Journal of Dynamical Systems & Geomatric Theories
ISSN : 1726-037X (Print) 2169-0057 (Online)
Vol. 17(2) November 2019, pp. 267–295
DOI : 10.1080/1726037X.2019.1668150

ON A MULTIFRACTAL PRESSURE FOR COUNTABLE MARKOV SHIFTS

ALEJANDRO MESÓN AND FERNANDO VERICAT
INSTITUTO DE FÍSICA DE LÍQUIDOS Y SISTEMAS BIOLÓGICOS (IFLYSIB)
CONICET–UNLP AND GRUPO DE APLICACIONES MATEMÁTICAS Y ESTADÍSTICAS
DE LA FACULTAD DE INGENIERÍA (GAMEFI) UNLP, LA PLATA, ARGENTINA
E-MAILS: MESON@IFLYSIB.UNLP.EDU.AR; VERICAT@IFLYSIB.UNLP.EDU.AR

(Received: 15 November 2018, Accepted: 18 March 2019)

ABSTRACT. In a recent article [*J. d´ Analyse Math* **131**, 207, 2017], Olsen intoduced a generalized notion of multifractal pressure, and also a multifractal dynamical zeta function, which essentilly consists in considering not all configurations, but those which are "multifractally relevant". In this way more precise information about the multifractal spectrum analyzed is encoded by the multifractal pressure and the multifratcal zeta function. He applied the theory for dynamical systems modelled by finite alphabet shifts, in particular for self conformal iterated systems. Here we continue with this line considering dynamical systems given by countable Markov shifts.

AMS Classification: 37B10, 37C30
Keywords: Topological pressure; Markov shift; zeta function

1. Introduction

The study of dimension of sets constitutes a major challenge in the area of Dynamical Systems. This is carried out by mean of adequate "characteristic dimensions", for instance the Hausdorff dimension and the topological entropy, which are important invariants to measure the complexity of the systems as well as the average amount of information. This quantities are particular cases of the general Carathéodory-Pesin dimension theory[22]. The general Theory of Dimension of Dynamical Systems is formulated considering a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ is a collection of subsets of X. This family satisfies some special conditions and it is called a $C-$structure.

The *Multifractal Analysis* is a relevant subject in the mentioned area. One of the main motivations was the study of the behavior of physical measures supported on strange attractors. To analyze these attractors is done a *fractal decomposition* of invariant sets with a complex mathematical structure. Such a decomposition is performed in the following way: let X be a set and $g : X \to [-\infty, +\infty]$, now X can be partitioned in level sets:

$$K_\alpha = K_\alpha (g) = \{x : g(x) = \alpha\}.$$

Let G be a function defined on sets, and let $F(\alpha) = G(K_\alpha)$, the map F is called *the multifractal spectrum* specified by the pair (g, G). In this real valuated case the most used spectra are defined by a limit process, for instance:

$$g(x) = D_\mu (x) := \lim_{r \to 0} \frac{\log (\mu (B_r (x)))}{-\log r}$$

, the *point wise dimension of the measure* μ, and $F(\alpha) = \dim_H K_\alpha$, the Hausdorff dimension. This is called the *point wise dimension spectrum.*

Dynamical examples are

Local entropies spectrum: Let $f : X \to X$, in this case

$$g(x) = h_\mu (f, x), \; F(\alpha) = h_{top} (f, K_\alpha),$$

where $h_\mu (f, x)$ is the point wise entropy of the measure μ and $h_{top} (f, .)$ is the topological entropy for non-compact nor invariant sets defined by Bowen.

Ergodic averages: here $g(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \varphi \left(f^i (x) \right)$, with $\varphi : X \to \mathbf{R}$ (often called potentials).

The problem of describing these spectra was extensively studied. Depending on the properties of the dynamics and the potentials the description can be given by a map which is the Legendre transform of the multifractal map $F(\alpha)$ or by a variational formula,[2]-[4],[6]-[10],[12],[18]-[20],[27],[28].

In [13], [14] were introduced zeta functions to analyze fractal objects. Dynamical zeta functions were initially introduced by Artin and Mazur [1] making an analogy with Number Theory, then Ruelle considered a kind of zeta functions which can related with distribution of periodic orbits and also to study Statistical Mechanics models. Parry and Pollicott[21] presented a zeta function by mean of closed geodesics, to obtain precisely a formula to counting closed geodesic in negatively curved manifolds, by an analogy by the counting of prime numbers. The novelty of the functions introduced in [13], [14] is that in the "partition function" of the definition are not considered all the configuration, but also that have a "multifractal relevance", In [20], Olsen introduced families of multifractal zeta functions to study diverse spectra. The setting for the dynamics are symbolic spaces with finite alphabet (we term this as the compact case), one case showed is a conformal iterated system which can be modelled for finite alphabet shift. In this article we continue with the idea of considering a multifractal pressure summing in the partition function over periodic configurations which are "multifractally relevant", the new here is that our dynamical setting will be symbolic spaces with a countable alphabet, which makes different the issue.

2. Multifractal pressure and zeta functions for Markov shifts

Let us consider a countable set Ω, for instance $\Omega = \mathbf{N}$, and a transition matrix $A = (A_{i,j})_{i,j\in\Omega}$, $A_{i,j} \in \{0,1\}$

$$\sum = \{x = x_0 x_1 ..., x_i \in \Omega,\ A_{x_i, x_i+1} = 1\}$$

The space $\sum$ is endowed with the product topology of the discrete topology in $\boldsymbol{\Omega}$, or equivalently the topology generated by cylinders

$$\mathcal{C}\left(i_1 i_2 ... i_n\right) = \{x : x_j = i_j,\ j = 1, ..., n\}.$$

If $\underline{i} = i_1 i_2 ... i_n \in \Omega^n$, its length is denoted by$|\ \underline{i}| = n$.A string $\underline{i} = i_1 i_2 ... i_n$ is called *admissible* if $\mathcal{C}\left(i_1 i_2 ... i_n\right) \neq \varnothing$. For any pair of strings $\underline{ij}$, by $\mathcal{C}\left(\underline{i}\right)\mathcal{C}\left(\underline{j}\right)$ is denoted the cylinder given by the concatenation of the strings, i.e. $\mathcal{C}\left(i_1 i_2 ... i_n j_1 j_2 ... j_n\right)$. The space $\sum$ is endowed with the metric $d(x,y) = t^{-N(x,y)}$ where $N(x,y)$ =first coordinate in which differ x and y, and $0 < t < 1$.

The dynamics are given by the shift $\sigma : \sum \to \sum$, with $(\sigma x)_i = x_{i+1}$, the dynamical system $(\sum, \sigma)$, with a transition matrix A and an alphabet Ω is called a *topological Markov shift.*

Definition: A topological Markov shift is topologically mixing if for any symbols $i, j \in \Omega$ there is an admissible string $\underline{a} = a_1 a_2 ... a_n$ with $a_1 = i$, $a_n = j$.

Let $\mathcal{A} = \{\mathcal{C}(i) : i \in \Omega\}$ be the natural partition by cylinders of $\sum$ and $\mathcal{A}^n = \{\mathcal{C}(i_1 i_2 ... i_n)\}$, let $A \subset \sigma^{-n}(\mathcal{C}(\underline{i})) \cap \mathcal{A}^n$, $B \subset \mathcal{C}(\underline{i}) \cap \mathcal{A}^m$, thus is denoted $A.B = \bigcup_{\substack{\mathcal{C}(\underline{i}) \subset A \\ \mathcal{C}(\underline{j}) \subset B}} \mathcal{C}(\underline{i}) \mathcal{C}(\underline{j})$

By $\mathcal{M}(\sum)$ is denoted he set of $\sigma-$invariant probability measure on Σ. The usual topological pressure, in the sense of Gurevic-Sarig, is defined in the following way: let us fix a symbol i_1 in the alphabet of $\sum$, then, for a potential $\varphi : \sum \to \mathbf{R}$ set

$$P(\varphi) = P(\varphi, \sigma) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\varphi, \sigma),$$

where Z_n is the partition function $Z_n(\varphi, \sigma) = \sum\limits_{x \in P_n(\sigma)} \exp(S_n(\varphi)(x)) I_{C(i_1)}$, while I_E is the characteristic function of the set E, $P_n(\sigma)$ is the set of $n-$periodic points of σ, i. e $P_n(\sigma) = \{x : \sigma^n(x) = x\}$ and $S_n(\varphi)(x) = \sum\limits_{i=0}^{n} \varphi(\sigma^i(x))$ (called the *statistical sum*). By $S_n(\varphi)$ is denote the map $\sum\limits_{i=0}^{n} \varphi \circ \sigma^i$. If the shift is topologically mixing the expression does not depend on the symbol considered.

Let

$$V_n(\varphi) = \sup_{i_1 i_2 \dots i_n \in \mathbf{N}^n} \sup_{x, y \in \mathcal{C}(i_1 i_2 \dots i_n)} \{|\varphi(x) - \varphi(y)|\}.$$

Definition: A potential $\varphi : \sum \to \mathbf{R}$, is *locally Hölder* if $V_n(\varphi) \leq A\theta^n$, for some constants $A > 0$ and $\theta \in (0, 1)$.

Let us call $\mathcal{H}$ the class of locally Hölder functions.

Let us denote $S_n(\varphi)(\mathcal{C}(\underline{i})) = \sup_{x \in \mathcal{C}(\underline{i})} S_n(\varphi)(x)$, the following properties hold for locally Hölder functions[24]:

- Let $W_n(\varphi) = \exp(\sum\limits_{k > n} V_k(\varphi))$, then for any $n \leq m$ is $V_n(S_m(\varphi)) \leq \log V_{m-n}$.

-For any $n \leq m$, $x = i_0 i_1 ...$ and $\underline{i} = i_0 i_1 ... i_{n-1}$ is $S_n(\varphi)(\mathcal{C}(\underline{i})) - logW_{m-n} \leq S_n(\varphi)(x) \leq S_n(\varphi)(\mathcal{C}(\underline{i})) + logW_{m-n}$.

