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CHARACTERISTIC OF POINTWISE-RECURRENT MAPS ON A DENDRITE

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Abstract. Let f be a continuous map on a dendrite D with $f(D) = D$. Denote by $R(f)$ and $AP(f)$ the set of recurrent points and the set of almost periodic points of f, respectively, and denote by $\omega(x, f)$, $\Lambda(x, f)$, $\Gamma(x, f)$ and $\Omega(x, f)$ the set of ω-limit points, the set of α-limit points, the set of γ-limit points and the set of weak ω-limit points of x under f, respectively. In this paper, we show that the following statements are equivalent: (1) $D = R(f)$. (2) $D = AP(f)$. (3) $\Omega(x, f) = \omega(x, f)$ for any $x \in D$. (4) $\Omega(x, f) = \Gamma(x, f)$ for any $x \in D$. (5) f is equicontinuous. (6) $[c, d] \not\subset \Omega(x, f)$ for any $c, d, x \in D$ with $c \neq d$. (7) $\Omega(x, f)$ is minimal for any $x \in D$. (8) $\mathrm{Card}(\Lambda^{-1}(x, f) \cap (D - \mathrm{End}(D))) < \infty$ for any $x \in D$, where $\Lambda^{-1}(x, f) = \{y : x \in \Lambda(y, f)\}$, $\mathrm{End}(D)$ is the set of endpoints of D and $\mathrm{Card}(A)$ is the cardinal number of set A. (9) If $x \in \Lambda(y, f)$ with $x, y \in D$, then $y \in \omega(x, f)$. (10) Map $h : x \longrightarrow \omega(x, f)$ $(x \in D)$ is continuous and for any $x, y \in D$ with $x \notin \omega(y, f)$, $\omega(x, f) \neq \omega(y, f)$. Besides, we also study characteristic of pointwise-recurrent maps on a dendrite with the number of branch points being finite.

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1. Introduction

Let f be a continuous map on a compact metric space X (briefly, $f \in C^0(X)$). If every point of X is recurrent, then f is said to be pointwise-recurrent. It is easy to see that if f is pointwise-recurrent, then $f(X) = X$. Recently there has been a great interest in studying pointwise-recurrent maps and flows (see [2,10,18]). In [13], Mai showed that if G is a graph and $f \in C^0(G)$, then f is pointwise-recurrent if and only if one of the following two statements holds: (1) G is a circle and f is a homeomorphism topologically conjugate to an irrational rotation of the unit circle. (2) f is a periodic homeomorphism. In [17], Naghmouchi showed that the following two statements hold: (1) if D is a dendrite whose set of endpoints $\mathrm{End}(D)$ is countable and $f \in C^0(D)$, then f is pointwise-recurrent if and only if f is a pointwise-periodic homeomorphism. (2) if D is a dendrite whose set of branch points $\mathrm{B}(D)$ is discrete, then f is pointwise-recurrent if and only if every point of $D-\mathrm{End}(D)$ is periodic. In [6], Blokh showed that if D is a uniquely arcwise connected locally arcwise connected space and $f \in C^0(D)$, then f is pointwise-recurrent if and only if every point of $D-\mathrm{End}(D)$ is periodic. In [19], Su and Qin studied the pointwise-recurrence and the equicontinuity of a dendrite flow and obtained some equivalent conditions of an equicontinuous dendrite flow. In [21], Sun et al. obtained some equivalent conditions of equicontinuous maps on a dendrite with the number of branch points being finite. For some results on discrete dynamical systems on dendrites, we refer the readers to see [1,3,4,8,9,12,14,16,20,22-24] and the related references therein. In this paper, we will study characteristic of pointwise-recurrent maps on a dendrite.

The rest of this paper is organized as follows. In Section 2 we give some definitions and notations and main result of this paper. In Section 3 we study the minimality of the set of weak ω-limit points of a continuous map on a compact metric space. In Section 4 we study the pointwise-recurrent maps on a dendrite and give the proof of the main result. In Section 5 we study the pointwise-recurrent maps on a dendrite with the number of branch points being finite.

