

ITERATED FUNCTION SYSTEMS WITH THE WEAK AVERAGE CONTRACTION CONDITIONS

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ABSTRACT. This paper concerns the chaos game of random iterated function systems. We consider a random iterated function system $\text{IFS}(\mathcal{F})$ generated by a finite family of Lipschitz maps on a compact ball of $\mathbb{R}^n$ with the weak average contraction condition and show that it admits a quasi-attractor satisfying the deterministic chaos game. In particular, these properties are preserved under small perturbations of the iterated function system $\text{IFS}(\mathcal{F})$ with respect to the Lipschitz topology.

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1. INTRODUCTION

The concept of fractals was coined by Mandelbrot in 1975 for certain types of fractured sets possessing self-similarity properties. Iterated function systems (IFSs) provide a method for both generating and characterizing fractal images. In [17], Hutchinson developed a theory for fractal sets and showed that a big class of fractals can be described in terms of reversed iterations of functions as the attractor of an IFS. The chaos game algorithm, popularized by Barnsley [1], is a simple algorithm

implementing the random method which allows one to produce pictures of fractal structures, using a computer [24, 2]. In [7] it was shown that, in proper metric spaces, attractors are limits of certain non-stationary stochastic chaos games. The chaos game algorithm was generalized in [9] to a weaker form of attractors the so-called quasi-attractors. Here we discuss random iterations of functions from an IFS based on weak average contraction conditions. This means that the contraction is non-uniform and it is specified in terms of Lyapunov exponents. We focus our study to the chaos game of quasi-attractors of such IFSs and show that they possess quasi-attractors satisfying the deterministic chaos game.

2. BACKGROUND, NOTATION AND MAIN RESULTS

We recall some standard definitions and then formulate our main result. Assume that X is a compact metric space and $\mathcal{F} = \{f_1, \dots, f_k\}$ is a finite family of continuous maps on X . Write $\langle \mathcal{F} \rangle^+$ for the semigroup generated by the collection $\mathcal{F}$ under functions composition. The action of the semigroup $\langle \mathcal{F} \rangle^+$ on X is called the *iterated function system* (IFS) associated to $\mathcal{F}$ and we denote it by $\text{IFS}(X; \mathcal{F})$ or briefly $\text{IFS}(\mathcal{F})$. For $x \in X$, we write the *total orbits* of the action of this semigroup by

$$\mathcal{O}_{\mathcal{F}}^+(x) := \{f(x) : f \in \langle \mathcal{F} \rangle^+\}.$$

Let us consider the symbol space $\Omega = \{1, \dots, k\}^{\mathbb{Z}}$. Take $\mathcal{P} = (p_1, \dots, p_k)$ a probability vector which gives us the probability with which we apply f_i . Equip Ω with the Bernoulli measure $\mathbb{P}$ corresponding to the given probability distribution: $\mathbb{P} = p_0^{\mathbb{Z}}$, where $p_0(i) = p_i$, $i = 1, \dots, k$.

Given $\omega = (\dots \omega_{-1} \omega_0 \omega_1 \omega_2 \dots) \in \Omega$ and $x \in X$, we will denote by

$$f_{\omega}^n := f_{\omega_{n-1}} \circ \dots \circ f_{\omega_0}, \text{ for every } n \in \mathbb{N}, \text{ and } \mathcal{O}_{\omega}^+(x) = \{f_{\omega}^n(x) : n \in \mathbb{N}\}$$

the *orbital branch* corresponding to ω and the ω -*fiberwise orbit* of x , respectively.

We recall that the infinite word $\omega = (\dots \omega_{-1} \omega_0 \omega_1 \omega_2 \dots) \in \Omega$ is *disjunctive* if it contains all possible finite words.

Also, we say that $\text{IFS}(X; \mathcal{F})$ is *forward minimal* if every invariant non-empty closed subset of X is the whole space of X , where $A \subset X$ is *invariant* for $\text{IFS}(X; \mathcal{F})$ if $f(A) \subseteq A$, for all $f \in \mathcal{F}$. That is equivalent to the following: for every point $x \in X$ and every open set U , there exists $h \in \langle \mathcal{F} \rangle^+$ so that $h(x) \in U$.