-Let $A \subset \sigma^{-n}(\mathcal{C}(\underline{i})) \cap \mathcal{A}^n$, $B \subset \mathcal{C}(\underline{i}) \cap \mathcal{A}^m$, then

$$W_1^{-2}\sum_{\mathcal{C}(\underline{k})\subset A.B\cap\mathcal{A}^{n+m}}\exp(S_n(\varphi)(\mathcal{C}(\underline{k})))$$
$$\leq\left[\sum_{\mathcal{C}(\underline{i})\cap A\cap\mathcal{A}^n}\exp(S_n(\varphi)(\mathcal{C}(\underline{i})))\right]\left[\sum_{\mathcal{C}(\underline{j})\subset B\cap\mathcal{A}^m}\exp(S_n(\varphi)\left(\mathcal{C}\left(\underline{j}\right)\right))\right]\leq$$
$$W_1^{2}\sum_{\mathcal{C}(\underline{k})\subset A.B\cap\mathcal{A}^{n+m}}\exp(S_n(\varphi)(\mathcal{C}(\underline{k}))).$$

Mauldin-Urbanki[16] and Sarig[24], independently, developed thermodynamic formalisms for countable shifts. The variational principle is in this case stated as: let φ be a locally Hölder map and let $\sum$ be a topologically mixing shift, then

$$P(\varphi)=\sup\left\{h_\mu(\sigma)+\int\varphi d\mu:\int\varphi d\mu<\infty\text{ and }\mu\in\ \mathcal{M}\left(\sum\right)\right\},$$

where $h_\mu(\sigma)$ is the measure-theoretic entropy of μ for the shift. Thus a version of [24], in [16] is not required locally Hölder but is imposed to the shift be finitely irreducible (see definition below).

Other result is the *approximation property*: let $\sum_m=\{x\in\sum:x_n\leq m\}$, this set if $\sigma-$invariant, then for potentials $\varphi\in\mathcal{H}$ an is valid

$$P(\varphi,\sigma)=\sup_m\{P(\varphi,\sigma\mid\textstyle\sum_m)\}.$$

Definition: A probability measure m is a Gibbs measure, or a Gibbs state, for a potential φ if there is a constant B such that for any string $\underline{i}=i_1i_2...i_n$ and for any $x\in\mathcal{C}(i_1i_2...i_n)$ holds

$$B^{-1}\leq\frac{m(\mathcal{C}(i_1i_2...i_n))}{\exp(-nP(\varphi,\sigma))\exp(S_n(\varphi)(x))}\leq B.$$

The existence of Gibbs states for locally Hölder maps is ensured for shifts with the *BIP.property (big images property),* .i.e. there is a string $\underline{i}=i_1i_2...i_n\in\Omega^n$ such that for any $a\in\Omega$ there exist r,s such that $A_{i_r,a}A_{a,i_s}=1$. A similar condition for the existence of Gibbs states is that the shift be *finitely irreducible, i.e.* if $E=\bigcup_n\Omega^n$, where Ω^n is the set of admissible strings of length n, then there is a finite set $\Lambda\subset E$ such that for $\underline{i},\underline{j}\in E$ there exists a string $\underline{a}\in\Lambda$ such that $\underline{iaj}\in E$.

The multifractal pressure for a "single target" is defined as follows: let $E:\mathcal{M}(\sum)\to\mathbf{R}$ be an affine functional, $\varphi:\sum\to\mathbf{R}$ and let $\mathcal{E}_n(x)$ be the empirical measure given as $\mathcal{E}_n(x)=\dfrac{1}{n}\sum\limits_{i=0}^{-1}\delta_{\sigma^i(x)}$, with δ the Dirac point mass measure, let now

$$Z_n^{\alpha}(\varphi, E, \sigma) = \sum_{\substack{x \in P_n(\sigma) \\ E\mathcal{E}_n x = \alpha}} \exp(S_n(\varphi)(x)) I_{C(i_1)}$$

and

$$P^{\alpha}(\varphi, E, \sigma) = \lim_{n \to \infty} \frac{1}{n} \log Z_n^{\alpha}(\varphi, E, \sigma),$$

here $E\mathcal{E}_n$ denotes the composition of E and $\mathcal{E}_n$.

The usual zeta function is defined by

$$\varsigma_{,\varphi,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n} \exp[\ \ Z_n(\varphi, \sigma)]$$

,the convergence radius of this series is denoted by $\sigma_{\varsigma,\varphi,f}$ and equals $exp(-P(\varphi, \sigma))$. In the same spirit of the multifractal pressure for a target α, the multifractal zeta function for a target α is

$$\varsigma_{,\varphi,\sigma,E}^{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n} \exp[\ Z_n^{\alpha}(\varphi, E, \sigma)],$$

with the convergence radius $\sigma_{\varsigma,\varphi,f}^{\alpha} = exp(-P^{\alpha}(\varphi, \sigma))$,

One the objectives is to perform a "shrinking target approach", in the usual Multifractal Analysis is considered as a "target" a level α, one approach is to enlarge the point, for instance , to an interval $I = [\alpha - r, \alpha + r]$, shrinking to α as $r \searrow 0$ and relate this behavior with the topological pressure, the zeta function and multifractal. spectra at α. The multifractal pressure and multifractal zeta function for the target I are defined similarly, considering in the sum the periodic points x such that $E\mathcal{E}_n(x) \in I$. More generally can be considered a functional $E : \mathcal{M}(\Sigma) \to X$, with X a metric space and C a subset of X, let C_r be the ball $B_r(C)$, thus the analysis is related to the shrinking to as $r \searrow 0$ and the definition of multifractal pressure and zeta functions with target C. If $\mathcal{E}_n : \Sigma \to \mathcal{M}(\Sigma)$ gives to any x the empirical measure $\mathcal{E}_n(x)$ then set

$$Z_n^{C}(E, \varphi, \sigma) = \sum_{\substack{x \in P_n(f) \\ E\mathcal{E}_n(x) \in C}} \exp(S_n(\varphi)(x)) I_{C(i_1)}, \tag{1}$$

The *upper and lower multifractal pressure for a potential* $\varphi \in \mathcal{H}$ are defined as

$$\overline{P}^{C}(E,\varphi,\sigma) = \limsup_{n\to\infty} \frac{1}{n} \log Z_n^C(E,\varphi,\sigma)$$

$$\underline{P}^{C}(E,\varphi,\sigma) = \liminf_{n\to\infty} \frac{1}{n} \log Z_n^C(E,\varphi,\sigma), \tag{2}$$

if the limit does exist then is denoted $P^C(E,\varphi,\sigma) = \overline{P}^C(E,\varphi,\sigma) = \underline{P}^C(E,\varphi,\sigma)$.

By the properties of locally Hölder maps displayed above we have

Lemma 1:

-For any $i, j \in \Omega$ there is a number M such that $Z_n^C(E,\varphi,\sigma) \leq M Z_{n+M}^C(E,\varphi,\sigma)$.

-For any $i \in \Omega$, for any n, m is valid $Z_n^C(E,\varphi,\sigma) \leq W_1^3 Z_{n+m}^C(E,\varphi,\sigma)$.

The multifractal zeta function is

$$\varsigma_{E,\varphi,\sigma}^C(z) = \sum_{n=0}^{\infty} \frac{z^n}{n} \exp\left[Z_n^C(E,\varphi,\sigma)\right],$$

and its convergence radius is denoted by $\sigma_{\varsigma_{E,\varphi,\sigma}}^C$ and equals $exp(-\overline{P}^C(E,\varphi,\sigma))$. For $C = \mathbf{R}$ are obtained the usual pressure and zeta function. If the functional E is taken as $E(\mu) = E_\varphi(\mu) = \int \varphi d\mu$ then $E\mathcal{E}_n$ is the Birkhoff average.

Definition: $\psi : \sum \to \mathbf{R}$ is *metric potential* if there is $K > 0$ such that if $x = x_0 x_1 ... \in \sum$ then

$$\frac{1}{K} \prod_{j=0} \overline{\psi}\left(\sigma^j(x)\right) \leq diam\left(\mathcal{C}(x_0 x_1 ... x_{n-1})\right) \leq K \prod_{j=0} \overline{\psi}\left(\sigma^j(x)\right).$$

The following properties for pressure map can be established in a direct way, following the thermodynamic formalisms of Mauldin-Urbanski or Sarig, for the usual case(see also [11]) . Let E be continuous, with respect to the weak topology, let us consider the maps $q \longmapsto \overline{P}^C(E,-q\varphi,\sigma)$ and $\lim_{r\searrow 0} \overline{P}^{C_r}(E,-q\varphi,\sigma)$, then there is a critical value $q^* \in (0,1]$ such that for $q < q^*$ is $\overline{P}^C(E,-q\varphi,\sigma) = \infty$, $\lim_{r\searrow 0} \overline{P}^{C_r}(E,-q\varphi,\sigma) = \infty$, and for $q > q^*$ we have that $\overline{P}^C(E,-q\varphi,\sigma) < \infty$, $\lim_{r\searrow 0} \overline{P}^{C_r}(E,-q\varphi,\sigma) < \infty$. Besides for $q > q^*$ the maps are real analytic. If $\overline{P}^C(E,0,\sigma) < \infty, \dim_H \sum < \infty$ then there are a values $F(C), G(C)$ such that

$\overline{P}^{C}(E,F(C),\sigma)=0$ and $\lim_{r\searrow 0}\overline{P}^{C_r}(E,G(C),\sigma)=0$ (multifractal Bowen equation) or equivalently $\sigma^{C}_{\varsigma\sigma}=1$ and $\lim_{r\searrow 0}\sigma^{C_r}_{\varsigma,G(C)\varphi,\sigma}=1$. The maps

$q\longmapsto \overline{P}^{C}(E,-q\varphi,\sigma\mid\sum_m)$ are real analytic, for any m.