2. Preliminaries and statement of main result

In the following, we let $\mathbb{N}$ denote the set of all positive integers. Set $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. Denote by $C^0(X)$ the set of all continuous self-maps on a compact metric space (X, d). For any $A \subset X$, denote by $\overline{A}$ and $\mathrm{Card}(A)$ the closure and the cardinal number of A, respectively. For any $x \in X$ and $\varepsilon > 0$, write $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. For any $f \in C^0(X)$ and $n \in \mathbb{N}$, write $f^n = f \circ f^{n-1}$, where f^0 is the identity map on X. For any $x \in X$, the set $O(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}^+\}$ and the set $\omega(x, f) = \{y \in X : \text{there exist } n_1 < n_2 < n_3 < \cdots \text{ such that } \lim_{k \longrightarrow \infty} f^{n_k}(x) = y\}$ are called the orbit and the set of ω-limit points of x under f, respectively. The set $P_n(f) = \{x \in X : f^n(x) = x \text{ and } f^i(x) \neq x \text{ for any } 1 \leq i \leq n-1 \text{ if } n \geq 2\}$, the set $R(f) = \{x : x \in \omega(x, f)\}$ and the set $AP(f) = \{x \in X : \text{for any } \varepsilon > 0,$ there exists an $N \in \mathbb{N}$ such that $B(x, \varepsilon) \cap \{f^{m+k}(x)) : 0 < k \leq N\} \neq \emptyset$ for any $m \in \mathbb{Z}^+\}$ are called the set of n-periodic points, the set of recurrent points and the set of almost periodic points of f, respectively. By Corollary 5.6 of [5] we see that $x \in AP(f)$ if and only if $x \in \omega(y, f)$ for every $y \in \omega(x, f)$. A sequence $\theta = (x_0, x_{-1}, \cdots, x_{-n}, \cdots)$ of points in X is called a negative orbit through x_0 of f if $f(x_{-n}) = x_{-n+1}$ for every $n \in \mathbb{N}$. Write $\alpha(\theta, f) = \{y \in X : \text{there exist } n_1 < n_2 < n_3 < \cdots \text{ such that } \lim_{k \longrightarrow \infty} x_{-n_k} = y\}$, which is called the set of α-limit points of θ under f. Note that in the general case, the negative orbit through x may not exist and, even if it does exist, it may not be unique. We also need the following notations:

$$
\begin{aligned}
S\Lambda(x, f) &= \cup\{\alpha(\theta, f) : \theta \ \text{ is a negative orbit through } x \text{ of } f\ \}. \\
\Lambda(x, f) &= \{y : \ \text{there exist } \ x_1, x_2, \cdots \in X \text{ and } \ n_1 < n_2 < \cdots \\
&\qquad \text{such that } \lim_{k \longrightarrow \infty} x_k = y \text{ and } f^{n_k}(x_k) = x\}. \\
\Gamma(x, f) &= \omega(x, f) \cap \Lambda(x, f). \\
\Omega(x, f) &= \{y : \ \text{ there exist } \ x_1, x_2, \cdots \in X \text{ and } \ n_1 < n_2 < \cdots \\
&\qquad \text{such that } \lim_{k \longrightarrow \infty} x_k = x \ \text{ and } \ \lim_{k \longrightarrow \infty} f^{n_k}(x_k) = y\}.
\end{aligned}
$$

$S\Lambda(x, f)$, $\Lambda(x, f)$, $\Gamma(x, f)$ and $\Omega(x, f)$ are called the set of special α-limit points, the set of α-limit points, the set of γ-limit points and the set of weak ω-limit points of x under f, respectively. $P(f) = \cup_{n=1}^{\infty} P_n(f)$ is called the set of periodic points of f. It is easy to see that

(1) $\Omega(x,f) \supset \omega(x,f) \supset \Gamma(x,f)$.

(2) $\Lambda(x,f) \supset \Gamma(x,f)$ and $\Lambda(x,f) \supset S\Lambda(x,f)$.

(3) If $y \in \Lambda(x,f)$, then $x \in \Omega(y,f)$.

(4) $R(f) \supset AP(f) \supset P(f)$.

(5) $\Omega(x,f)$, $\omega(x,f)$ and $\alpha(\theta,f)$ are closed, and $f(\Omega(x,f)) = \Omega(x,f)$, $f(\omega(x,f)) = \omega(x,f)$ and $f(\alpha(\theta,f)) = \alpha(\theta,f)$.

In this paper, we write

$$\begin{aligned} \Omega E\omega(f) &= \{x \in X : \Omega(x,f) = \omega(x,f)\}, \\ \Omega E\Gamma(f) &= \{x \in X : \Omega(x,f) = \Gamma(x,f)\}. \end{aligned}$$

A locally connected and compact connected metric space is called a dendrite if it contains no simple closed curve. For any nondegenerate dendrite D and $x \in D$. Denote by $V(x)$ the number of connected components of $D - \{x\}$. x is called an endpoint of D if $V(x) = 1$ and a branch point of D if $V(x) \geq 3$. Denote by $\mathrm{End}(D)$ and $\mathrm{B}(D)$ the set of all endpoints and the set of all branch points of D, respectively. A space is called an arc if it homeomorphic to $[0,1]$. By [15] we see that for any $a, b \in D$ with $a \neq b$, there is a unique arc in D, denoted by $[a,b]$, such that $\mathrm{End}([a,b]) = \{a,b\}$. Write $[a,b) = (b,a] = [a,b] - \{b\}$ and $(a,b) = [a,b) - \{a\}$. In addition, we write $[a,a] = \{a\}$.

Let D be a dendrite. For any $x \in D$ and any closed subset A of D, write $d(x,A) = \inf_{a \in A} d(x,a)$. Let U and V be two closed subsets of D, we define the Hausdorff metric of U and V by

$$H_d(U,V) = \max\{\sup_{u \in U} d(u,V), \sup_{v \in V} d(v,U)\}.$$

Now we state the main result of this paper.

