

ON A MULTIFRACTAL PRESSURE FOR SUSPENSION FLOWS

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ABSTRACT. In a previous article [*JDSGT* **17**, 267-295, 2019] we have extended to countable shifts the notion of multifractal pressure previously introduced by Olsen [*J. d'Analyse Math* **131**, 207-253, 2017]. This kind of pressure is defined by considering in the "partition function" only those configurations which are "multifractally relevant". In this article we continue working in this direction and consider a pressure of this nature, but for suspension flows over countable shifts.

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1. INTRODUCTION

Dimension Theory plays a relevant role within the discipline of Dynamical Systems [17]. Its main aim is to characterize sets by mean of adequate "dimensions" like, for instance, the Hausdorff dimension and the topological entropy. These are important invariants used to measure the complexity of the systems as well as the average amount of information. A general theory to study dimension of sets is the Carathéodory-Pesin dimension theory[22]. Central objects in this theory are the

so called *C-structures*, say pairs $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ is a collection of subsets of X which, in addition, satisfies some special conditions.

An important subject in the Dimension Theory of Dynamical Systems is the *Multifractal Analysis*. One of the main motivations for its development was the study of the behavior of physical measures supported on strange attractors. A way to analyze these attractors is to do a *fractal decomposition* of the involved invariant sets which generally have a complex mathematical structure. Such a decomposition is performed in the following way: let X be a set and let $g : X \rightarrow [-\infty, +\infty]$. The set X can be partitioned in level sets:

$$K_\alpha = K_\alpha(g) = \{x : g(x) = \alpha\}.$$

Now, let G be a function defined on sets, and let $F(\alpha) = G(K_\alpha)$, the map F is called *the multifractal spectrum* specified by the pair (g, G) . In this real valued case the most used spectra are defined by a limit process. For instance, in the so called *point wise dimension spectrum* we have as $g(x)$ the *point wise dimension of the measure* μ :

$$g(x) = D_\mu(x) := \lim_{r \rightarrow 0} \frac{\log(\mu(B_r(x)))}{-\log r}$$

and for F :

$$F(\alpha) = \dim_H K_\alpha,$$

the Hausdorff dimension.

Other examples are:

a) *Local entropies spectrum*. In this case

$$g(x) = h_\mu(f, x), \quad F(\alpha) = h_{top}(f, K_\alpha),$$

where $h_\mu(f, x)$, with $f : X \rightarrow X$, is the point wise entropy of the measure μ and $h_{top}(f, \cdot)$ is the topological entropy for non-compact nor invariant sets as defined by Bowen.

b) *Ergodic averages*. Here $g(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \varphi(f^i(x))$, with $\varphi : X \rightarrow \mathbf{R}$ (often called the potential).

The problem of describing these spectra has been extensively studied. A complete description can be done by means of a map which is the Legendre transform of the

multifractal map $F(\alpha)$ or by a variational formula [10], [1], [2], [3], [6], [24], [25], [18], [11], [19], [21], [13].

In reference [21], Olsen presented a notion of *multifractal pressure*. This concept was inspired in previous articles [15], [16], where zeta functions were introduced to analyze fractal objects. The novelty in these articles lies in the fact that the "partition function" used to define the pertinent functions considers not all the configurations, but just those which have "multifractal relevance". From this partition function the mentioned multifractal pressure can be obtained as well as the radius of convergence of the series defining the dynamical zeta function. Thus, the terminology of "multifractal" is motivated by the fact that special cases can lead to multifractal descriptions of particular spectra. The setting considered by Olsen in [21] for the dynamics, involves symbolic spaces with a finite alphabet. In reference [20] we made a similar analysis but for countable alphabet shifts. The objective of the present article is to obtain results like those, but for suspension flows over countable shifts.