Now, consider the hyperspace $\mathcal{K}(X)$ consists of all non-empty compact subsets of X endowed with the Hausdorff metric d_H . It is known that if X is compact, so is $\mathcal{K}(X)$ (see [23]). Associated with the set $\mathcal{F}$, we define the *Hutchinson operator* $\mathcal{L} = \mathcal{L}_{\mathcal{F}}$ as follows:

$$\mathcal{L} : \mathcal{K}(X) \rightarrow \mathcal{K}(X), \quad \mathcal{L}(A) = \bigcup_{i=1}^k f_i(A).$$

We also say that A is a *self-similar* set if $A = \mathcal{L}(A) = f_1(A) \cup \cdots \cup f_k(A)$, i.e., A is a fixed point of $\mathcal{L}$.

A compact set $A \in \mathcal{K}(X)$ is a *strict attractor* of $\text{IFS}(X; \mathcal{F})$ if there exists an open neighbourhood $U(A) \supset A$ such that in the Hausdorff metric

$$(1) \quad \mathcal{L}^n(B) \rightarrow A \text{ as } n \rightarrow \infty, \text{ for } U(A) \supset B \in \mathcal{K}(X).$$

The basin $B(A)$ of an attractor A is the union of all open neighborhoods U for which (1) holds. Given an attractor A and for each $\omega \in \Omega$, take

$$(2) \quad A_\omega := \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A).$$

The set A_ω is called a *fibre* of A . If A_ω is a singleton for all $\omega \in \Omega$, then A is said to be *point-fibred*. A is *strongly-fibred* means that if $\mathcal{U}$ is an open cover of A and $U \in \mathcal{U}$ then there is $\omega \in \Omega$ such that $A_\omega \subset U$. Notice that strongly-fibred is weaker than point-fibred which is weaker than the situation where A is the attractor of a contractive IFSs, see [4].

Given a compact set K of X , the Ls-limit set of K for $\mathcal{L}$ is the set

$$(3) \quad Ls(\mathcal{L}^n(K), \mathcal{F}) := \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m} \mathcal{L}^n(K)}.$$

Observe that $Ls(K, \mathcal{F})$ is always closed. However it can be non-compact.

(H1) Let A be a compact subset of X and $x \in X$. We say that x satisfies condition (H1) if for any neighborhood U of A and each disjunctive sequence ω , there exists $n_0 \in \mathbb{N}$ such that $f_\omega^n(x) \in U$ for every $n \geq n_0$.

Now, assume that A is a compact set. So, we define the *pointwise basin of Ls-attraction* of A for $\mathcal{L}$ as the set

$$\mathcal{B}_p^*(A) := \{x \in X : Ls(\{x\}, \mathcal{F}) = A \text{ and (H1) holds}\}.$$

Definition 2.1. We say that A is a quasi-attractor of the IFS generated by $\mathcal{F}$ if it is a forward minimal self-similar compact set, i.e., if

$$A \in \mathcal{K}(X), \mathcal{L}(A) = A \text{ and } A = Cl(\mathcal{O}_{\mathcal{F}}^+(x)) \text{ for all } x \in A.$$

By Theorem 1.2 of [9], a compact set A is a quasi-attractor if and only if $A \subset \mathcal{B}_p^*(A)$. Moreover, in this case $Ls(K, \mathcal{F}) = A$ for all nonempty compact set $K \subset A$.

Definition 2.2. Following [9], we say that a quasi-attractor A of $IFS(X; \mathcal{F})$ is well-fibred if for every compact set K in A so that $K \neq A$ and for any open cover $\mathcal{U}$ of A , there exist $g \in \langle \mathcal{F} \rangle^+$ and $U \in \mathcal{U}$ such that $g(K) \subset U$.

It is not difficult to see that, in the metric space case, a quasi-attractor A is well-fibred if and only if for every compact set K in A so that $K \neq A$, there is a sequence $h_n \in \langle \mathcal{F} \rangle^+$ such that the diameter $\text{diam}(h_n(K))$ converges to zero as $n \rightarrow \infty$. On the other hand, it is easy to show that [9] strongly-fibred implies well-fibred.