Theorem 4.6 Let D be a dendrite and $f \in C^0(D)$ with $f(D) = D$. Then the following statements are equivalent.

(1) $D = R(f)$.

(2) $D = AP(f)$.

(3) $D = \Omega E\Gamma(f)$.

(4) $D = \Omega E\omega(f)$.

(5) $f \in EC(D)$ (see Definition 4.1).

(6) $[c,d] \not\subset \Omega(x,f)$ for any $c, d, x \in D$ with $c \neq d$.

(7) $\Omega(x,f)$ is minimal for any $x \in D$.

(8) $\mathrm{Card}(\Lambda^{-1}(x,f)\cap(D-\mathrm{End}(D)))<\infty$ for any $x\in D$, where $\Lambda^{-1}(x,f)=\{y: x\in\Lambda(y,f)\}$.

(9) If $x\in\Lambda(y,f)$ with $x,y\in D$, then $y\in\omega(x,f)$.

(10) Map $h: x\longrightarrow\omega(x,f)$ $(x\in D)$ is continuous and for any $x,y\in D$ with $x\notin\omega(y,f)$, $\omega(x,f)\neq\omega(y,f)$.

3. Minimality of the set of weak ω-limit points

In this section, we will obtain some equivalent conditions of minimality of the sets of weak ω-limit points of maps on a compact metric space X. Let $f\in C^0(X)$ and $A\subset X$ be non-empty, A is called a minimal set under f if $\overline{O(x,f)}=A$ for any $x\in A$.

Lemma 3.1 (see [11]) Let $f\in C^0(X)$. If $y\in X$ and $x\in S\Lambda(y,f)$, then $\overline{O(x,f)}\subset S\Lambda(y,f)$.

Lemma 3.2 Let $f\in C^0(X)$ and $f(X)=X$.

(1) If $x\in\omega(y,f)$ for some $y\in X$, then $x\in\Omega(x,f)$.

(2) If $X=R(f)$, then $\Lambda(x,f)\subset\omega(x,f)$.

(3) If $X=\Omega E\omega(f)$, then $X=AP(f)$ and $\omega(x,f)=\Lambda(x,f)=S\Lambda(x,f)=\alpha(\theta,f)$ for any $x\in X$, where θ is any negative orbit through x of f.

Proof (1) If $x\in\omega(y,f)$ for some $y\in X$, then there exists a sequence of positive integers $r_1<r_2<\cdots$ with $r_{n+1}-r_n>n$ for any $n\in\mathbb{N}$ such that $f^{r_n}(y)\to x$. Then $f^{r_{n+1}-r_n}(f^{r_n}(y))=f^{r_{n+1}}(y)\to x$. Thus $x\in\Omega(x,f)$.

(2) If $X=R(f)$ and $y\in\Lambda(x,f)$, then there exist a sequence of points $x_1,x_2,\cdots\in X$ and a sequence of positive integers $n_1<n_2<\cdots$ such that $\lim_{k\longrightarrow\infty}x_k=y$ and $f^{n_k}(x_k)=x$. Thus $x_k\in\omega(x_k,f)=\omega(f^{n_k}(x_k),f)=\omega(x,f)$ for every $k\in\mathbb{N}$, which implies $y\in\omega(x,f)$ since $\omega(x,f)$ is closed.

(3) Let $x_0\in X$. Then there exists a negative orbit $\theta=(x_0,x_{-1},\cdots,x_{-n},\cdots)$ since $f(X)=X$. It is easy to see $x_0\in\Omega(y,f)=\omega(y,f)$ for any $y\in\alpha(\theta,f)$. By (1) we see $x_0\in\Omega(x_0,f)=\omega(x_0,f)$, which implies $x_0\in R(f)$. Hence $X=R(f)$.

For any $x\in X=R(f)$ and any $u\in\omega(x,f)$, choose $r_n,s_n\in\mathbb{N}$ satisfying $r_n-s_n>n$ and $s_n\longrightarrow\infty$ such that $f^{r_n-s_n}(f^{s_n}(x))=f^{r_n}(x)\longrightarrow x$ and $f^{s_n}(x)\longrightarrow u$, which implies $x\in\Omega(u,f)=\omega(u,f)$. Thus $\omega(x,f)$ is minimal and $x\in AP(f)$.

Let $x\in X$ and $\theta=(x_0,x_{-1},\cdots,x_{-n},\cdots)$ be a negative orbit through $x_0=x$ of f. Then $\omega(x,f)$ is minimal. Since $x_{-k}\in\omega(x_{-k},f)=\omega(x,f)$ for any $k\in\mathbb{N}$, we have $\alpha(\theta, f) \subset \omega(x, f)$. Note that $\alpha(\theta, f)$ is closed with $f(\alpha(\theta, f)) = \alpha(\theta, f)$ and $\omega(x, f) \supset \Lambda(x, f) \supset S\Lambda(x, f) \supset \alpha(\theta, f)$. Thus $\omega(x, f) = \Lambda(x, f) = S\Lambda(x, f) = \alpha(\theta, f)$. This completes the proof of the Lemma.