As we mentioned one the aims is to analyze the pressure $\overline{P}^{C_r}(E,\varphi,\sigma)$ as $r\searrow 0$, in particular we prove, for countable Markov maps, the variational principle for this pressure. In this case we previously prove the approximation property for the multifractal pressure. As a consequence of the variational principle for the multifractal pressure can have an expression for the possible zero of the pressure map. As it was observed by Mauldin-Urbanski[17] the function $\overline{P}^{C}(E,-q\varphi,\sigma)$ in general may have not a root, this occurs in the conditions above mentioned, recall $\overline{P}^{C}(E,0,f)<\infty,\dim\sum<\infty$. But in case of existing a zero its value, as well of convergence radius of the multifractal zeta function, can be estimated from the variational formula. This result will be used for the variational principle.

The analysis is complemented with the description of the following spectra

$$K_C=\{x:A(E\mathcal{E}_n(x))\subset C\},$$

where $A(x_n)$ denotes the set of accumulation points of the sequence x_n, and C a closed convex subset of X. In this case X is a vector space equipped also with a linearly compatible metric. The study of this spectrum was done in [5] for the compact case, with specification as main hypothesis. For the single case $C=\{\alpha\}$ is obtained the description of [10]. The general description is done by a variational formula. To obtain a description by mean of Legendre transform can be considered the spectrum of the point wise measure, i.e. as we pointed out earlier, with the special functional $E\mu=\dfrac{\int\varphi d\mu}{\int\psi d\mu}$,and a potential metric ψ such that $D_\mu(x)=\lim_{r\to 0}\dfrac{\log(\mu(B_r(x)))}{\log r}=\lim_{n\to\infty}\dfrac{\log(\mu(\mathcal{C}(x_0x_1...x_{n-1})))}{\log\left(\prod_{i=0}^{n-1}\psi(\sigma^i(x))\right)}$. Thus the spectrum to be considered is

$K_{C,\mu}=\left\{x:A_{r\searrow 0}(\dfrac{\log(\mu(B_r(x)))}{\log r})\subset C\right\}$, $C\subset\mathbf{R}$. The Haussdorf dimension of $K_{C,\mu}$ can be obtained as the Legendre transform of an adequate "free energy" map.

3. Variational principle for the multifractal pressure of countable Markov maps.

In this Section we shall work in the following setting:

-The potentials being in the class $\mathcal{H}$ of locally Hölder maps.

-The shifts $\sum$ satisfying the $BIP-$property of being finitely reducible, to ensure the existence of Gibbs states for potentials in $\mathcal{H}$.

- $\dim_H \sum < \infty$.

-$P(0,\sigma) < \infty$ and $\overline{P}^C(E,0,\sigma) < \infty$.

Theorem 1 (approximation theorem for the multifractal pressure): Let $\sum_m = \{x \in \sum : x_n \leq m\}$, then is valid

$$\overline{P}^C(E,\varphi,\sigma) = \sup_m \left\{ \overline{P}^C\left(E,\varphi \mid \sum_m,\ \sigma \mid \sum_m\right)\right\}$$

Proof: We follow[24], let $Z^C_{n,m}(E,\varphi,\sigma) = Z^C_{n,m}(\ E,\varphi \mid \sum_m,\ \sigma \mid \sum_m)$ and so

$$\overline{P}^C(E,\varphi,\sigma) = \limsup_{n\to\infty} \frac{1}{n}\log Z^C_{n,m}(E,\varphi,\sigma)$$

. To prove that $\overline{P}^C(E,\varphi,\sigma) \leq \sup_m \left\{\overline{P}^C(E,\varphi \mid \sum_m,\ \sigma \mid \sum_m)\right\}$ it must see that for any $\varepsilon > 0$, there is an m with $\overline{P}^C(E,\varphi,\sigma) - \varepsilon < \overline{P}^C(E,\varphi \mid \sum_m,\ \sigma \mid \sum_m)$. let k enough big such and $\varepsilon > 0$, that $\overline{P}^C(E,\varphi,\sigma) < \frac{1}{k}\log Z^C_k(E,\varphi,\sigma) + \varepsilon$, let us choose an m such that
$\frac{1}{k}\log Z^C_k(E,\varphi,\sigma) < \frac{1}{k}\log Z^C_{k,m}(E,\varphi,\sigma)$.
By the lemma 1 we have

$$\log Z^C_{n,m}(E,\varphi,\sigma) + \log Z^C_{k,m}(E,\varphi,\sigma) \leq 3\log W_1 + \log\log Z^C_{k+n,m}(E,\varphi,\sigma)$$

.

Let $n = sk + r$, $r = 0,1,...,m-1$, thus

$$\begin{aligned}
&\frac{s\log Z^C_{k,m}(E,\varphi,\sigma) + \log Z^C_{r,m}(E,\varphi,\sigma)}{n} \\
\leq\ & \frac{s\log Z^C_{sk+r,m}(E,\varphi,\sigma) + (3s+1)\ \log W_1}{sk+r} \\
\leq\ & \frac{\log Z^C_{n,m}(E,\varphi,\sigma)}{sk+r=n} + \varepsilon\left(\frac{s+1}{s}\right).
\end{aligned}$$

Therefore

$$\overline{P}^{C}\left(E,\varphi\mid\sum_{m},\ \sigma\mid\sum_{m}\right)=\limsup_{n\to\infty}\frac{1}{n}\log Z_{n,m}^{C}\left(E,\varphi,\sigma\right)\geq\overline{P}^{C}\left(E,\varphi,\sigma\right)-\varepsilon.$$

□

Corollary 1: If C is closed then

$$\overline{P}^{C}\left(E,\varphi,\sigma\right)\leq\left\{\sup h_{\mu}\left(\sigma\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}\left(\Sigma\right),E\mu\in C\ \right\}.$$

Proof: A measure $\mu\in\mathcal{M}\left(\sum\right)$ is *finitely supported* if there is a m such that $\mu\left(\sum_{m}\right)=1$.By [20] for C closed, holds

$$\overline{P}^{C}\left(E,\varphi\mid\sum_{m},\ \sigma\mid\sum_{m}\right)\leq\sup\left\{\ h_{\mu}\left(f\right)+\int\varphi d\mu:E\mu\in C,\ \mu\left(\sum_{m}\right)=1\right\},$$

so that by the **Theorem 1**, we have

$$\overline{P}^{C}\left(E,\varphi,\sigma\right)\leq\sup\left\{\ h_{\mu}\left(\sigma\right)+\int\varphi d\mu\right\},$$

where the supremum is taken over all the measures μ finitely supported with $E\mu\in C$. Thus

$$\overline{P}^{C}\left(E,\varphi,\sigma\right)\leq\sup\left\{h_{\mu}\left(\sigma\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}\left(\Sigma\right),\ E\mu\in C\right\}$$

□

We now can state a variational principle in our setting of countable Markov maps. Similar one was established in [20] for the compact case (finite alphabet shifts). For the proof we follow the idea of [20], consisting in using the large deviation theory to estimate the size of sets of measures which by the application of the functional E fall in the considered sets C. These kind of sets appear in the sum of the partition function, thus a large deviation theory is quite suitable to bound the average limit of the partition function. In the context of countable alphabet shifts large deviations principle were formulated by Sarig[25] and in recent manuscript by Takahasi[26].

Main grounds of the theory are the following: let $\varphi, \psi \in \mathcal{H}$, consider a Gibbs state for ψ, denoted by $m = m_\psi$, let $\{\nu_n\}$ be a sequence of probability measures in $\mathcal{M}(\Sigma)$, where any ν_n is the distribution of the random variable $x \to \mathcal{E}_n(x)$ with respect to m, i.e., $\nu_n = m \circ \mathcal{E}_n^{-1}$, thus the sequence $\{\nu_n\}$ satisfies a large deviation principle for a rate function $I_\psi(\mu) = -P(\psi, \sigma) + h_\mu(\sigma) + \int \psi d\mu$, in particular if U is an open set, in the weak topology, of $\mathcal{M}(\sum)$ holds

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(U) \geq -\inf\{I_\psi(\mu) : \mu \in U\}.$$

If , in particular, $\psi \equiv 0$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(U) \geq -\inf\{-P(0,\sigma) + h_\mu(\sigma) : \mu \in U\}$$

and, in this case, ν_n is the distribution of the random variable with respect to the Gibbs state for $\psi \equiv 0$. We are interested particularly in $U = C_r := B_r(C)$, and in the deviation of the measures of C_r. The above large deviation principle is the so called level-2 principle and estimates deviation for sets in $\mathcal{M}(\mathcal{M}(\Sigma))$. A consequence of the level 2- large deviation principle is the Varadhan principle

$$\lim_{n\to\infty} \frac{1}{n} \log \left[\int \exp\left(n \int_{\mu \in \mathcal{M}(\Sigma)} \varphi d\mu \right) d\nu_n \right] = \sup\left\{ \int \varphi d\mu + P(0,\sigma) - h_\mu(\sigma) \right\}$$

.

Let us consider the sequence of measures in $\mathcal{M}(\mathcal{M}(\Sigma))$ given by

$$\Omega_n(Z) = \frac{\int_Z \exp\left(n \int \varphi d\mathcal{E}_n\right) d\nu_n}{\int \exp\left(n \int \varphi d\mathcal{E}_n\right) d\nu_n}.$$

To do the estimations which are of our interest we can consider a level-1 large deviation process by mean of the contraction principle, let η_n be the sequences of measures in X obtained by the push-forward of Ω_n by the functional E, i.e. $\eta_n(C_r) = \Omega_n\left(E^{-1}(C_r)\right)$, let $J : X \to \mathbf{R}$ defined by

$$J(x) = \inf\{I_0(\mu) = P(0,\sigma) + h_\mu(\sigma) : E\mu = x\}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \eta_n(C_r) = \liminf_{n\to\infty} \frac{1}{n} \log \nu_n\left(E^{-1}(C_r)\right) \geq -\inf\{J(x) : x \in C\}$$

.