The simple and main tool for approximation of attractors is the chaos game. We will focus on a definition of a kind of chaos game which is independent of start points.

Definition 2.3. Let A be a quasi-attractor of the IFS generated by $\mathcal{F}$. Based on [9] we say that A satisfies the deterministic chaos game if there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$A \subset Cl(\mathcal{O}_{\omega}^+(x)) \text{ for all } \omega \in \Omega_0 \text{ and } x \in \mathcal{B}_p^*(A).$$

A simple argument shows that [13] the deterministic chaos game is justified for the case of attractors of contractive IFSs. The deterministic chaos game was obtained in [4] for strongly-fibred attractors. Then in [9] the authors proved that every well-fibred quasi-attractor of an IFS of continuous maps of a Hausdorff first countable topological space satisfies the deterministic chaos game. The objective of this paper is to give a new sufficient condition for a quasi-attractor of an iterated function system that satisfies the deterministic chaos game.

Definition 2.4. Let $\mathcal{F} = \{f_1, \dots, f_k\}$ be a family of Lipschitz maps on $\mathbb{R}^n$ with the map f_i having Lipschitz constant C_i . We form the random iterated function system (abbreviated RIFS) $IFS(X; f_1, \dots, f_k : p_1, \dots, p_k)$ by choosing f_i with probability p_i .

We say that $IFS(X; \mathcal{F})$ has the average contraction condition if

$$(4) \quad \sum_{i=1}^k p_i \log C_i < 0.$$

For each Lipschitz map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we define the norm $\|\cdot\|$ by

$$(5) \quad \|f\| := \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|}$$

and we take the metric

$$(6) \quad \rho(f, g) := \|f - g\| + \sup_{x \in \mathbb{R}^n} |f(x) - g(x)|,$$

where f and g are Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that

$$(7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|f_\omega^n\|$$

exists for $\mathbb{P}$ a.e. $\omega \in \Omega$, where $f_\omega^n := f_{\omega_n} \circ f_\omega^{n-1}$; $\forall n \in \mathbb{N}$. If there exists a real number λ such that $\lambda(\omega) = \lambda$ for a.e. ω then λ is called the *maximal Lyapunov exponent* of $IFS(\mathbb{R}^n; \mathcal{F})$.

Definition 2.5. Following [11], $IFS(\mathbb{R}^n; \mathcal{F})$ has the weak average contraction condition if all the maps in $\mathcal{F}$ are Lipschitzian and $IFS(\mathbb{R}^n; \mathcal{F})$ has negative maximal Lyapunov exponent.

Remark 2.6. In [11, 15], the authors illustrated the examples of random iterated function systems which are not contractive on average, in the most commonly used sense, but they satisfy the weak average contraction condition.

Now we can state the main results of this paper. In what follows, X will stand for a compact ball of $\mathbb{R}^n$. The following theorem adds a new class of systems with the deterministic chaos game.

Theorem A. Assume $IFS(X; \mathcal{F})$ has the weak average contraction condition. Then $IFS(X; \mathcal{F})$ admits a well-fibred quasi-attractor satisfying the deterministic chaos game.

Theorem B. Assume $IFS(X; \mathcal{F})$ has the weak average contraction condition. Then there exists an open set $\mathcal{U}$ of iterated function systems in the Lipschitz topology containing $IFS(X; \mathcal{F})$ so that each $IFS(X; \mathcal{G}) \in \mathcal{U}$ has the weak average contraction condition. Furthermore, $IFS(X; \mathcal{G})$ admits a well-fibred quasi-attractor satisfying the deterministic chaos game.

Definition 2.7. Let A be a quasi-attractor of $\text{IFS}(X; \mathcal{F})$. We say that A is fiberwise minimal if there is $\omega \in \Omega$ such that

$$A \subset \text{Cl}(\mathcal{O}_\omega^+(z)) \text{ for all } z \in \mathcal{B}_p^*(A).$$

Notice that the choice of ω does not depend on the point z .