2. MULTIFRACTAL PRESSURE FOR SUSPENSION FLOWS

Let us consider a countable set Ω , for instance $\Omega = \mathbf{N}$, and a transition matrix $A = (A_{i,j})_{i,j \in \Omega}$ with $A_{i,j} \in \{0, 1\}$. Let

$$(1) \quad \sum = \{x = x_0 x_1 \dots, x_i \in \Omega, A_{x_i, x_{i+1}} = 1\}$$

The space $\sum$ is endowed with the product topology of the discrete topology in Ω , or equivalently, the topology generated by cylinders

$$\mathcal{C}(i_1 i_2 \dots i_n) = \{x : x_j = i_j, j = 1, \dots, n\}.$$

Let $\underline{i} = i_1 i_2 \dots i_n \in \Omega^n$, we denote its length by $|\underline{i}| = n$. Such a string is called *admissible* if $\mathcal{C}(i_1 i_2 \dots i_n) \neq \emptyset$. We denote by $\mathcal{C}(\underline{i})\mathcal{C}(\underline{j})$ the cylinder corresponding to the concatenation of the strings $\underline{i}$, $\underline{j}$, i.e. $\mathcal{C}(i_1 i_2 \dots i_n j_1 j_2 \dots j_n)$. The space $\sum$ is besides endowed with the metric $d(x, y) = t^{-N(x, y)}$ where $N(x, y) =$ first coordinate in which differ x and y , and $0 < t < 1$.

We shall consider dynamics given by the shift $\sigma : \sum \rightarrow \sum$, with $(\sigma x)_i = x_{i+1}$. The dynamical system $(\sum, \sigma)$, with a transition matrix A and an alphabet Ω , is called a *topological Markov shift*.

Definition: A topological Markov shift is topologically mixing if for any symbols $i, j \in \Omega$ there is an admissible string $\underline{a} = a_1 a_2 \dots a_n$ with $a_1 = i$, $a_n = j$.

Denote $\mathcal{A}$ the natural partition by cylinders of Σ : $\{\mathcal{C}(i) : i \in \Omega\}$ and $\mathcal{A}^n = \{\mathcal{C}(i_1 i_2 \dots i_n)\}$. Let $A \subset \sigma^{-n}(\mathcal{C}(\underline{i})) \cap \mathcal{A}^n$ and $B \subset \mathcal{C}(\underline{j}) \cap \mathcal{A}^n$, we use $A.B$ for $\bigcup_{\substack{\mathcal{C}(\underline{i}) \subset A \\ \mathcal{C}(\underline{j}) \subset B}} \mathcal{C}(\underline{i}) \mathcal{C}(\underline{j})$.

Definition: A potential $\varphi : \Sigma \rightarrow \mathbf{R}$, is *locally Hölder* if $V_n(\varphi) \leq A\theta^n$, for some constants $A > 0$ and $\theta \in (0, 1)$, where

$$V_n(\varphi) = \sup_{i_1 i_2 \dots i_n \in \mathbf{N}^n} \sup_{x, y \in \mathcal{C}(i_1 i_2 \dots i_n)} \{|\varphi(x) - \varphi(y)|\}$$

Let us call $\mathcal{H}$ the class of locally Hölder functions.

For countable shifts Σ we have considered in [20] a *multifractal pressure* in the sense of Gurevic[8] as follows: let $\mathcal{E}_n(x)$ be the empirical measure given as $\mathcal{E}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\sigma^i(x)}$, and let $E : \mathcal{M}(\Sigma) \rightarrow X$, be an affine functional with X a metric space. Let C be a subset of X , and φ be a locally Hölder functions and set

$$Z_n^C(E, \varphi, \sigma, i) = \sum_{\substack{x \in P_n(\sigma) \\ E\mathcal{E}_n(x) \in C}} \exp(S_n(\varphi)(x)) I_{C(i)},$$

where I_E means the characteristic function of the set E , $P_n(\sigma)$ is the set of n -periodic points of σ , i. e. $P_n(\sigma) = \{x : \sigma^n(x) = x\}$ and $S_n(\varphi)(x) = \sum_{i=0}^{n-1} \varphi(\sigma^i(x))$ (called the *statistical sum*). Besides $E\mathcal{E}_n$ denotes the composition of E and $\mathcal{E}_n$.

The *multifractal pressure* for a potential $\varphi \in \mathcal{H}$ and for topologically mixing countable shift Σ are defined as

$$P_{Gur}(E, \varphi, \sigma) = \lim_{t \rightarrow \infty} \frac{1}{n} \log Z_n^C(E, \varphi, \sigma, i).$$

When the shift is topologically mixing and $\varphi \in \mathcal{H}$ the limit in the r.h.s. does not depend on the symbol i considered.