Theorem 2: Let $E:\mathcal{M}\left(\sum\right)\rightarrow X$ be an affine functional, with X a compact metric space. Let φ be a potential in the class $\mathcal{H}$ and $C\subset X$. If C_r denotes the ball $B_r\left(C\right)$ then

$$\begin{aligned}\lim_{r\rightarrow 0}\overline{P}^{C_r}\left(E,\varphi,f\right) &= \lim_{r\rightarrow 0}\quad \underline{P}^{C_r}\left(E,\varphi,f\right)\\ &= \sup\left\{h_\mu\left(\sigma\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}\left(\sum\right)\ E\mu\in C\right\}\end{aligned}$$

.

Proof:.

Like in [20], we arrive to the desired result proving

$$sup\left\{h_\mu\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}_f,\ E\mu\in C\right\}\leq\inf_{r\searrow 0}\underline{P}^{C_r}\left(E,\varphi,f\right)$$

and

$$sup\left\{h_\mu\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}_f,\ E\mu\in C\right\}\geq\inf_{r\searrow 0}\underline{P}^{C_r}\left(E,\varphi,f\right).$$

Now we shall see that

$$\underline{P}^{C_r}\left(E,\varphi,f\right)\geq\sup\left\{h_\mu\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}\left(\Sigma\right),\ E\mu\in C_r\right\}.$$

The behavior of $Z_n^C\left(E,\varphi,\sigma\right)$ may be analyzed by mean of the study of the following expression: fix two cylinders of length 1, namely $\mathcal{C}\left(\ i\right),\mathcal{C}\left(\ \ j\right)$ with $i,j<m$, for some integer m, and set

$$P_n^C=P_n^C\left(\mathcal{C}\left(\ i\right),\mathcal{C}\left(\ \ j\right)\right)=\sum_{\substack{\mathcal{C}(\underline{i}),.\ |\underline{i}|=n\\ \mathcal{C}(\underline{i})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)\\ E\mathcal{E}_n(\mathcal{C}(\underline{i}))\subset C}}\sup_{x\in\mathcal{C}(\underline{i})}\left\{\exp S_n\left(\varphi\right)\left(x\right)\right\},$$

by [24] this expression behaves, for large n, as the pressure independently of the cylinders considered for $i,j<m$.

Let

$$T_n^C\left(\underline{\ i}=i_1i_2...i_n\right)=T_{n.\varphi,E}^C\left(\underline{i}\right)=\sup_{x\in\mathcal{C}(\underline{i})}\left\{\exp S_n\left(\varphi\right)\left(x\right)\right\}. \tag{3}$$

All the cylinders $\mathcal{C}(\underline{\ s})$ considered, for any string $\underline{\ s}$, must satisfy $\mathcal{C}(\underline{\ s}) \subset \mathcal{C}(i) \cap \sigma^{-n}\mathcal{C}(j)$, for the chosen $i, j < m$. Thus

$$
\begin{aligned}
&\int_{\left\{\underline{\ j}\in\mathbf{\Omega}^n : E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C,\ \right\}} T^C_{n.\varphi,E}\left(\underline{\ i}\right) dm\left(\underline{\ i}\right) \\
&= \sum_{\substack{|\underline{\ s}|=n \\ \mathcal{C}(\underline{i})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}} \int_{\left\{\mathcal{C}(\underline{\ s})\cap\left\{\underline{j}\in\Omega^n ;: E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C\right\}\right\}} T^C_{n.\varphi,E}\left(\underline{\ i}\right) dm\left(\underline{\ i}\right) \\
&= \sum_{\substack{|\underline{\ s}|=n \\ \mathcal{C}(\underline{s})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}} T^C_{n.\varphi,E}\left(\ \underline{s}\right) m\left(\mathcal{C}\left(\underline{\ s}\right)\cap\left\{\underline{\ j} : E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C\right\}\right) \\
&= \sum_{\substack{|\underline{\ s}|=n \\ \mathcal{C}(\underline{\ s})\cap\left\{\underline{\ j}: E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C\right\}\neq\varnothing \\ \mathcal{C}(\underline{s})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}} T^C_{n.\varphi,E}\left(\ \underline{s}\right) m\left(\mathcal{C}\left(\underline{\ s}\right)\cap\left\{\underline{\ j} : E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C\right\}\right) \\
&= \sum_{\substack{|\underline{\ s}|=n \\ \mathcal{C}(\underline{\ s})\cap\left\{\underline{\ j}: E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C\right\}\neq\varnothing \\ E\mathcal{E}_n(\mathcal{C}(\ js))\subset C \\ \mathcal{C}(\underline{s})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}} T^C_{n.\varphi,E}\left(\ \underline{s}\right) m\left(\mathcal{C}\left(\underline{\ s}\right)\cap\left\{\underline{\ j} : E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C\right\}\right)
\end{aligned}
$$

.

Thus

$$
\begin{aligned}
&\int_{\left\{\underline{\ j}:: E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C,\ \right\}} T^C_{n.\varphi,E}\left(\ \underline{i}\right) dm\left(\underline{\ i}\right) \\
&= \sum_{\substack{|\underline{\ s}|=n \\ E\mathcal{E}_n(\mathcal{C}(\underline{\ s}))\subset C \\ \mathcal{C}(\underline{s})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}} T^C_{n.\varphi,E}\left(\ \underline{s}\right) m\left(\mathcal{C}\left(\underline{\ s}\right)\cap\left\{\underline{\ j} : E\mathcal{E}_n\left(\mathcal{C}\left(\underline{\ j}\right)\right)\subset C\right\}\right) \\
&= \sum_{\substack{|\underline{\ s}|=n \\ E\mathcal{E}_n(\mathcal{C}(\ js))\subset C \\ \mathcal{C}(\underline{s})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}} T^C_{n.\varphi,E}\left(\ \underline{s}\right) m\left(\underline{\ s}\right)
\end{aligned}
$$

Since $\varphi \in \mathcal{H}$ we can choose a number $K > 0$ such that $\sum_{n=0}^{\infty} V_n(\varphi) < K$ and so

$$
\frac{1}{K} \sup_{x\in\mathcal{C}(\underline{i})} \{\exp S_n(\varphi)(x)\} \leq T^C_{n.\varphi,E}\left(\underline{\ i}\right) \leq K \sup_{x\in\mathcal{C}(\underline{i})} \{\exp S_n(\varphi)(x)\},
$$

and since m is a Gibbs state for the identically zero map we have

$$(KB\ )^{-1}\exp(-nP(0,\sigma))\times\int_{\{\underline{\ j\in\Omega^n}::E\mathcal{E}_n(\mathcal{C}(\underline{\ j}))\subset C,\ \}}\exp\left(n\int\varphi d\mathcal{E}_n\right)m(\underline{\ s})$$
$$\leq\sum_{\substack{|\underline{\ s}|=n\\ E\mathcal{E}_n(\mathcal{C}(\underline{\ s}))\subset C\\ \mathcal{C}(\underline{s})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}}T^C_{n.\varphi,E}(\ \underline{s})\leq$$
$$KB\ \exp(-nP(0,\sigma))\times\int_{\{\underline{\ j\in\Omega^n}::E\mathcal{E}_n(\mathcal{C}(\underline{\ j}))\subset C,\ \}}\exp\left(n\int\varphi d\mathcal{E}_n\right)m(\underline{\ s}).$$

Therefore, by the definition of the measures ν_n

$$\begin{aligned}&\sum_{\substack{|\underline{\ s}|=n\\ E\mathcal{E}_n(\mathcal{C}(\underline{\ s}))\subset C\\ \mathcal{C}(\underline{s})\subset\mathcal{C}(i)\cap\sigma^{-n}\mathcal{C}(j)}}T^C_{n.\varphi,E}(\ \underline{s})\\ \geq\ &(KB\ )^{-1}\exp(-nP(0,\sigma))\times\int_{\{\mathcal{E}_n\in E^{-1}C\ \}}\exp\left(n\int\varphi d\mathcal{E}_n\right)d\nu_n\\ =\ &(KB\ )^{-1}\exp(-nP(0,\sigma))\times\Omega_n\ \left(E^{-1}C\right)\int\exp\left(n\int\varphi d\mathcal{E}_n\right)d\nu_n.\end{aligned}$$

Then considering, instead of C, the open neighborhood C_r of C, setting $P=P(0,\sigma)$and applying the Varadhan result above mentioned and the contraction principle, we get

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{n}\log P_n^{C_r}\\ \geq\ &-P-\inf_{\mu\in\mathcal{M}(\Sigma),\ \mu\in E^{-1}(C_r)}\left\{I_0(\mu)-\int\varphi d\mu\right\}=\\ &\sup\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}(\Sigma),\ E\mu\in C_r\right\}.\end{aligned}$$

Thus we have

$$\underline{P}^{C_r}(E,\varphi,f)\geq\ \sup\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}(\Sigma),\ E\mu\in C_r\right\}\geq$$

$$\sup\left\{h_{\mu}\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}\left(\Sigma\right),\ E\mu\in C\right\},$$

and so

$$\inf_{r\searrow 0}\underline{P}^{C_r}\left(E,\varphi,f\right)\geq\sup\left\{h_{\mu}\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mu\in\mathcal{M}\left(\Sigma\right),\ E\mu\in C\right\}.$$

.