Corollary C. Assume $\text{IFS}(X; \mathcal{F})$ is an iterated function system having the weak average contraction condition with the quasi-attractor A . Then the attractor A is fiberwise minimal.

3. CHAOS GAME OF IFSs WITH THE WEAK AVERAGE CONTRACTION CONDITION

In this section, we prove the main results of the paper. Throughout this section, we assume that $X \subset \mathbb{R}^n$ is a compact ball containing the origin, $f_i : X \rightarrow X$, $i = 1, \dots, k$, are Lipschitz maps, $\mathcal{F} = \{f_1, \dots, f_k\}$ and take $\text{IFS}(X; \mathcal{F})$ the iterated function system generated by $\mathcal{F}$. Assume that $\text{IFS}(X; \mathcal{F})$ satisfies the weak average contraction condition, this means that $\text{IFS}(X; \mathcal{F})$ admits a negative Lyapunov exponent $\lambda < 0$ given by (7).

Following [21], we take the subset $\mathcal{S} \subset \Omega$ of *weakly hyperbolic sequences* defined by

$$(8) \quad \mathcal{S} = \{\omega \in \Omega : \lim_{n \rightarrow +\infty} \text{diam}(f_{\omega_0} \circ f_{\omega_1} \circ \dots \circ f_{\omega_n}(X)) = 0\}.$$

Consider the coding map

$$(9) \quad \pi : \mathcal{S} \rightarrow X, \text{ by } \pi(\omega) = \lim_{n \rightarrow +\infty} (f_{\omega_0} \circ f_{\omega_1} \circ \dots \circ f_{\omega_n}(x))$$

where x is any point of X . We call $A := \pi(\mathcal{S})$ the *target set* whenever $\mathcal{S} \neq \emptyset$. Moreover, the closure of A is denoted by $\text{Cl}(A)$.

Notice that if f_i , $i = 1, \dots, k$, are strict contractions then $\mathcal{S} = \Omega$ and the set of all limit points is the attractor of the IFS. For the IFS with the average contraction condition, the limit exists for almost all $\omega \in \Omega$ and the attractor may therefore not in general be well defined for such systems. We refer to [3, 25] for a nice discussion and further information.

In what follows, take $\Omega = \{1, \dots, k\}^{\mathbb{Z}}$ equipped with the Bernoulli measure $\mathbb{P}$. Let $\sigma : \Omega \rightarrow \Omega$ denote the left (full) Bernoulli shift. It is well known that σ is an ergodic transformation preserving the probability $\mathbb{P}$, see [26].

An IFS can be embedded in a single dynamical system, the skew product F over the full shift $\sigma : \Omega \rightarrow \Omega$ which is defined by

$$(10) \quad F : \Omega \times \mathbb{R}^n \rightarrow \Omega \times \mathbb{R}^n, \quad F(\omega, x) = (\sigma\omega, f(\omega, x)) = (\sigma\omega, f_{\omega_0}(x)),$$

for each $x \in \mathbb{R}^n$ and $\omega = (\dots\omega_{-n}\dots\omega_{-1}\omega_0\omega_1\dots\omega_n\dots) \in \Omega$. Then for all $n \in \mathbb{N}$, $F^n(\omega, x) = (\sigma^n\omega, f^n(\omega, x))$, where $f^n(\omega, x) = f_{\omega_{n-1}} \circ \dots \circ f_{\omega_0}(x)$.

Theorem 3.1. *Assume $\text{IFS}(X; \mathcal{F})$ has the weak average contraction condition. Then the subset $\mathcal{S}$ of weakly hyperbolic sequences is nonempty and the limit given by (9) exists for almost all $\omega \in \Omega$. Moreover, $\text{Cl}(A)$ is a well-fibred quasi-attractor of $\text{IFS}(X; \mathcal{F})$, where $A = \pi(\mathcal{S})$ is the target set.*

