS $(X, f_{1,\infty})$ is sensitive, then there exists a positive ϵ such that $Asym_\epsilon(f_{1,\infty})$ is a first category subset of $X \times X$.*

Proof. It is sufficient to assume that if $Asym_\epsilon(f_{1,\infty})$ is not of first category, then NDS $f_{1,\infty}$ is not sensitive. For this, by he Baire category Theorem there exists nonempty open subset $U \subset A_{k,\epsilon}$ for some k, where $A_{k,\epsilon} = \bigcap_{i=k}^{\infty} (f_1^{-i} \times f_1^{-i})(\bar{V_\epsilon})$ and $Asym(f_{1,\infty}) = \bigcap_{\epsilon>0} Asym_\epsilon(f_{1,\infty})$. Then we have $U \times U \subset A_{k,2\epsilon}$ (By using the triangle inequality). Now, by shrinking U to an nonempty open subset $V \subset X$, we have $d(f_1^i(x), f_1^i(y)) \leq 2\epsilon$ for all $i \geq 0$ and all $x, y \in V$ that implies NDS $f_{1,\infty}$ is not sensitive.

$\square$

Proof of Theorem 3.5.

Proof. $f_{1,\infty}$ has PWMP then by Lemma 3.6, $Prox(f_{1,\infty})(x)$ is dense in X and by Lemma 3.7 is sensitive. Hence by Theorem 3.8 $Asym_\epsilon(f_{1,\infty})$ is a first category subset of $X \times X$. Then $Prox(f_{1,\infty}) \setminus Asym_\epsilon(f_{1,\infty})$ is dense by the Baire category theorem. Hence $f_{1,\infty}$ with PWMP, is Li-Yorke sensitive. □

Another main result of this paper is the following theorem.

Theorem 3.9. *Let $(X, f_{1,\infty})$ be a nonautonomous dynamical system that has PWMP, then $(X, f_{1,\infty})$ is densely Li-Yorke chaotic.*

In order to prove the Theorem 3.9, it is neccessary to have the following lemmas.

Lemma 3.10. [8] *Assume that X is a complete separable metric space without isolated points. If R is a symmetric relation with the property that there is a dense G_δ subset A of X such that for each $x \in A$, $R(x)$ contains a dense G_δ subset., then there is a dense subset B of X with uncountably many points such that $B \times B \setminus \triangle \subset R$.*

Lemma 3.11. [8] *Let R be a relation on a complete separable metric space X, which contains a dense G_δ subset of $X \times X$. Then there is a dense G_δ subset A of X such that for each $x \in A$, there exists a dense G_δ subset A_x of X with $\{(x, y) : x \in A, y \in A_x\} \subset R$.*

Proof of Theorem 3.9

Proof. Since $(X, f_{1,\infty})$ has PWMP, then by Lemma 3.6, $Prox(f_{1,\infty})$ is a dense G_δ subset of $X \times X$ and by Lemma 3.7 is sensitive. Hence by Theorem 3.8 $Asym(f_{1,\infty})$ is a first category subset of $X \times X$, then $LYR = Prox(f_{1,\infty}) \setminus Asym(f_{1,\infty})$ includs a dense G_δ subset of $X \times X$. By Lemma 3.11, there is a dense G_δ subset A of X such that for any $x \in X$, there is a dense G_δ set A_x with $\{(x, y) : x \in A, y \in A_x\} \subset LYR$. Finally, a dense, uncountable, scrambled set for $f_{1,\infty}$ can be founded using Lemma 3.10.

□

References

[1] Ethan Akin, Sergiy Kolyada, Li-Yorke sensitivity, Nonlinearity, vol. 16, 1421-1433 (2003) .

[2] John Banks, J. Brooks, Grant Cairns, G. Davis and P. Stacey ,On Devaneys definition of chaos, Amer. Math. Monthly, vol. 99, Number 4, 332-334 (1989).

[3] Francisco Balibrea, Piotr Oprocha, Weak mixing and chaos in nonautonomous discrete systems, Applied Mathematics letters, vol. 25, 1135-1141 (2012).

[4] Jose S. Cnovas, Li-Yorke chaos in a class of nonautonomous discrete systems, Journal of Difference Equations and Applications, vol. 17, Number 4, 479-486 (2011).

[5] Robert L. Devaney, An introduction to chaotic dynamical systems, Addison-Wesley (1989).

[6] Jana Dvorakova, Chaos in nonautonomous discrete dynamical systems, Commun Nonlinear Sci Numer Simulat vol.17, 4649-4652 (2012).

[7] Jana Hantakova, Li-Yorke sensitivity does not imply Li-Yorke chaos, Ergod Theory and Dynamical Systems, vol. 39, 1-9 (2018).

[8] Wen Huang, Xiangdong Ye, Devaney's chaos or 2-scattering implies Li-Yorke's chaos, Topology and it's Application vol. 117, 259-272 (2002).

[9] Anzelm Iwanik, Independent sets of transitive points, Dynamical Systems and Ergodic Theory, Banach Center Publ. vol. 23, 277-282 (1989).

[10] Tien-Yien Li, James A. Yorke, Period three implies chaos, Amer Math Monthly, vol. 82, 985-92 (1975).

[11] O. V. Rybak, Li-Yorke sensitivity for semigroup actions, Ukrainian Mathematical Journal, vol. 65, 752-759 (2013).

[12] Yuming Shi, Guanrong Chen, Chaos of time-varying discrete dynamical systems, Journal of Difference Equations and Applications, vol. 15, 429-449 (2009).

[13] Marta Tefnkov, Inheriting of chaos in uniformly convergent nonautonomous dynamical systems on the interval. Discrete and Continuous Dynamical Systems, vol. 36 , 3435-3443 (2016).