By the **Corollary 1** we have that

$$\begin{aligned}&\overline{P}^{C_r}\left(E,\varphi,f\right)\\ \leq\ &\overline{P}^{\overline{C_r}}\left(E,\varphi,f\right)\\ &\leq\sup\left\{h_{\mu}\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}\left(\Sigma\right),\ E\mu\in\overline{C_r}\ \mu\text{ finitely supported }\right\},\end{aligned}$$

hence

$$\begin{aligned}&\inf_{r\searrow 0}\overline{P}^{C_r}\left(E,\varphi,f\right)\leq\\ &\inf_{r\searrow 0}\sup\left\{h_{\mu}\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}\left(\Sigma\right),\ E\mu\in\overline{C_r}\ \mu\text{ finitely supported }\right\}\end{aligned}$$

To prove the other inequality, let us consider a sequence $r_n\searrow 0$ such that

$$\begin{aligned}&\inf_{r\searrow 0}\sup\left\{h_{\mu}\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}\left(\Sigma\right),\ E\mu\in\overline{C_r}\ \mu\text{ finitely supported }\right\}\\ &=\\ &\inf_{n\nearrow\infty}\sup\left\{h_{\mu}\left(f\right)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}\left(\Sigma\right),E\mu\in\overline{C_{r_n}}\ \mu\text{ finitely supported}\right\}.\end{aligned}$$

Let E_{fin} be the functional E restricted to the set of finitely supported measures in $\mathcal{M}\left(\Sigma\right)$, thus $E_{fin}^{-1}\left(\overline{C_{r_n}}\right)$ is compact and since the map $\mu\longmapsto h_{\mu}\left(f\right)+\int\varphi d\mu$ is upper- semi-continuous, we have[20]

$$\inf_{n\nearrow\infty}\sup\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}(\Sigma),E_{fin}\mu\in\overline{C_{r_n}}\right\}$$
$$=\sup\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}(\Sigma),\mu\in\cap_n E_{fin}^{-1}\left(\overline{C_{r_n}}\right)\right\}.$$

Also

$$\bigcap_n E_{fin}^{-1}\left(\overline{C_{r_n}}\right)\subset E_{fin}^{-1}\left(\bigcap_n\left(\overline{C_{r_n}}\right)\right)=E_{fin}^{-1}\left(\left(\overline{C_r}\right)\right).$$

Therefore

$$\inf_{n\nearrow\infty}\sup\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}(\Sigma),E_{fin}\mu\in\overline{C_{r_n}}\right\}$$
$$=\sup\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}(\Sigma),\mu\in\cap_n E_{fin}^{-1}\left(\overline{C_{r_n}}\right)\right\}\leq$$

$$\sup\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}(\Sigma),\mu\in E_{fin}^{-1}\left(\overline{C_r}\right)\right\}$$
$$\leq\left\{h_\mu(f)+\int\varphi d\mu:\int\varphi d\mu<\infty:\mathcal{M}(\Sigma),\ E\mu\in\overline{C_r}\right\}.$$

It finishes the proof of the theorem

□

Recall that the conditions for existence of a zero for the pressure map are $\overline{P}^C(E,0,\sigma)<\infty,\dim_H\sum<\infty$. Thus from the variational principle the zero value is

$$F(C)=\sup_{\substack{\mu\in\mathcal{M}(\Sigma)\\ E\mu\in C}}\frac{h_\mu(f)}{\int\varphi d\mu}. \tag{4}$$

With respect to the zeta function as an immediate consequence of the variational principle

(5)
$$\lim_{r\searrow 0} -\log \sigma^{C_r}_{E,\varphi,\sigma}(z) = \sup\left\{ h_\mu(f) + \int \varphi d\mu : \int \varphi d\mu < \infty : \mu \in \mathcal{M}(\Sigma),\ E\mu \in C \right\}.$$

4. Multifractal Analysis

We begin this section studying the multifractal spectrum of point-wise dimension of a measure. which corresponds to the particular case of the functional $E\mu = \frac{\int \varphi d\mu}{\int \psi d\mu}$. A kind of spectrum like this was described by Iommi[11] for the single case. Thus the objective is to analyze the following multifractal spectrum

$$K_{C,\mu} = \left\{ x : A_{r\searrow 0}\left(\frac{\log(\mu(B_r(x)))}{-\log r}\right) \subset C \right\}, \tag{6}$$

where $C \subset \mathbf{R}$ and μ is a Gibbs state for the potential φ.If ψ is a metric potential the can be considered a metric such that[11]

$$D_\mu(x = i_0 i_1 ...) = \lim_{n\to\infty} \frac{\log(\mu(\mathcal{C}(i_0 i_1 ... i_{n-1})))}{\log\left(\prod_{i=0}^{n-1} \psi(\sigma^i(x))\right)}. \tag{7}$$

The idea is that used in [11] which is to apply the approximation principle. In our case for the single case $C = \{\alpha\}$ the spectrum is

$$K_{\alpha,\mu} = \left\{ x \in \Sigma : \alpha \in A_{r\searrow 0}\left(\frac{\log(\mu(B_r(x)))}{-\log r}\right) \right\},$$

which is a bit different form that of [11], so we must do the following considerations, for finite alphabet subsfhits, to apply the approximation principle, like in [11]. If ψ is a metric potential, defined on a finite alphabet subshift Σ^N, then it can be modelled by "self-similarities" via Moran constructions or geometric constructions[23]. Specifically, let K be a compact subset of $\mathbf{R}^d$ and let $S_1, ..., S_N$ be contractive similarity maps, i.e. for any $x \in K$ there is a constant $0 < \beta_i = \beta_i(x) < 1$ such that $|D_x S_i(u) - D_x S_i(v)| < \beta_i |u - v|$, for any $u, v \in \mathbf{R}^d$. The subset K satisfies $K = \cup S_i(K)$. If $\underline{i} = i_0 ... i_{n-1}$ then denotes $S_{\underline{i}} = S_{i_{n-1}} \circ ... S_{i_0}$, and $K_{\underline{i}} = S_{\underline{i}}(K)$. There is a code map $\pi : \Sigma^N \to K$ defined by

$$\pi(x = i_0 i_1 ...) = \bigcap_{n\geq 1} \bigcup_{\underline{i} = i_0 i_1 ... i_{n-1}} K_{\underline{i}},$$

the intersection is a single point. Thus we consider $\psi(x) = \log|D_x S_{i_0}(\pi(x))|$, and results $S_n(\psi)(x) = S_{\underline{i}}(\pi(x))$. These maps will be used to define geometric constructions for the compact case we shall used in turn to approximate the countable case.

Let $T : \mathbf{R} \to \mathbf{R}$ be a convex map, the *Legendre transform of* T is a map $\mathcal{E}$ such that

$$\mathcal{E}(\alpha) = \sup_{q \in \mathbf{R}} \{q\alpha + T(q)\}. \tag{8}$$

For $\varphi \in \mathcal{H}$, and ψ a metric potential, let us consider the map, introduced in [11], to describe the spectrum,

$$T(q) = \sup_{m} \{t : P_m(t\psi + q\varphi) = 0\}, \tag{9}$$

where $P_m = P \mid \sum_m$, with $\sum_m = \{x \in \sum : x_n \leq m\}$. A sequence of maps $\widetilde{T}_m :$ $\mathbf{R} \to \mathbf{R}$ can be defined implicitly by

$$P_m\left(\widetilde{T}_m(q)\,\psi + q(\varphi - P_m(\varphi))\right) = 0.$$

The map $T(q)$ is convex and decreasing, the functions $\widetilde{T}_m$ are analytic, decreasing and convex, being valid

$$T(q) = \sup_{m} \left\{\widetilde{T}_m(q)\right\}, [11].$$

This sequence $\widetilde{T}_m$ converges *infimally* to $T(q)$[11]. this means that if $\mathcal{E}_m$ is the Legendre transform of $\widetilde{T}_m$ and if for $r > 0$ denotes $\widetilde{T}_m^{(r)}(q) = \inf\left\{\widetilde{T}_m(s) : |q - s| < r\right\}$ then

$$\lim_{r \to 0} \lim_{m \to \infty} \inf \widetilde{T}_m^{(r)}(q) = \lim_{r \to 0} \lim_{m \to \infty} \sup \widetilde{T}_m^{(r)}(q) = T(q).$$

By the Wijsman theory[29] for convex functions is valid that if the sequence $\widetilde{T}_m(q)$ infimally converges to $T(q)$ then $\mathcal{E}_m$ infimally converges to the Legendre transform $\mathcal{E}(\alpha)$ of $T(q)$. Besides, by the mentioned theory, if $\mathcal{E}_p$ denotes the pointwise limit of the sequence $\mathcal{E}_m$ then $\mathcal{E} \leq \mathcal{E}_p$.

Let $\varphi_q := -T(q)\psi + q\varphi$, this potential has a Gibbs state μ_q for $q > \overline{q} := \inf\{q : \text{there is a real } t \text{ such that } P(-t\psi + q\varphi) \leq 0\}$. If $\dim_H \Sigma < \infty$ then $\overline{q} = 0$ or $\overline{q} = \infty$[11] Besides $P(\varphi_q) \leq 0$.

Let

$$\alpha(q) = \frac{\int \varphi d\mu_q}{\int \psi d\mu_q}, \tag{10}$$

it holds $\alpha(q) = -T'(q)$.

Definition: Let X be a compact metric space and let

$G(X, \alpha, \varepsilon) = \inf_{\mathcal{G}} \left\{ \sum_{Y \in \mathcal{G}} |Y|^{\alpha} \right\}$, where the infimum is taken over all the coverings $\mathcal{G}$ of X with $diam\mathcal{G} \leq \varepsilon$, and $|Y|$ denotes the diameter of Y. The outer $\alpha-$Hausdorff measure of X is $\mu_\alpha(X) = \lim_{\varepsilon \to 0} G(X, \alpha, \varepsilon)$ and finally the Hausdorrff dimension of X is

$\dim_H X = \inf\{\alpha : \mu_\alpha(X) = 0\}$

Theorem 3: Let $\varphi \in \mathcal{H}$, and ψ be a metric potential, let $\sum$ satisfying the $BIP-$property and with $\dim_H \Sigma < \infty$. Let $C \subset \mathbf{R}$, then if $\mathcal{E}(\alpha)$ is the Legendre transform of the map $T(q)$ defined by eq. (10), if

$$K_{C,\mu} := \dim_H \left\{ x \in \Sigma : A_{r \searrow 0}\left(\frac{\log(\mu(B_r(x)))}{\log r}\right) \subset C \right\} \tag{11}$$

then

$$\dim_H K_{C,\mu} = \inf_{\alpha \in C} \{\mathcal{E}(\alpha)\}.$$

To prove one inequality we reformulate the result of [23] (lemma 2), for countable alphabet subshits (in that article is obtained for finite alphabet subshifts). For the other we adapt the techniques of [11], like the infimally convergence and the approximation principle.