Proof. Consider an iterated function system $\text{IFS}(X; \mathcal{F})$ with the weak average contraction condition. Then, by definition, the maximal Lyapunov exponent of $\text{IFS}(X; \mathcal{F})$, say λ , is negative. Take the associated skew product

$$F : \Omega \times X \rightarrow \Omega \times X, \quad F(\omega, x) = (\sigma\omega, f(\omega, x)) = (\sigma\omega, f_{\omega_0}(x)).$$

For each $\omega \in \Omega$, write $f_\omega^n := f^n(\omega, \cdot) = f_{\omega_{n-1}} \circ \dots \circ f_{\omega_0}$. By hypothesis, for $\mathbb{P}$ a.e. $\omega \in \Omega$, one has

$$(11) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|f_\omega^n\| = \lambda < 0.$$

Notice that for each ω , the family of functions $\alpha_n(\omega) := \log \|f_{\sigma^n(\omega)}\|$ is subadditive, since $\alpha_{n+\ell}(\omega) \leq \alpha_n(\omega) + \alpha_\ell(\omega)$ for all $n, \ell > 0$. Now according to Theorem 5.4 of [20]

$$(12) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|f_\omega^n\| = \inf_n \frac{1}{n} \log \|f_\omega^n\|.$$

As a consequence, given small enough $\varepsilon > 0$ (with $\lambda + \varepsilon < 0$), for $\mathbb{P}$ almost every $\omega \in \Omega$ there exists a measurable function $K : \Omega \rightarrow \mathbb{R}^+$ such that

$$(13) \quad \|f_\omega^n\| \leq K(\omega)e^{(\lambda+\varepsilon)n}, \quad \forall n > 0.$$

Note that since the base map σ is ergodic and invertible then σ^{-1} is also ergodic. Hence σ^{-1} is ergodic with respect to $\mathbb{P}$ and has the same spectrum of Lyapunov exponents by Furstenberg-Kesten Theorem [16], see also [14, Pro. 2]. We define

$$h_n(\omega, x) := f^n(\sigma^{-n}(\omega), x) = f_{\sigma^{-n}(\omega)}^n(x).$$

The above observations with together (11) ensure that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|h_n(\omega, \cdot)\| = \lambda + \epsilon < 0, \text{ for } \mathbb{P} \text{ a.e. } \omega \in \Omega,$$

where $\lambda < 0$ is the maximal Lyapunov exponent of $\text{IFS}(X; \mathcal{F})$. Hence there exists $\ell(\omega)$ such that

$$\|h_n(\omega, \cdot)\| < e^{n(\lambda + \epsilon)} \quad \forall n \geq \ell(\omega).$$

Thus given $\epsilon > 0$ there exists a measurable function $c : \Omega \rightarrow \mathbb{R}^+$ such that for $\mathbb{P}$ a.e. $\omega \in \Omega$, we have

$$(14) \quad \|h_n(\omega, \cdot)\| < c(\omega)e^{(\lambda + \epsilon)n}, \text{ for all } n > 0.$$

Now the approach used in the proof of [10, Thm. 3.1] implies that for fixed $x \in X$ and a.e. $\omega \in \Omega$, the sequence $\{h_\ell(\omega, x)\}$ is a Cauchy sequence. Therefore, for all $x \in X$ and a.e. $\omega \in \Omega$, the limit $\lim_{n \rightarrow +\infty} h_n(\omega, x)$ exists. Furthermore, by construction, for a.e. $\omega \in \Omega$ one has

$$\lim_{n \rightarrow +\infty} |f^n(\omega, x) - f^n(\omega, 0)| = 0.$$

This is because

$$|f^n(\omega, x) - f^n(\omega, 0)| \leq \|f^n(\omega, \cdot)\| |x|$$

and $\|f^n(\omega, \cdot)\| \rightarrow 0$ as $n \rightarrow +\infty$. Thus, there exists a subset $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ so that for each $\omega \in \tilde{\Omega}$,

$$(15) \quad \lim_{n \rightarrow +\infty} f_{\omega_0} \circ f_{\omega_{-1}} \circ \dots \circ f_{\omega_{-n}}(X) = \lim_{n \rightarrow +\infty} f_{\omega_0} \circ f_{\omega_{-1}} \circ \dots \circ f_{\omega_{-n}}(0).$$