As we mentioned earlier the term multifractal alludes to the possibility of getting multifractal description of spectra obtained as particular cases. For instance the multifractal spectrum of point-wise dimension of a measure corresponds to the

particular case of the functional $E\mu = \frac{\int \varphi d\mu}{\int \psi d\mu}$ and the objective is to analyze the following multifractal spectrum:

$$K_{C,\mu} = \left\{ x : A_{r \searrow 0} \left(\frac{\log(\mu(B_r(x)))}{-\log r} \right) \subset C \right\},$$

where here $A(x_n)$ denotes the set of accumulation points of the sequence x_n , $C \subset \mathbf{R}$ and μ is a Gibbs state for the potential φ . The idea is to apply the approximation principle and the variational principle.

In our case, for the single case $C = \{\alpha\}$ the spectrum is

$$K_{\alpha,\mu} = \left\{ x \in \Sigma : \alpha \in A_{r \searrow 0} \left(\frac{\log(\mu(B_r(x)))}{-\log r} \right) \right\},$$

whereas the usual spectrum of point-wise dimension of measures is

$$K_{\alpha,\mu} = \left\{ x \in \Sigma : \lim_{r \rightarrow 0} \left(\frac{\log(\mu(B_r(x)))}{-\log r} \right) = \alpha \right\}.$$

Let $\tau : \Sigma \rightarrow \mathbf{R}^+$ be a continuous map (called the roof function) with Σ a countable alphabet shift and let us form the space

$$\Sigma_\tau = \{(x, t) : x \in \Sigma, 0 \leq t \leq \tau(x)\},$$

with the identifications $(x, \tau(x)) \sim (\sigma x, 0)$. The *suspension flow* over the shift (Σ, σ) is defined as

$$(2) \quad \begin{aligned} \text{Susp} &= \{S_t : \Sigma_\tau \rightarrow \Sigma_\tau\} \\ S_t(x, s) &= (x, t + s). \end{aligned}$$

Let $\mathcal{M}_{\text{Susp}}(\Sigma_\tau)$ be the set of *Susp*-invariant probability measure on Σ_τ , with topologically mixing countable shift Σ , and let $E : \mathcal{M}(\Sigma_\tau) \rightarrow \mathbf{R}$ be an affine functional and continuous with respect to weak-topology in $\mathcal{M}(\Sigma_\tau)$. For a continuous map $\varphi : \Sigma_\tau \rightarrow \mathbf{R}$ we consider the empirical measure given as $\mathcal{E}_t(x, s) = \frac{1}{t} \int_0^t \delta_{S_r(x, s)} dr$, with δ the Dirac point mass measure. Let $\varphi : \Sigma_\tau \rightarrow \mathbf{R}$ be a continuous map such that the function

$$\Delta_{\varphi}(x) = \int_0^{\tau(x)} \varphi(x, t) dt,$$

be locally Hölder. For a function φ such that $\Delta_{\varphi} \in \mathcal{H}$, $C \subset X$, set

$$(3) \quad Z_t^C(E, \varphi, Susp, i) = \sum_{\substack{x: S_r(x, 0) = (x, 0) \\ E\mathcal{E}_t(x) \in C}} \exp\left(\int_0^s (S_r(x, 0) dr) I_{C(i)}\right).$$

Then

$$(4) \quad P_{Gur}^C(E, \varphi, Susp) = \lim_{t \rightarrow \infty} \frac{1}{t} \log Z_t^C(E, \varphi, Susp, i)$$

is the *multifractal Gurevic pressure* for the suspension flow and for the potential φ . Like in the definition for shifts, if it is assumed that the shift of the basis of the suspension space is topologically mixing and φ is such that $\Delta_{\varphi} \in \mathcal{H}$, then the symbol is omitted in the expression for the pressure.

As in the case of shifts a main aim here is to enlarge the "shrinking target approach" done in the usual Multifractal Analysis, where a single "target" at level α is considered. As we mentioned in our earlier article[20] one approach is to enlarge the point, for instance, to an interval $I = [\alpha - r, \alpha + r]$, shrinking to α as $r \searrow 0$ and relate this behavior with the topological pressure, the zeta function and multifractal spectra at α . If C_r is the ball $B_r(C)$, thus the analysis is related to the shrinking as $r \searrow 0$.