Theorem 3.3 Let $f \in C^0(X)$ and $f(X) = X$. Then the following statements are equivalent.

(1) $\Omega(x, f)$ is minimal for any $x \in X$.

(2) $X = \Omega E\omega(f)$.

(3) $X = \Omega E\Gamma(f)$.

(4) $X = R(f)$ and $\Omega(x, f) = \alpha(\theta, f)$ for any negative orbit $\theta = (x, y_{-1}, \cdots, y_{-n}, \cdots)$.

(5) $X = R(f)$ and $\Omega(x, f) = S\Lambda(x, f)$ for any $x \in X$.

(6) $X = R(f)$ and $\Omega(x, f) = \Lambda(x, f)$ for any $x \in X$.

(7) For any $x \in X$ and any negative orbit $\theta = (y_0, y_{-1}, \cdots, y_{-n}, \cdots)$ with $y_0 \in \Omega(x, f)$, $\alpha(\theta, f) = \Omega(x, f)$.

Proof (2) $\Rightarrow$ (1) It follows from Lemma 3.2.

(1) $\Rightarrow$ (2) If $\Omega(x, f)$ is minimal for every $x \in X$, then by $\Omega(x, f) \supset \omega(x, f)$ and $f(\omega(x, f)) = \omega(x, f)$ and $\omega(x, f)$ being closed we see $\Omega(x, f) = \omega(x, f)$ for any $x \in X$.

(2) $\Rightarrow$ (3) If $\Omega(x, f) = \omega(x, f)$ for any $x \in X$, then by Lemma 3.2 (3) we see $\omega(x, f) = \Lambda(x, f)$, which implies $\Gamma(x, f) = \omega(x, f) = \Omega(x, f)$ for any $x \in X$.

(3) $\Rightarrow$ (2) If $\Gamma(x, f) = \Omega(x, f)$ for any $x \in X$, then by $\Gamma(x, f) \subset \omega(x, f) \subset \Omega(x, f)$ we have $\omega(x, f) = \Omega(x, f)$ for any $x \in X$.

(2) $\Rightarrow$ (4) It follows from Lemma 3.2 (3).

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (2) It follows from Lemma 3.2 (2) and $\alpha(\theta, f) \subset S\Lambda(x, f) \subset \Lambda(x, f)$ and $\omega(x, f) \subset \Omega(x, f)$.

(2) $\Rightarrow$ (7) Suppose that $X = \Omega E\omega(f)$. Let $\theta = (y_0, y_{-1}, \cdots, y_{-n}, \cdots)$ with $y_0 \in \Omega(x, f) = \omega(x, f)$. Then by Lemma 3.2 we see $X = AP(f)$ and $y_{-n} \in \omega(y_{-n}, f) = \omega(y_0, f) = \omega(x, f)$. Thus $\alpha(\theta, f) \subset \omega(x, f)$. Since $\alpha(\theta, f)$ is closed and $f(\alpha(\theta, f)) = \alpha(\theta, f)$ and $\omega(x, f)$ is minimal, we have $\alpha(\theta, f) = \omega(x, f) = \Omega(x, f)$.

(7) $\Rightarrow$ (2) Suppose that (7) holds. Let $x \in X$ and $y_0 \in \omega(x, f)$. Then there exists a negative orbit $\theta = (y_0, y_{-1}, \cdots, y_{-n}, \cdots)$ with $y_{-n} \in \omega(x, f)$ for every $n \in \mathbb{N}$. Thus $\alpha(\theta, f) \subset \omega(x, f)$, which follows that $\omega(x, f) = \Omega(x, f)$. This completes the proof of the Theorem.

4. Pointwise-recurrent maps on a dendrite

In this section, we will obtain some equivalent conditions of pointwise-recurrent maps on a dendrite.

Definition 4.1 (see [13]) Let X be a compact metric space. $f \in C^0(X)$ is called equicontinuous (briefly, $f \in EC(X)$) if for any $\varepsilon > 0$, there exists an $\delta > 0$ such that $f^n(B(x, \delta)) \subset B(f^n(x), \varepsilon)$ for any $x \in X$ and any $n \in \mathbb{N}$.

The main result of this paper is established through the following lemmas.

Lemma 4.2 (see [7]) If X is a compact metric space and $f \in EC(X)$ with $f(X) = X$, then $X = \Omega E\omega(f)$.

Lemma 4.3 (see [6]) Let D be a dendrite and $f \in C^0(D)$. Then $D = R(f)$ if and only if $D-\mathrm{End}(D) \subset P(f)$.