Lemma 2: Let $\varphi \in \mathcal{H}$, and ψ be a metric potential, if $\dim_H \Sigma < \infty$, if

$$K_{\alpha,\mu} = \left\{ x \in \Sigma : \alpha \in A_{r \searrow 0}\left(\frac{\log(\mu(B_r(x)))}{-\log r}\right) \right\} \tag{12}$$

then

$$\dim_H K_{\alpha,\mu} \leq \mathcal{E}(\alpha).$$

Proof: By the Birkhoff ergodic theorem we can write

$$\alpha(q) = \lim_{n\to\infty} \frac{\sum\limits_{i=0}^{n-1} \varphi\left(\sigma^i(x)\right)}{\sum\limits_{i=0}^{n-1} \psi\left(\sigma^i(x)\right)},$$

for μ_q-almost $x \in \Sigma$. Therefore if $x \in K_{\alpha(q),\mu}$, then for any $\varepsilon > 0$,there exist a sequence $\{n_k\}$ and a number $N = N(x)$ such that

$$\alpha(q) - \varepsilon \leq \frac{\sum\limits_{i=0}^{n_k-1} \varphi\left(\sigma^i(x)\right)}{\sum\limits_{i=0}^{n_k-1} \psi\left(\sigma^i(x)\right)} \leq \alpha(q) + \varepsilon,$$

for any k with $n_k \geq N$. Let $m > 0$ and let $Z_m = \left\{x \in K_{\alpha(q),\mu} : N \leq m\right\}$, so that $Z_m \subset Z_{m+1}$ and $K_{\alpha(q),\mu} = \bigcup\limits_{m\geq 1} Z_m$.Thus, there is a $m_0 > 0$ such that $\mu_q(Z_m) > 0$ for $m > m_0$. By [23] can be found points $x_j \in Z_m$ in such a way that are determined cylinders $\mathcal{C}_m^j = \mathcal{C}(i_0 i_1 ... i_{n-1}(x_j))$. This cylinders correspond to a Moran covering (for more details see [23]) and if the Moran coefficient r is enough small holds that $n(x_j) \geq m$ for any j (for more details see [23]),Since μ_q is a Gibbs state for the potential $\varphi_q := -T(q)\psi + q\varphi$ we have that

$$B^{-1} \leq \frac{\mu_q\left(\mathcal{C}(i_0 i_1 ... i_{n_k-1})\right)}{\prod\limits_{i=0}^{n_k-1} \psi\left(\sigma^i(x)\right)^{-T(q)} \varphi\left(\sigma^i(x)\right)} \leq B,$$

for some constant B. The inequality holds since $P(\varphi_q) \leq 0$. Hence for $n_k \geq N$, $x \in Z_m$ holds[23]

$$\mu_q\left(Z_m \cap B_r(x)\right) \leq D r^{T(q)+q(\alpha(q)-\varepsilon)},$$

for some constant D, and

$$\mu_q\left(B_r(x)\right) \leq 2\mu_q\left(Z_m \cap B_r(x)\right).$$

Thus, for $m > m_0$ and $x \in Z_m$ holds

$$\begin{aligned} D_{\mu_q}(x) &= \lim_{r\to 0}\frac{\log\left(\mu_q\left(B_r\left(x\right)\right)\right)}{\log r} \\ &= \lim_{r\to 0}\frac{\log\mu_q\left(Z_m\cap\mu\left(B_r\left(x\right)\right)\right)}{\log r}\geq T(q)+q\left(\alpha\left(q\right)-\varepsilon\right), \end{aligned}$$

so that

$$\mu_q\left(B_{2Dr}\left(x\right)\right)\geq\mu_q\left(\mathcal{C}\left(i_0 i_1 ... i_{n_k-1}\right)\right)\geq B^{T(q)+q(\alpha(q)-\varepsilon)},$$

for any $x \in Z_m$. Finally

$$D_{\mu_q}(x)\leq T(q)+q\left(\alpha\left(q\right)-\varepsilon\right),$$

for any $x \in Z_m$ and for arbitrary $\varepsilon > 0$, so $D_{\mu_q}(x) \leq T(q) + q\alpha(q)$, for any μ_q-almost $x \in K_{\alpha(q),\mu}$ and any $q > \overline{q}$. From this is obtained that

$$\dim_H K_{\alpha(q),\mu}\leq T(q)+q\alpha\left(q\right)$$

□

Proof of the **Theorem 3**: for any $\alpha \in C$ we have that $K_{C,\mu} \subset K_{\alpha,\mu}$, therefore,

$$\dim_H K_{C,\mu}\subset\dim_H K_{\alpha,\mu},$$

for any $\alpha \in C$.So that, by lemma 2,

$$\dim_H K_{C,\mu}\leq\inf_{\alpha\in C}\left\{\mathcal{E}\left(\alpha\right)\right\}.$$

To prove the opposite inequality, let $P_m = P \mid \Sigma_m$, recall that there is a finite family $t_m(\alpha)$ such that $P_m(t_m(\alpha)\psi) = 0$, where $t_m(\alpha) = \dim_H(K_{\alpha,\mu} \cap \Sigma_m)$. (Bowen equation for the finite alphabet case). Consequently, there is a sequence $t_m(C)$ such that $P_m(t_m(C)\psi) = 0$, where $t_m(C) = \dim_H(K_{C,\mu} \cap \Sigma_m)$. Recall that $T(q) = \sup_m \left\{\widetilde{T}_m(q)\right\}$, with $\widetilde{T}_m(q)$ implicitly defined by $P_m\left(\widetilde{T}_m(q)\psi + q(\varphi - P_m(\varphi))\right) = 0$. *Re*call also that if $\mathcal{E}_m$ is the Legendre transform of $\widetilde{T}_m$,then $\mathcal{E}_m$ infimally converges to the Legendre transform $\mathcal{E}(\alpha)$ of $T(q)$, and if $\mathcal{E}_p$ denotes the point-wise limit of the sequence $\mathcal{E}_m$ then $\mathcal{E} \leq \mathcal{E}_p$. For any finite alphabet subshift Σ_m can be obtained fractal sets $F_m \subset K_{C,\mu} \cap \Sigma_m$ and with the property that if α_0 is the point of C in which the inf $\{\mathcal{E}_m(\alpha)\}$ is attained then $\dim_H F_m = \mathcal{E}_m(\alpha_0)$ ([15],[23]). The construction of such a fractal sets is done by mean of self-similar geometric constructions with basic sets $\Delta_{\underline{i}} = S_{\underline{i}}(K)$, $\underline{i} = i_0 ... i_{n-1}$, for a compact, simply connected domain K (for details see the above ref.). Thus we have

$$\inf\left\{\mathcal{E}_m\left(\alpha\right)\right\}=\mathcal{E}_m\left(\alpha_0\right)=\dim_H F_m\leq\dim_H\left(K_{C,\mu}\cap\Sigma_m\right)\leq\dim_H K_{C,\mu}$$

Since $\mathcal{E}\leq\mathcal{E}_p$ we obtain $\inf_{\alpha\in C}\left\{\mathcal{E}\left(\alpha\right)\right\}\leq\dim_H K_{C,\mu}$.

□

Next we analyze the following spectrum

$$K_C=\left\{x\in\Sigma : A(E\mathcal{E}_n\left(x\right))\subset C\right\}. \tag{13}$$

In this case we consider a general continuous affine functional $E:\mathcal{M}\left(\sum\right)\rightarrow X$, with X a vector space with a linearly compatible metric and C closed , convex subset of X, unlike in the case of point-wise dimension of a measure, for which are taken $E\mu=\dfrac{\int\varphi d\mu}{\int\psi d\mu}$ and $X=\mathbf{R}$. The condition are more general although the description is not given a Legendre transform but a variational formula. For the special case $E\mu=E_{\varphi}\mu=\int\varphi d\mu$, is obtained the Birkhoff average spectrum for the potential φ.

For finite alphabet subshifts Lyapunov exponents,for the measure μ may be defined. This is done by

$\lambda\left(\mu\right)=\int\ \psi d\mu$, with $\psi\left(x\right)=\log\left|D_xS_{i_0}\left(\pi\left(x\right)\right)\right|$, $x\in\sum^N$. The description of the spectrum like (14) for finite alphabet subshifts is given by the quotient of the entropy and the Lyapunov exponent defined as above.

For countable shifts we shall consider the so called Expanding Markov Renyi maps (*EMR*-maps).

Definition: Let $I=[0,1]$, let $\{I_n\}$ be a countable family of closed subintervals of I, with the property that $int\left(I_n\right)\cap int\left(I_m\right)=\varnothing$, if $n\neq m$,a map $f:\cup I_n\rightarrow I$ is an $EMR-$map if

i) $f\mid\bigcup\limits_{n-1}^{\infty}I_n$ is C^2

ii)$\sup_{x\in I_n}\leq\sup_{x\in I_{n+1}}$, for each $n\in\mathbf{N}$

iii) $\left|\left(f^n\right)^{\acute{}}\left(x\right)\right|>B^n$, for some $B>0$ and for any x.

iv)There is number $D>0$ such that $\sup\limits_n\inf\limits_{x,y,z}\dfrac{\left|f^{\acute{}}(x)\right|}{\left|f^{\acute{}}(y)\right|\left|f^{\acute{}}(z)\right|}\leq D$ (Renyi condition).

The set $\Lambda = \left\{ x \in \cup I_n : f^i(x) \ \in \cup I_n, \text{ for any } i \in \mathbf{N} \right\}$ is called the *repeller* of the map f. One interesting example of $EMR-$map is the Gauss map $f(x) = \frac{1}{x} - \left(\frac{1}{x}\right)$, whose repeller is $\Lambda = I - \mathbf{Q}$.