One objective is now to prove for suspension flows, the results obtained for shifts in [20]. This is done by transferring the results obtained for the shift on the base Σ to the suspension flow on Σ_{τ} . For this it will be convenient to introduce a definition of multifractal pressure based on that given by Barreira and Iommi[4] and to prove the equivalence between both definitions, like is done for the usual pressure by Kempton[14].

Let $\mathcal{M}_{\sigma}(\Sigma)$ be the set of σ -invariant probability measures on Σ and for a map $\varphi: \Sigma \rightarrow \mathbf{R}$, let us denote by $\mathcal{M}(\sigma, \varphi)$ the subset of $\mathcal{M}_{\sigma}(\Sigma)$ consistent of those measures ν such that $-\int \varphi d\nu < \infty$. There is a natural identification between $\mathcal{M}_{Susp}(\Sigma_{\tau})$ and $\mathcal{M}(\sigma, -\tau)$ [4];

$$R: \mathcal{M}(\sigma, -\tau) \rightarrow \mathcal{M}_{Susp}(\Sigma_\tau)$$

$$R(\nu) = \frac{(\nu \times m) \mid \Sigma_\tau}{(\nu \times m) \mid (\Sigma_\tau)},$$

where m is the Lebesgue measure on $\mathbf{R}$. This map is bijective and holds

$$\int_{\Sigma_\tau} \varphi dR(\nu) = \frac{\int_{\Sigma} \Delta_\varphi d\nu}{\int_{\Sigma} \tau d\nu}.$$

The Abramov formula establishes that if $\mu = R(\nu) \in \mathcal{M}_{Susp}(\Sigma_\tau)$ then

$$h_\mu(Susp) = \frac{h_\nu(\sigma)}{\int_{\Sigma} \tau d\nu},$$

where $h_\mu(Susp)$, $h_\nu(\sigma)$ denote the entropies of the suspension flow for μ and the shift for ν respectively.

For potentials φ in the class $\mathcal{H}$ of locally Hölder maps and for topologically mixing shifts Σ satisfying the *BIP*-property (which means finitely reducible) the following results have been proved[20]:

- *Approximation theorem for the multifractal pressure:*

$$P_{Gur}^C(E, \varphi, \sigma) = \sup \left\{ P_{Gur}^C(E, \varphi \mid K, \sigma \mid K) \right\},$$

with the supremum taken over all compact subset K of Σ , i.e. over all finite alphabet subshifts

- *Variational principle:* Let $E: \mathcal{M}_\sigma(\Sigma) \rightarrow X$ be an affine functional, with X a compact metric space. Let φ be a potential in the class $\mathcal{H}$ and $C \subset X$, and Σ be a topologically mixing shift satisfying the *BIP*-property

If C_r denotes the ball $B_r(C)$ then

$$(5) \quad \lim_{r \rightarrow 0} P_{Gur}^{C_r}(E, \varphi, \sigma) =$$

$$\sup \left\{ h_\mu(\sigma) + \int \varphi d\mu : - \int \varphi d\mu < \infty : \mu \in \mathcal{M}_\sigma(\Sigma) \text{ } E\mu \in C \right\}.$$

The condition of the *BIP*-property is imposed to ensure the existence of Gibbs measure in Σ . This was essential to develop a large deviation theory used to prove the variational principle for shifts in [20].

If $E : \mathcal{M}_\sigma(\Sigma) \rightarrow X$ is an affine functional, then we denote by $\tilde{E} : \mathcal{M}_{Susp}(\Sigma_\tau) \rightarrow X$ the natural extension to the suspension space.