Remark 4.4 Let D be a dendrite and $f \in C^0(D)$. If $c \in \Omega(x, f)$ for some $x \in D$, then we may choose $d \in \omega(x, f)$, $x_n \in D$ with $x_n \to x$ and $k_1 < k_2 < \cdots$ such that $f^{k_n}(x_n) \to c$ and $f^{k_n}(x) \to d$. Furthermore, if $c \neq d$, then $d(c, d) > 0$. It is easy to verify that for any $\tau \in (c, d)$, there exists an $N(\tau) \in \mathbb{N}$ such that $\tau \in (f^{k_n}(x_n), f^{k_n}(x))$ for any $n \geq N(\tau)$. Thus for every $n \geq N(\tau)$, there exists an $y_n \in (x_n, x)$ such that $f^{k_n}(y_n) = \tau$, which implies $\tau \in \Omega(x, f)$ and $[c, d] \subset \Omega(x, f)$.

Lemma 4.5 Let D be a dendrite and $f \in C^0(D)$ with $f(D) = D$. If $D = \Omega E\omega(f)$, then $f \in EC(D)$ and for any $c, d, x \in D$ with $c \neq d$, we have $[c, d] \not\subset \Omega(x, f)$

Proof Assume that $D = \Omega E\omega(f)$. Then it follows from Theorem 3.3 and Lemma 4.3 that $D = AP(f)$ and $D-\mathrm{End}(D) \subset P(f)$ and $\Omega(x, f)$ is minimal for any $x \in D$. If $[c, d] \subset \Omega(x, f)$ for some $c, d, x \in D$ with $c \neq d$, then $(c, d) \subset D-\mathrm{End}(D) \subset P(f)$, from which it follows that $\Omega(x, f)$ is not a minimal set. This is a contradiction. Now we show that $f \in EC(D)$. Indeed, if $f \notin EC(D)$, then there exist $x, x_n \in D$ with $x_n \to x$ and $k_n \to \infty$ such that $f^{k_n}(x_n) \to c \in \Omega(x, f)$ and $f^{k_n}(x) \to d \in \omega(x, f)$ with $d(c, d) > 0$. By Remark 4.4 we see that $[c, d] \subset \Omega(x, f)$, which also leads to a contradiction. This completes the proof of the Lemma.

Now we show the main result of this paper.

Theorem 4.6 Let D be a dendrite and $f \in C^0(D)$ with $f(D) = D$. Then the following statements are equivalent.

(1) $D = R(f)$.

(2) $D = AP(f)$.

(3) $D = \Omega E\Gamma(f)$.

(4) $D = \Omega E\omega(f)$.

(5) $f \in EC(D)$.

(6) $[c,d] \not\subset \Omega(x,f)$ for any $c,d,x \in D$ with $c \neq d$.

(7) $\Omega(x,f)$ is minimal for any $x \in D$.

(8) $\mathrm{Card}(\Lambda^{-1}(x,f) \cap (D - \mathrm{End}(D))) < \infty$ for any $x \in D$, where $\Lambda^{-1}(x,f) = \{y : x \in \Lambda(y,f)\}$.

(9) If $x \in \Lambda(y,f)$ with $x,y \in D$, then $y \in \omega(x,f)$.

(10) Map $h : x \longrightarrow \omega(x,f)$ $(x \in D)$ is continuous and for any $x,y \in D$ with $x \notin \omega(y,f)$, $\omega(x,f) \neq \omega(y,f)$.

Proof (2) $\Leftrightarrow$ (4) It follows from Theorem 3.3.

(4) $\Leftrightarrow$ (5) It follows from Lemma 4.2 and Lemma 4.5.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) It follows from Theorem 3.3.

(2) $\Rightarrow$ (1) It is obvious.

(4) $\Rightarrow$ (6) It follows from Lemma 4.5.

(6) $\Rightarrow$ (4) Suppose that (6) holds. If $D \neq \Omega E\omega(f)$, then there exists some $x \in D$ such that $\omega(x,f) \neq \Omega(x,f)$ and by Remark 4.4 we see that there exist $c \in \Omega(x,f) - \omega(x,f)$ and $d \in \omega(x,f) \subset \Omega(x,f)$ with $d(c,d) > 0$ such that $[c,d] \subset \Omega(x,f)$. This will lead to a contradiction.

(1) $\Rightarrow$ (4) If $D = R(f)$, then by Lemma 4.3 we see that $(c,d) \subset P(f)$ for any $c,d \in D$ with $c \neq d$ since $(c,d) \subset D - \mathrm{End}(D)$. Assume on the contrary that $\omega(x,f) \neq \Omega(x,f)$ for some $x \in D$. Then by Remark 4.4 we see that there exist $x_n \in D$ with $x_n \to x$ and $k_n \to \infty$ such that $f^{k_n}(x_n) \to c \in \Omega(x,f) - \omega(x,f)$ and $f^{k_n}(x) \to d \in \omega(x,f)$. Furthermore, for any $a \in (c,d)$, there exist $y_n \longrightarrow x$ with $y_n \in (x_n, x)$ such that $f^{k_n}(y_n) = a$.