.Let $\sum$ be a countable full shift, the$EMR-$maps can be coded by a function $\pi : \sum \to I$, which is defined in the following way: any $EMR-$map f has infinite inverse branches i.e, maps $\phi_n : I \to I_n$ such that $(\phi_n \circ f) \mid I_n = id \mid I_n$. If $x = (x_n)_{n \in \mathbf{N}} \in \sum$ then $x \mid n$ denotes the truncated sequence $(x_1 x_2 ... x_n) \in \mathbf{N}^n$ and let $\phi_{x|n} = \phi_{x_n} \circ ... \circ \phi_{x_1}$. In this way the code map is defined as

$$\pi(x) = \bigcap_{n \in \mathbf{N}} \phi_{x|n}.$$

We have that $\pi(\sum) = \Lambda$, except at most a countable set of points, this is due that the intervals in the definition of f have disjoint interiors. Besides$(f \circ \pi)(x) = (\pi \circ \sigma)(x)$, for any $x \in \pi^{-1}(\Lambda)$. Thus we have a coding $\pi : \sum \to \Lambda$ which is injective and surjective except at most a countable set of points. The $EMR-$maps were considered by Iommi and Jordan[10] to give a precise description of multifractal spectra of ergodic averages corresponding to our single case $C = \{\alpha\}$. Thus if we describe the spectrum (14) we shall have a generalization of the result of [10].

If f is a *EMR*-map then the map $x \in \sum \mapsto \log \left| f'(\pi(x)) \right|$ is locally Hӧlder[10]. In this way we define the Lyapunov exponent of μ by $\lambda(\mu) = \lambda(\mu, f) = \int \log \left| f'(\pi(x)) \right|$ $d\mu$.

Theorem 4: :Let $E : \mathcal{M}(\Sigma) \to X$, be a continuous affine functional with X a vector space with a linearly compatible metric. Let C be a closed , convex subset of X. Under these conditions, if

$$K_C = \{x \in \Sigma : A(E\mathcal{E}_n(x)) \subset C\}$$

then

$$\dim_H K_C = \sup \left\{ \frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}(\Sigma), E\mu \in C, \ \lambda(\mu) < \infty \right\}, \tag{14}$$

where $h(\mu)$ is the measure-theoretic entropy of μ and $\lambda(\mu)$ is the Lyapunov exponent defined from a given *EMR*-map.

If $\varphi \in \mathcal{H}$, and let Σ satisfying the $BIP-$property and with $\dim_H \Sigma < \infty$ then with $E_\varphi \mu = \int \varphi d\mu$ and $X = \mathbf{R}$ is obtained, as a particular case, the dimension description of the Birkhoff average spectrum

$$K_C = \left\{ x \in \Sigma : A(\frac{1}{n} S_n(\varphi)(x)) \subset C \right\}.$$

Proof: Let ν be an ergodic measure with $E\nu \in C$, $\lambda(\nu) < \infty$, so $\nu(K_C) = 1$ and since ν is ergodic holds

$$\dim_H \nu = \frac{h(\nu)}{\lambda(\nu)},$$

where $\dim_H \nu = \inf\{\dim_H Z : \nu(Z) = 1\}$. Hence

$$\dim_H K_C \geq \frac{h(\nu)}{\lambda(\nu)},$$

for any ergodic measure ν, thus

$$\dim_H K_C \geq \sup\left\{ \frac{h(\nu)}{\lambda(\nu)} : \nu \in \mathcal{M}(\Sigma), E\nu \in C,\ \nu \text{ ergodic}, \lambda(\nu) < \infty \right\}. \tag{15}$$

If $\mu \in \mathcal{M}(\Sigma)$ then it can be approximated by ergodic measures in the following sense, for any $\mu \in \mathcal{M}(\Sigma)$ and for any arbitrary $\gamma > 0$ there is an ergodic measure ν such that

$h(\nu) \geq h(\mu) - \gamma$, $\lambda(\nu) \geq \lambda(\mu) - \gamma$ and also $d(E\mu, E\nu) < \delta$ for $\delta > 0$. From this deduces

$$\frac{h(\nu)}{\lambda(\nu)} \geq \frac{h(\mu)}{\lambda(\mu)} - A(\delta), \tag{16}$$

with $A(\delta) \to 0$ as $\delta \to 0$. Therefore

$$\sup\left\{ \frac{h(\nu)}{\lambda(\nu)},\ \nu \text{ ergodic}, \lambda(\nu) < \infty,\ d(E\nu, C) < 2\delta \right\} \geq$$

$$\sup\left\{ \frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}(\Sigma),\ \nu \text{ ergodic}, \lambda(\nu) < \infty,\ d(E\mu, C) < 2\delta \right\} + A(\delta),$$

with $\delta \to 0$. Thus

$$\dim_H K_C \geq \sup\left\{ \frac{h(\mu)}{\lambda(\mu)} : \mu \in \mathcal{M}(\Sigma), E\mu \in C,\ \lambda(\mu) < \infty \right\}. \tag{17}$$

Now prove the opposite inequality.

We have the following bounded distortion property: if

$$d_n\left(x,y\right):=\max d\left(\sigma^i\left(x\right),\sigma^i\left(y\right)\right)<\varepsilon$$

then $d\left(\mathcal{E}_n\left(x\right),\mathcal{E}_n\left(y\right)\right)<\epsilon$. In fact, if we have

$$d\left(\mu,\nu\right).;=\sup_{\varphi\in Lip(1)}\left\{\left|\int\varphi d\mu-\int\varphi d\nu\right|\right\},$$

with $Lip(1)$ the set of Lipschitz maps with constant $=1$, then

$$\begin{aligned}d\left(\mathcal{E}_n\left(x\right),\mathcal{E}_n\left(y\right)\right) &= \sup_{\varphi\in Lip(1)}\left\{\left|\int\varphi d\mathcal{E}_n\left(x\right)-\int\varphi d\mathcal{E}_n\left(y\right)\right|\right\}\\ &= \sup_{\varphi\in Lip(1)}\left\{\left|\frac{1}{n}S_n\left(\varphi\right)\left(x\right)-\frac{1}{n}S_n\left(\varphi\right)\left(y\right)\right|\right\}\\ &\leq \frac{1}{n}\sum_{i=0}^{n}\left|\varphi\left(\sigma^i\left(x\right)\right)-\varphi\left(\sigma^i\left(y\right)\right)\right|<\frac{1}{n}n\varepsilon=\varepsilon.\end{aligned}$$

Let $V\left(E,\varepsilon\right)=\sup\left\{d\left(E\mu,E\nu\right):\mu,\nu\in\mathcal{M}\left(\Sigma\right):d\left(\mu,\nu\right)<\varepsilon\right\}$, then $V\left(E,\varepsilon\right)\to 0$ as $\varepsilon\to 0$.

Let us recall the definition of the Hausdorff distance between sets in a metric space Y,

$$D_H(A,B):=\max\left\{\sup_{x\in A}\inf_{y\in B}d\left(x,y\right),\ \sup_{y\in B}\inf_{x\in A}d\left(x,y\right)\right\}. \tag{18}$$

Let $Conv=\{C\subset X,\ \text{convex and closed}\}$ endowed with the Hausdorff distance. Let us consider the map

$$F:Conv\to\mathbf{R}$$
$$F(C)=\sup\left\{\frac{h\left(\mu\right)}{\lambda\left(\mu\right)}:\mu\in\mathcal{M}\left(\Sigma\right),E\mu\in C,\ \lambda\left(\mu\right)<\infty\right\}. \tag{19}$$

We shall see that this function is continuous, let C_0 be a convex, closed subset of X and let (C_n) with $D_H(C_n,C_0)\to 0$. For any $\varepsilon>0$ there is a sequence of measures $\left(\mu_n\right)$, with $\lambda\left(\mu\right)<\infty$, and $E\mu_n\in C_n$ such that
$F(C_n)-\varepsilon\leq\dfrac{h\left(\mu_n\right)}{\lambda\left(\mu_n\right)}$. Let μ be the limit of the sequence $\left(\mu_n\right)$, so that $E\mu\in C_0$.

Let $m_1,\ m_2\in\mathcal{M}\left(\Sigma\right),\ \lambda\left(m_i\right)<\infty$, and such that $Em_1,\ Em_2\in C_0$, We can consider convex combinations $\nu_n=p\mu_n+(1-p)m_i,\ i=1,2$. So that $E\nu_n=$

$pE\mu_n+(1-p)Em_i$. Thus for enough large n we get $E\nu_n\in C_0$. Besides the sequence ν_n can be considered for enough large n hold $\dfrac{h(\mu_n)}{\lambda(\mu_n)}=\dfrac{h(\nu_n)}{\lambda(\nu_n)}$. Therefore we obtain

$\limsup(F(C_n)-\varepsilon)\leq\limsup\dfrac{h(\mu_n)}{\lambda(\mu_n)}=\limsup\dfrac{h(\nu_n)}{\lambda(\nu_n)}\leq F(C_0)$, since ε is arbitrary, we have

$$\limsup_{n\to\infty}F(C_n)\leq F(C_0) \tag{20}$$

Let $m\in\mathcal{M}(\Sigma)$, with $\lambda(m)<\infty$, and. $Em\in C_0$, Let (C_n) with $D_H(C_n,C_0)\to 0$, For $\varepsilon>0$ we may consider family of measures (ν_n) with $E\nu_n\in C_n$, $d(E\nu_n,C_0)\to 0$, and such that

$F(C_0)-\varepsilon\leq\lim\limits_{n\to\infty}\dfrac{h(\nu_n)}{\lambda(\nu_n)}$. Then let us consider convex combinations of ν_n with m, namely $\eta_n=t\nu_n+(1-t)m$, $t\in(0,1)$, thus $d(E\eta_n,C_0)\to 0$, with $t\to 1$. From all this, and arbitrary small $\varepsilon>0$

$$F(C_0)\leq\liminf_{n\to\infty}F(C_n). \tag{21}$$

Let

$$K_{C,n,\varepsilon}=\{x:E\mathcal{E}_n(x)\in B_\varepsilon(C)\}, \tag{22}$$

let us consider, for $\varepsilon>0$, a decomposition of K_C with these sets as $K_C=\bigcup\limits_{n=1}^{\infty}K_{C,n,\varepsilon}$ and cover any $K_{C,n,\varepsilon}$ by cylinders

$$C_n=\{\mathcal{C}(i_1i_2...i_n):\mathcal{C}(i_1i_2...i_n)\cap K_{C,n,\varepsilon}\neq\varnothing\}. \tag{23}$$

We may consider a finite subset A_n of each C_n ,

Let (s_n) be a sequence of real numbers such that

$$\sum_{\mathcal{C}(i_1i_2...i_n)\in A_n}|\mathcal{C}(i_1i_2...i_n)|^{s_n}=1. \tag{24}$$

If $s=\limsup_{n\to\infty}s_n$ then $\dim_H K_{C,n.\varepsilon}\leq s$.