The approximation theorem for multifractal Gurevic pressure for the suspension flow is formulated as

$$P_{Gur}^C(\tilde{E}, \varphi, Susp) = \sup \left\{ P_{Gur}^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) : K_\tau \subset \Sigma_\tau \text{ compact} \right\}.$$

In reference [4], Barreira and Iommi introduced a pressure for suspension flows, from this concept we give this definition of multifractal pressure for suspension flows as follows:

$$P_{BI}^C(\tilde{E}, \varphi, Susp) = \inf \left\{ t \in \mathbf{R} : P^C(E, \Delta_\varphi(x) - t\tau, \sigma) \leq 0 \right\}.$$

Proposition 1: If it is assumed that the approximation theorem for suspension flows is valid then

$$(6) \quad P_{Gur}^C(\tilde{E}, \varphi, Susp) = P_{BI}^C(\tilde{E}, \varphi, Susp).$$

Proof: The demonstration is similar to that given in reference [14]. By the assumption

$$\begin{aligned} P_{Gur}^C(\tilde{E}, \varphi, Susp) &= \sup \left\{ P_{Gur}^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) : K_\tau \subset \Sigma_\tau \text{ compact} \right\} \\ &= \inf \left\{ t \in \mathbf{R} : P_{Gur}^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) \leq t : \forall K_\tau \subset \Sigma_\tau \text{ compact} \right\}. \end{aligned}$$

The compact subsets K_τ of Σ_τ can be naturally identified by subcompacts K_τ of Σ . From this and the result $P^C(E, \Delta_\varphi - P_{Gur}^C(E, \varphi, Susp)\tau, \sigma) = 0$, which is originated in a result by Bowen and Ruelle[5], we can write

$$\begin{aligned}
 & P_{Gur}^C(\tilde{E}, \varphi, Susp) \\
 = & \inf \{t \in \mathbf{R} : P_{Gur}^C(E, (\Delta_\varphi(x) - t\tau) \mid K, \sigma) \leq 0 : \forall K \subset \Sigma \text{ compact}\} \\
 = & \inf \{t \in \mathbf{R} : P_{Gur}^C(E, (\Delta_\varphi(x) - t\tau), \sigma) \leq 0\} \\
 = & P_{BI}^C(\tilde{E}, \varphi, Susp).
 \end{aligned}$$

The second equality follows for the approximation theorem of the pressure for shifts by compact invariant subsets. ■

In the next section we shall prove the approximation theorem.

3. VARIATIONAL PRINCIPLE AND APPROXIMATION THEOREM FOR SUSPENSION FLOWS.

The following properties can be proved in analogue way as for shifts. This was done in [20] for the multifractal pressure, following [23].

Lemma 1: Let $\sum$ be a topologically mixing shift with countable alphabet Ω and let φ such that $\Delta_\varphi \in \mathcal{H}$ then

i) For any $i, j \in \Omega$ there is a number s such that

$$Z_t^C(E, \varphi, Susp, i) \leq s Z_{t+s}^C(E, \varphi, Susp, j).$$

ii) For any $i \in \Omega$, and any t, s is

$$Z_t^C(E, \varphi, Susp, i) \leq W_1^3 Z_{t+s}^C(E, \varphi, Susp, i),$$

where $W_n(\varphi) = \exp(\sum_{k>n} V_k(\varphi))$, with

$$V_n(\varphi) = \sup_{i_1 i_2 \dots i_n \in \mathbf{N}^n} \sup_{x, y \in \mathcal{C}(i_1 i_2 \dots i_n)} \{|\varphi(x) - \varphi(y)|\}.$$

From these properties follows that the expression $\lim_{t \rightarrow \infty} \frac{1}{t} \log Z_t^C(E, \varphi, Susp, i)$ does not depend on the limit considered. Indeed let

$a^C(t) = \log Z_t^C(E, \varphi, Susp, i)$, then if $r = 3 \log W_1$ gets $a^C(t+s) + r > a^C(t) + a^C(s)$, thus the limit $\lim_{t \rightarrow \infty} \frac{1}{t} a^C(t)$ exists and is independent of the symbol.

Theorem 1 (approximation theorem): Let $\sum$ be a topologically mixing shift and Σ_τ be the corresponding suspension space. Let $E : \mathcal{M}_\sigma(\sum) \rightarrow X$ be an

affine functional, and $\tilde{E} : \mathcal{M}_{Susp}(\Sigma_\tau) \rightarrow X$ its natural extension to the suspension space. Finally, let φ be such that Δ_φ is locally Hölder, then

$$(7) \quad P_{Gur}^C(\tilde{E}, \varphi, Susp) = \sup \left\{ P_{Gur}^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) : K_\tau \subset \Sigma_\tau \text{ compact} \right\}.$$