Choose $p,q \in [c,d] \cap P(f)$ such that $O(p,f) \cap O(q,f) = \emptyset$. Let $p_n \longrightarrow x$ and $q_n \longrightarrow x$ with $f^{k_n}(p_n) = p$ and $f^{k_n}(q_n) = q$. Since $p_n \in \omega(p_n,f) = \omega(f^{k_n}(p_n),f) = \omega(p,f) = O(p,f)$ and $q_n \in \omega(q_n,f) = \omega(f^{k_n}(q_n),f) = \omega(q,f) = O(q,f)$, we see $x \in O(p,f)$ and $x \in O(q,f)$. This is a contradiction. Thus $D = \Omega E\omega(f)$.

(8) $\Rightarrow$ (4) Suppose that $\mathrm{Card}(\Lambda^{-1}(x,f) \cap (D - \mathrm{End}(D))) < \infty$ for any $x \in D$. Assume on the contrary that $D \neq \Omega E\omega(f)$. Then there exists some $x \in D$ such that $\omega(x,f) \neq \Omega(x,f)$ and by Remark 4.4 we see that there exist $c \in \Omega(x,f) - \omega(x,f)$ and $d \in \omega(x,f) \subset \Omega(x,f)$ with $d(c,d) > 0$ such that $(c,d) \subset \Lambda^{-1}(x,f)$. This is a contradiction.

(4) $\Rightarrow$ (8) If $D = \Omega E\omega(f)$, then it follows from Lemma 3.2 and Lemma 4.3 that $\omega(x,f) = \Lambda(x,f)$ for any $x \in D$ and $D-\mathrm{End}(D) \subset P(f)$. Let $y \in \Lambda^{-1}(x,f) \cap (D-\mathrm{End}(D))$. Then $x \in \Lambda(y,f) = \omega(y,f) = O(y,f)$, from which it follows that $x \in P(f)$ and $y \in O(x,f)$. Thus $\Lambda^{-1}(x,f) \cap (D-\mathrm{End}(X)) \subset O(x,f)$.

(4) $\Rightarrow$ (9) If $D = \Omega E\omega(f)$, then by Lemma 3.2 we have $D = AP(f)$ and $\omega(x,f) = \Lambda(x,f)$ for any $x \in D$. Let $x, y \in D$ satisfy $x \in \Lambda(y,f)$, then $x, y \in \omega(y,f)$ and $y \in \omega(x,f)$.

(9) $\Rightarrow$ (4) Suppose that (9) holds. Assume on the contrary that $D \neq \Omega E\omega(f)$. Then there exists some $x \in D$ such that $\omega(x,f) \neq \Omega(x,f)$ and by Remark 4.4 we see that there exist $c \in \Omega(x,f) - \omega(x,f)$ and $d \in \omega(x,f) \subset \Omega(x,f)$ with $d(c,d) > 0$ such that $x \in \Lambda(y,f)$ for any $y \in (c,d)$. Thus $(c,d) \subset \omega(x,f)$, from which it follows $c \in \omega(x,f)$. This will lead to a contradiction.

(4) $\Rightarrow$ (10) If $D = \Omega E\omega(f)$, then $D = AP(f)$ and $f \in EC(D)$. Thus for any $\varepsilon > 0$, there exists an $\delta > 0$ such that $f^n(B(x,\delta)) \subset B(f^n(x), \varepsilon/3)$ for any $x \in X$ and any $n \in \mathbb{N}$. For any $w \in \omega(x,f)$ and $y \in B(x,\delta)$, there exist $z \in \omega(y,f)$ and $k_n \to \infty$ satisfying $f^{k_n}(x) \to w$ and $f^{k_n}(y) \to z$. Thus there exists an $r \in \mathbb{N}$ such that

$$f^{k_r}(x) \in B(w, \varepsilon/3), \;\; f^{k_r}(y) \in B(z, \varepsilon/3), \;\; f^{k_r}(x) \in B(f^{k_r}(y), \varepsilon/3).$$

So $w \in B(z,\varepsilon)$, which implies that $d(w, \omega(y,f)) < \varepsilon$. In a similar fashion, we can prove that $d(z, \omega(x,f)) < \varepsilon$ for any $z \in \omega(y,f)$. Thus

$$\begin{aligned} H_d(h(x), h(y)) &= H_d(\omega(x,f), \omega(y,f)) \\ &= \max\{\sup_{z \in \omega(y,f)} d(z, \omega(x,f)), \sup_{w \in \omega(x,f)} d(w, \omega(y,f))\} \\ &\leq \varepsilon \end{aligned}$$

and h is continuous.

For any $x, y \in D$ with $x \notin \omega(y,f)$, we have $x \in \omega(x,f)$ and $y \in \omega(y,f)$. Thus $\omega(x,f) \neq \omega(y,f)$.