In each cylinder $\mathcal{C}(i_1 i_2 ... i_n)$ of A_n, select a point $\overline{x}_{i_1 i_2 \ldots i_n}$ and let $T_{i_1 i_2 \ldots i_n} = \left\{ \overline{x}_{i_1 i_2 \ldots i_n} \right\}$, therefore

$$cardT_{i_1 i_2 \ldots i_n} = cardA_n$$

Let us assign to any cylinder $\mathcal{C}(i_1 i_2 ... i_n)$ the probability $p_{i_1 i_2 \ldots i_n} = |\mathcal{C}(i_1 i_2 ... i_n)|^{s_n}$ and let us construct measures with this probabilities as weights in this way

$$\nu_n = \frac{1}{cardT_{i_1 i_2 \ldots i_n}} \sum_{\overline{x} \in T_{i_1 i_2 \ldots i_n}} p_{i_1 i_2 \ldots i_n} \mathcal{E}_n(\overline{x}). \tag{25}$$

For any $\overline{x}_{i_1 i_2 \ldots i_n} \in T_{i_1 i_2 \ldots i_n}$, let us consider a point $y = y\left(\overline{x}_{i_1 i_2 \ldots i_n}\right) \in \mathcal{C}(i_1 i_2 ... i_n) \cap K_{C,n,\varepsilon}$, and a pint $z = z\left(\overline{x}_{i_1 i_2 \ldots i_n}\right) \in C$ such that $d\left(E\mathcal{E}_n(y), z\right) < \varepsilon$. Since C is convex, we have that $\widehat{z} := \dfrac{1}{cardT_{i_1 i_2 \ldots i_n}} \sum_{\overline{x} \in T_{i_1 i_2 \ldots i_n}} z\left(\overline{x}_{i_1 i_2 \ldots i_n}\right) \in C$.

Then

$$d\left(E\nu_n, \widehat{z}\right) \leq$$

$$\frac{1}{cardT_{i_1 i_2 \ldots i_n}} \sum_{\overline{x} \in T_{i_1 i_2 \ldots i_n}} \left[d\left(p_{i_1 i_2 \ldots i_n} \mathcal{E}_n(\overline{x}), E\mathcal{E}_n(y)\right) + d\left(E\mathcal{E}_n(y), z\left(\overline{x}_{i_1 i_2 \ldots i_n}\right)\right) \right]$$

$$\leq V(E, \varepsilon) + \varepsilon.$$

The last inequality is due to the bounded distortion property described earlier.

So that $d(E\nu_n, C) \to 0$, as $n \to \infty$ and if μ is the weak-limit of the sequence $\{\nu_n\}$, since C is closed, then $E\mu \in C$. To finish the proof we should show that $\lim\limits_{n\to\infty} \dfrac{h(\nu_n)}{\lambda(\nu_n)} = \lim\limits_{n\to\infty} s_n$. Once established this fact and with $\dim_H K_{C,n.\varepsilon} \leq s$, $E\mu \in C$ we can conclude that

$$\dim_H K_C \leq \lim_{n\to\infty} F(B_\varepsilon(C)),$$

then the the desired inequality is obtained by the continuity of the map F.

The numbers s_n are related with the dimension and by the relationship among dimension, Lyapunov exponents and entropy, there is a $K > 0$ such that

$$\left| -\lambda(\nu_n, f^n) + \frac{h(\nu_n)}{s_n} \right| < K, \tag{26}$$

and so

$$\frac{s_n\left(\lambda\left(\nu_n, f^n\right) - K\right)}{\lambda\left(\nu_n, f^n\right)} \leq \frac{h\left(\nu_n\right)}{\lambda\left(\nu_n, f^n\right)} \leq \frac{s_n\left(\lambda\left(\nu_n, f^n\right) + K\right)}{\lambda\left(\nu_n, f^n\right)}, \tag{27}$$

by the expanding property of the $EMR-$maps (condition iii) of the definition) we have that $\left|(f^n)^{\prime}\right| > B^n$, for some $B > 0$ and so $\lambda\left(\nu_n, f^n\right) > B^n$. From this concludes that $\lim_{n\to\infty}\left(\frac{h\left(\nu_n\right)}{\lambda\left(\nu_n\right)} - s_n.\right) = 0.$

$\square$

References

[1] M. Artin and B. Mazur, On periodic points, Ann. of Math, Vol. 81, 82-99 (1965).

[2] L. Barreira, Y. Pesin and J. Schmeling, On a general goncept of multifractality: Multifractal spectra for dimension, entropies and Lyapunov exponents. Multifractal rigidity, Chaos, Vol. 7, Number 1, 27-38 1997.

[3] L. Barreira and B. Saussol, Multifractal Analysis of hyperbolic flows, Commun. Math. Phys, Vol. 214, 339-371, (2000),

[4] L. Barreira and B. Saussol, Variational principle and mixed multifractal spectra, Trans. Amer. Math. Soc., 116, 3919-3944 (2001).

[5] C. Ercai, T. Küper, S. Lin, Topological entropy for divergent points, Ergod. Th. and Dynam. Sys., Vol. 25, 1173-1208 (2005).

[6] K. J. Falconer, Techniques in Fractal Geometry, John Wiley & sons, 1997.

[7] A. Fan, J. Schmeling and M. Wu, The multifractal spectra of $V-$statistics. Further developments in fractals and related fields. Mathematical foundations and connections. Outgrowth of the 2nd international conference on fractals and related fields, Porquerolles Island, France, June 2011

[8] T.C. Halsey, M. Jensen, L. Kadanoff, I, Procaccia and B. Shraiman, Fractal measures and their singularities: The characterization of strange sets, Phys. Rev. A, Vol. 33, 1141-1151 (1986).

[9] H. G. E. Hentschel and I. Procaccia, The infinite number of generalized dimensions of fractals and strange attractors, Physica D, Vol.8, 435-444 (1983).

[10] G. Iommi and T. Jordan, Multifractal analysis and Birkhoff averages for countable Markov maps, Ergod. Th. and Dynam. Sys., Vol. 35, Number 8, 2559-2586 (2015).

[11] G. Iommi, Multifractal analysis for countable Markov shifts, Ergod. Th. and Dynam. Sys., Vol. 23, 1881-1907 (2005).

[12] T. Jordan and M. Pollicott, Multifractal analysis and the variance of Gibbs measures, Journal of the London Mathematical Society, Vol. 76, 57-72, (2007).

[13] M. Lapidus, J. Lévy-Vehel and J. Rock, Fractal strings and multifractal zeta functions, Lett. Math. Phys, Vol. 88, 101-129 (2009).

[14] M. Lapidus and J. Rock, Towards zeta functions and complex dimensions of multifractals, Complex Var Elliptic Equ. Vol. 54, 545–559, (2009).

[15] J. Li, M. Wu and Y. Xiong, Hausdorff dimensions of the divergence points of self-similar measures with the open set condition, Nonlinearity, Vol. 25 93-105 (2012).

[16] R. D. Mauldin and M. Urbanski, Gibbs states on the symbolic space over an infinite alphabet, Israel J. of Math., Vol. 125, 93-130, (2001).

[17] D. Mauldin and M. Urbanski, Dimension and measures in infinite iterated function systems., Proceedings London Math. Soc., Vol. 73, 105-154 (1996).

[18] A. Mesón and F. Vericat, On the topological entropy of the irregular part of $V-$statistics multifractal spectra, J. Dynam. Sys. and Geom. Theories, Vol. 11, 1-12, (2013).

[19] A. Mesón and F. Vericat, Multifratcal spectrum for barycentric averages, J. Dynam. and Control Sys., Vol. 22, 623-635 (2016).

[20] L. Olsen, Dynamical multifractal zeta functions, multifractal pressure and fine multifractal spectrum, J. d´ Analyse Math., Vol. 131, 207-253., (2017)

[21] W. Parry and M. Pollicott, An analogue of the prime number theorem for closed orbits of Axiom-A flows, Ann. of Math., Vol. 118, 573-591 (1983).

[22] Y. Pesin, Dimension theory in dynamical systems, contemporary views and applications, The University of Chicago Press, 1997.

[23] Y. Pesin and H. Weiss, A Multifractal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric gonstructions, J. Statatistical Phys., Vol. 86, 233-275 (1997).

[24] O. Sarig, Existence of Gibbs measures for countable Markov shifts, Proc. Amer. Math. Soc., Vol. 131, 1751-1758 (2003).

[25] Sarig, O.: Thermodynamic formalism for countable Markov shifts, *Proceedings of Symposia in Pure Mathematics.* **89**, 81–117.(2015).

[26] H. Takahasi, Large deviations principles for countable Markov shifts, Preprint, (2018)

[27] F. Takens and E. Verbitski, Multifractal analysis of local entropies for expansive homeomorphisms with specifications, Commun. Math., Vol. 203, 593-612 (1999).

[28] F. Takens and E. Verbitski, On the variational principle for the topological entropy of certain non-compact sets, Fundamenta Mathematicae, Vol. 105, 203-237 (2000)

[29] R.A. Wijsman, Convergence of sequences of convex sets, cones and functions,Trans. Amer. Mat. Soc., Vol. 123, 32-45 (1966).