Proof: Let $\sum_m = \{x \in \Sigma : x_n \leq m\}$, and let us consider the suspension space over the subshift $\sum_m \cap \Sigma$ and with roof function τ . We denote this space $(\sum_m \cap \Sigma)_\tau$. Let

$$Z_{t,m}^C(\tilde{E}, \varphi, Susp) = Z_t^C(\tilde{E}, \varphi \mid (\Sigma \cap \sum_m)_\tau, Susp \mid (\Sigma \cap \sum_m)_\tau),$$

and let $r > 0$ enough big and $\varepsilon > 0$ such that

$$\frac{1}{r} \log Z_r^C(\tilde{E}, \varphi, Susp) < \frac{1}{r} \log Z_{r,m}^C(\tilde{E}, \varphi, Susp).$$

It may be added states (symbols) to the alphabet of $\sum_m$ until to obtain a compact suspension space $K_\tau \subset \sum_\tau$ over a topologically mixing shift. We have

$$\frac{1}{r} \log Z_r^C(\tilde{E}, \varphi, Susp) < \frac{1}{r} \log \left(Z_t^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) \right).$$

Let $g^C(t) = \log Z_t(E, \varphi \mid K_\tau, Susp \mid K_\tau)$, then, by the above lemma, we have $g^C(t) + g^C(s) < g^C(t+s) + 3 \log W_1$. Let $t = sw + r$, $r \in [0, w-1]$, thus

$$\frac{wg^C(s) + g^C(r)}{t} \leq \frac{g^C(sw+r) + (3w+1) \log W_1}{sw+r} \leq$$

$$\frac{g^C(t)}{sw+r=t} + \left(\frac{w+1}{w} \right) \varepsilon.$$

For $t \rightarrow \infty$ we have

$$P_{Gur}^C(\tilde{E}, \varphi, Susp) \leq P_{Gur}^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) + 3\varepsilon,$$

and so

$$(8) \quad P_{Gur}^C(\tilde{E}, \varphi, Susp) \leq \sup \left\{ P_{Gur}^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) : K_\tau \subset \Sigma_\tau \text{ compact} \right\}.$$

To prove the other inequality, we have that for any compact subset K_τ of Σ_τ holds $P_{Gur}^C(\tilde{E}, \varphi \mid K_\tau, Susp \mid K_\tau) \leq P_{Gur}^C(\tilde{E}, \varphi, Susp)$, hence

$$P_{Gur}^C(E, \varphi, Susp) = \sup \{ P_{Gur}^C(E, \varphi \mid K_\tau, Susp \mid K_\tau) : K_\tau \subset \Sigma_\tau \text{ compact} \}.$$

■

Theorem 2 (variational principle): Let Σ be a topologically mixing shift satisfying the *BIP*-property, with Σ_τ the corresponding suspension space. Let $\varphi : \Sigma_\tau \rightarrow \mathbf{R}$ be a continuous map such that Δ_φ is locally Hölder. Let $E : \mathcal{M}(\Sigma_\tau) \rightarrow X$ be an affine functional, with the natural extension to the suspension space $\tilde{E} : \mathcal{M}_{Susp}(\Sigma_\tau) \rightarrow X$. Then

$$\begin{aligned} P_{Gur}^C(\tilde{E}, \varphi, Susp) &= \\ &= \sup \left\{ h_\mu(Susp) + \int \varphi d\mu : - \int \varphi d\mu < \infty : \mu \in \mathcal{M}(\Sigma_\tau), \tilde{E}\mu \in C \right\}. \end{aligned}$$

Proof: Using the Barreira-Iommi definition for the multifractal Gurevic pressure we have that if $t > P_{Gur}^C(\tilde{E}, \varphi, \sigma)$ then $P^C(E, \Delta_\varphi(x) - t\tau, \sigma) \leq 0$, and so, applying the variational principle in the shift base [20],

$$\sup \left\{ h_\nu(\sigma) + \int \Delta_\varphi d\nu - t \int_\Sigma \tau d\nu : \nu \in \mathcal{M}(\sigma, -\tau) \cap \mathcal{M}(\sigma, \Delta_\varphi), E\nu \in C \right\} \leq 0,$$

we have

$$\begin{aligned} \sup \left\{ \int_\Sigma \tau d\nu \left[\frac{h_\nu(\sigma)}{\int_\Sigma \tau d\nu} + \frac{\int_\Sigma \Delta_\varphi d\nu}{\int_\Sigma \tau d\nu} - t \right] : \nu \in \mathcal{M}(\sigma, -\tau) \cap \mathcal{M}(\sigma, \Delta_\varphi), E\nu \in C \right\} \\ \leq 0. \end{aligned}$$