(10) $\Rightarrow$ (4) Suppose that (10) holds. Assume on the contrary that $\omega(x,f) \neq \Omega(x,f)$ for some $x \in D$. Then by Remark 4.4 we see that there exist $x_n \in D$ with $x_n \to x$ and $k_n \to \infty$ such that $f^{k_n}(x_n) \to w \in \Omega(x,f) - \omega(x,f)$ and $f^{k_n}(x) \to z \in \omega(x,f) \subset \Omega(x,f)$ with $d(w,z) > 0$. It is easy to verify that $w \notin \omega(z,f) \subset \omega(x,f)$. Thus $\omega(w,f) \neq \omega(z,f)$ and $H_d(\omega(w,f), \omega(z,f)) > 0$.

Since h is continuous, for $\varepsilon = H_d(\omega(w, f), \omega(z, f))/6 = H_d(h(w), h(z))/6$, there exists an $\delta > 0$ such that when $d(u, v) < \delta$,

$$H_d(h(u), h(v)) < \varepsilon/3.$$

Let $N \in \mathbb{N}$ such that

$$f^{k_N}(x_N) \in B(w, \delta), \quad f^{k_N}(x) \in B(z, \delta), \quad x_N \in B(x, \delta).$$

Then we have

$$H_d(h(f^{k_N}(x_N)), h(w)) < \varepsilon/3, \; H_d(h(f^{k_N}(x)), h(z)) < \varepsilon/3, \; H_d(h(x_N), h(x)) < \varepsilon/3,$$

which implies that

$$\begin{aligned}
H_d(h(w), h(z)) &\le H_d(h(w), h(f^{k_N}(x_N))) \\
&\quad + H_d(h(f^{k_N}(x_N)), h(f^{k_N}(x))) \\
&\quad + H_d(h(f^{k_N}(x)), h(z)) \\
&= H_d(h(w), h(f^{k_N}(x_N))) \\
&\quad + H_d(h(x_N), h(x)) \\
&\quad + H_d(h(f^{k_N}(x)), h(z)) \\
&< \varepsilon = H_d(h(w), h(z))/6.
\end{aligned}$$

This is a contradiction. This completes the proof of the Theorem.

Example 4.7 In $\mathbb{R}^2$, for any $k \in \mathbb{N}$, set

$$L_k = \{(x, x/k) : 0 \le x \le 1/k\}$$

and

$$L_{-k} = \{(x, -x/k) : -1/k \le x \le 0\}.$$

Then $D = \cup_{k=1}^{\infty}(L_k \cup L_{-k})$ is a dendrite. Now we define $f \in C^0(D)$ as following:

(1) For any $k \ge 2$ and $P = (x, x/k) \in L_k$, $f(P) = (kx/(k-1), kx/(k-1)^2) \in L_{k-1}$.

(2) For any $P = (x, x) \in L_1$, $f(P) = (-x, x) \in L_{-1}$.

(3) For any $k \in \mathbb{N}$ and $P = (x, -x/k) \in L_{-k}$, $f(P) = (kx/(k+1), -kx/(k+1)^2) \in L_{-k-1}$.

Write $O = (0, 0)$. It is easy to see that $f(D) = D$, $\omega(P, f) = \{O\}$ for any $P \in D$ and $\Omega(O, f) = D$. From Theorem 4.6 we see that $f \notin EC(D)$.

From Example 4.7 we see that $\Omega(x, f)$ of Theorem 4.6 (7) can not be replaced by $\omega(x, f)$ and (10) of Theorem 4.6 can not be replaced by "Map $h : x \longrightarrow \omega(x, f)$ ($x \in D$) is continuous".

Remark 4.8 After this paper was accepted, we see that there are some equivalences of Theorem 4.6 has been proved in [7] (for example, (4) $\Leftrightarrow$ (5) has been proved in [7, Theorem 4.12], etc.).

5. Pointwise-recurrent maps on dendrites D with Card$(\mathrm{B}(D)) < \infty$

In this section, we study pointwise-recurrent maps on dendrites D with Card $(\mathrm{B}(D)) < \infty$. In the following, let $\mathbb{S} = \{k_1 < k_2 < \cdots\}$ be a subsequence of $\mathbb{N}$.

Definition 5.1 (see [24]) Let X be a compact metric space. $f \in C^0(X)$ is called $\mathbb{S}$-equicontinuous (briefly, $f \in \mathbb{S}EC(X)$) if for any $\varepsilon > 0$, there exists an $\delta > 0$ such that $f^{k_n}(B(x, \delta)) \subset B(f^{k_n}(x), \varepsilon)$ for any $x \in X$ and any $n \in \mathbb{N}$.

Lemma 5.2 Let (X, d) be a compact metric space. If $f \in \mathbb{S}EC(X)$ with $f(X) = X$, then $X = \cup_{x \in X}\omega(x, f)$.

Proof Let $x_0 \in X$. Then we can choose a negative orbit $\theta = (x_0, x_{-1}, \cdots, x_{-n}, \cdots)$ since $f(X) = X$ and a point $y \in X$ and a subsequence $r_1 < r_2 < \cdots$ of $\mathbb{S}$ such that $\lim_{n\longrightarrow\infty} x_{-r_n} = y$. Let ε and δ be the same as Definition 5.1. Take x_{-r_n} satisfying $d(x_{-r_n}, y) < \delta$. Then $d(f^{r_n}(y), f^{r_n}(x_{-r_n})) = d(f^{r_n}(y), x_0) < \varepsilon$, which implies $x_0 \in \omega(y, f)$. This completes the proof of the Lemma.