For any measure ν with $E\nu \in C$, we consider the measure $\mu = R(\nu)$ with $\tilde{E}\mu \in C$, thus by the Abramov formula, the relationship $\int_{\Sigma_\tau} \varphi dR(\nu) = \frac{\int_\Sigma \Delta_\varphi d\nu}{\int_\Sigma \tau d\nu}$ and $\int_\Sigma \tau d\nu > 0$, we obtain

$$\sup \left\{ h_\mu(Susp) + \int \varphi d\mu : - \int \varphi d\mu < \infty : \mu \in \mathcal{M}(\Sigma_\tau), \tilde{E}\mu \in C \right\} \leq t,$$

and since $t > P_{Gur}^C(\tilde{E}, \varphi, \sigma)$ we have

$$\begin{aligned} P_{Gur}^C(\tilde{E}, \varphi, Susp) &\geq \\ &\geq \sup \left\{ h_\mu(Susp) + \int \varphi d\mu : - \int \varphi d\mu < \infty : \mu \in \mathcal{M}(\Sigma_\tau), \tilde{E}\mu \in C \right\}. \end{aligned}$$

To prove the other inequality we recall this concept: A measure $\mu \in \mathcal{M}(\Sigma_\tau)$ is *compactly supported* if there is a compact set $K_\tau \subset \Sigma_\tau$ such that $\mu(K_\tau) = 1$. By the approximation theorem

$$P_{Gur}^C(\tilde{E}, \varphi, Susp) \leq \sup \left\{ h_\mu(Susp) + \int \varphi d\mu \right\}.$$

where the supremum is taken over all the measures compactly supported measures $\mu \in \mathcal{M}(\Sigma_\tau)$ with $\tilde{E}\mu \in C$. Thus

$$\begin{aligned} P_{Gur}^C(\tilde{E}, \varphi, Susp) &\leq \\ &\leq \sup \left\{ h_\mu(Susp) + \int \varphi d\mu : - \int \varphi d\mu < \infty : \mu \in \mathcal{M}(\Sigma_\tau), \tilde{E}\mu \in C \right\}. \end{aligned}$$

■

Regularity properties of the map $q \mapsto P_{Gur}^C(\tilde{E}, q\varphi, Susp)$ can be studied from the map $q \mapsto P_{Gur}^C(E, (\Delta_\varphi(x) - t\tau), \sigma)$, which in turn can be analyzed from the functions $q \mapsto P_{Gur}^C(E, -t\tau, \sigma)$. To do this it must be assumed that the shift has the *BIP*-property and the map is bounded and such that Δ_φ is locally Hölder. Recall [20] that for the the map $q \mapsto P_{Gur}^C(E, -t\tau, \sigma)$ there is a critical value $q^* \in (0, 1]$ such that for $q < q^*$ holds $P_{Gur}^C(E, -t\tau, \sigma) = \infty$, and for $q > q^*$ we have that $P_{Gur}^C(E, -t\tau, \sigma) < \infty$ Besides for $q > q^*$ that map is real analytic This facts are established in a direct way, following the thermodynamic formalisms of Mauldin-Urbanski or Sarig, for the usual case(see also [12]). If $P_{Gur}^C(E, -t\tau, \sigma) < \infty, \dim_H \Sigma < \infty$ then there are a value $F(C)$ such that $P_{Gur}^C(E, -F(C)\tau, \sigma) = 0$ (multifractal Bowen equation).

Further from these results, in a similar way than in [4], may be proved that there is value $F(C)$ such that $P_{Gur}^C(E, \Delta_\varphi - F(C)\tau, \sigma) = 0$. Also can be established that

if the map $q \mapsto P_{Gur}^C(E, \Delta_\varphi - q\tau, \sigma)$ has a root and then is finite. Therefore this pressure map is finite and so the map $q \mapsto P_{Gur}^C(\tilde{E}, q\varphi, Susp)$ is real analytic.

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