Lemma 5.3 Let (X, d) be a compact metric space and $m \in \mathbb{N}$. If $f \in \mathbb{S}EC(X)$, then there exists a subsequence $\mathbb{S}_1$ of $\mathbb{N}$ such that $g = f^m \in \mathbb{S}_1EC(X)$.

Proof By taking a subsequence we may assume that there exist $0 \leq r < m$ and a subsequence $s_1 < s_2 < \cdots$ of $\mathbb{S}$ such that $s_n = r_n m + r$ for any $n \in \mathbb{N}$. Since X is compact and $f \in \mathbb{S}EC(X)$, we see that for $\varepsilon > 0$, there exist $0 < \delta < \delta_1 \leq \varepsilon$ such that $f^{r_n m+r}(B(u, \delta_1)) \subset B(f^{r_n m+r}(u), \varepsilon)$ for any $u \in X$ and any $n \in \mathbb{N}$, and $f^{m-r}(B(x, \delta)) \subset B(f^{m-r}(x), \delta_1)$ for any $x \in X$, which implies that $g^{r_n+1}(B(x, \delta)) \subset B(g^{r_n+1}(x), \varepsilon)$ for any $x \in X$. Thus $g \in \mathbb{S}_1EC(X)$ with $\mathbb{S}_1 = \{r_1 + 1, r_2 + 1, \cdots\}$. This completes the proof of the Lemma.

Lemma 5.4 Let D be a dendrite and $f \in C^0(D)$. If there exist $p \in P_1(f)$ and $x \in D$ such that $x \in (p, f(x))$, then $f \notin \mathbb{S}EC(X)$.

Proof Since $f([p, x]) \supset [p, x]$, we see that there exist a negative orbit $\theta = (x_0, x_{-1}, \cdots, x_{-n}, \cdots)$ and $q \in P_1(f) \cap [p, x)$ satisfying $x_0 = x$, $[q, x_{-n}] \supset$

$[q, x_{-n-1}]$ and $\lim_{n\longrightarrow\infty} x_{-n} = q$. Then $d(f^{k_n}(x_{-k_n}), f^{k_n}(q)) = d(x, q)$ for any $n \in \mathbb{N}$. Thus $f \notin \mathbb{S}EC(X)$. This completes the proof of the Lemma.

Theorem 5.5 Let D be a dendrite with $\text{Card}(\text{B}(D)) < \infty$ and $f \in C^0(D)$ with $f(D) = D$. Then $f \in \mathbb{S}EC(X)$ if and only if $D = R(f) = P(f)$.

Proof $\Longrightarrow$ Assume that $f \in \mathbb{S}EC(X)$.

We claim that $\text{End}(D) \subset P(f)$. Indeed, if $\text{End}(D) \not\subset P(f)$, then let $q \in \text{End}(D) - P(f)$ and by Lemma 5.2 there exist $p \in P_1(f)$ and $y \in D$ and $n, m \in \mathbb{N}$ with $n < m$ such that $f^n(y), f^m(y) \in (p, q)$ and $f^m(y) \in (f^n(y), q)$. Write $g = f^{m-n}$. Then it follows from Lemma 5.4 that $g \notin \mathbb{S}_1EC(X)$ for any subsequence $\mathbb{S}_1$ of $\mathbb{N}$ since $g([p, f^n(y)] \supset [p, f^m(y)]$, which contradicts Lemma 5.3.

Now we show that $D = P(f)$. Indeed, if $D \neq P(f)$, then let $w \in D - P(f)$. We see that there exist $p \in P_1(f)$ and $q \in \text{End}(D)$ with $q \in P_m(f)$ such that $w \in (p, q)$. Thus $w \in (q, f^m(w))$ if $f^m(w)$ is in the connected component of $D - \{w\}$ containing p and $w \in (p, f^m(w))$ if $f^m(w)$ is not in the connected component of $D - \{w\}$ containing p. Then it follows from Lemma 5.4 that $g = f^m \notin \mathbb{S}_1EC(X)$ for any subsequence $\mathbb{S}_1$ of $\mathbb{N}$, which also contradict Lemma 5.3.

$\Longleftarrow$ It follows from Theorem 4.6. This completes the proof of the Theorem.

In the end of this section, we propose the following question.

Question 5.6 Obtain some equivalences for $f \in \mathbb{S}EC(X)$ if we omit the condition $\text{Card}(\text{B}(D)) < \infty$ in Theorem 5.5.

6. Conclusion

In this paper, we study the minimality of the sets of weak ω-limit points of maps on a compact metric space and obtain some equivalent conditions of pointwise-recurrent maps on a dendrite. Furthermore, we study characteristic of pointwise-recurrent maps on a dendrite with the number of branch points being finite.

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