In particular, for each $\omega \in \tilde{\Omega}$ one has

$$(16) \quad \lim_{n \rightarrow +\infty} \text{diam}(f_{\omega_0} \circ f_{\omega_{-1}} \circ \dots \circ f_{\omega_{-n}}(X)) = 0.$$

Let us consider the mapping $q : \Omega \rightarrow \Omega$ that is defined by

(17)

$$q(\omega) = \theta, \text{ where } \theta = (\dots, \theta_{-i}, \dots, \theta_{-1}, \theta_0, \theta_1, \dots, \theta_i, \dots) \text{ and } \forall i \in \mathbb{Z}, \theta_i = \omega_{-i}.$$

Clearly, $q : \Omega \rightarrow \Omega$ is a homeomorphism. It is easy to see that for each cylinder set $C = [m; \omega_m, \omega_{m+1}, \dots, \omega_n]$ one has that $\mathbb{P}(C) = \mathbb{P}(q(C))$; hence, $\mathbb{P}(\tilde{\Omega}) = \mathbb{P}(q(\tilde{\Omega})) = 1$.

Let $\theta \in q(\tilde{\Omega})$ and $q^{-1}(\theta) = \omega$. Then $\omega \in \tilde{\Omega}$ and hence by (16)

$$\lim_{n \rightarrow +\infty} \text{diam}(f_{\theta_0} \circ f_{\theta_1} \circ \dots \circ f_{\theta_n}(X)) = \lim_{n \rightarrow +\infty} \text{diam}(f_{\omega_0} \circ f_{\omega_{-1}} \circ \dots \circ f_{\omega_{-n}}(X)) = 0.$$

In particular, $\mathcal{S} \neq \emptyset$, where $\mathcal{S}$ is given by (8), and hence the target set $A \neq \emptyset$. Let $x \in A$. Then there exists $\theta \in \Omega$ such that $x = \lim_{n \rightarrow +\infty} f_{\theta_0} \circ f_{\theta_1} \circ \dots \circ f_{\theta_n}(X) = \lim_{n \rightarrow +\infty} f_{\theta_0} \circ f_{\theta_1} \circ \dots \circ f_{\theta_n}(0)$. By continuity of f_i , for all $1 \leq i \leq k$, $\lim_{n \rightarrow +\infty} \text{diam}(f_i \circ f_{\theta_0} \circ f_{\theta_1} \circ \dots \circ f_{\theta_n}(X)) = 0$. Hence, $f_i(x) = \lim_{n \rightarrow +\infty} f_i \circ f_{\omega_0} \circ f_{\omega_{-1}} \circ \dots \circ f_{\omega_{-n}}(0)$. In particular, $f_i(x) \in A$. Thus, the target set A and therefore its closure $\text{Cl}(A)$ are $\mathcal{F}$ -invariant. On the other hand, if $x \in A$, then $x = \lim_{n \rightarrow +\infty} f_{\theta_0} \circ f_{\theta_1} \circ \dots \circ f_{\theta_n}(X) = \lim_{n \rightarrow +\infty} f_{\theta_0} \circ f_{\theta_1} \circ \dots \circ f_{\theta_n}(0)$, for some $\theta \in \Omega$. Thus, $\lim_{n \rightarrow +\infty} \text{diam}(f_{\theta_0} \circ f_{\theta_1} \circ \dots \circ f_{\theta_n}(X)) = 0$ which implies that $\lim_{n \rightarrow +\infty} \text{diam}(f_{\theta_1} \circ \dots \circ f_{\theta_n}(X)) = 0$. Hence, if $x' = \lim_{n \rightarrow +\infty} f_{\theta_1} \circ \dots \circ f_{\theta_n}(0)$, then $x' \in A$ and $f_{\theta_0}(x') = x$. By these observations, $\mathcal{L}(A) = A$ and hence A is a self similar set. By [5], the Hutschinson operator $\mathcal{L}$ is continuous, hence $\text{Cl}(A)$ is also a self-similar set.

Furthermore, according to [9, Thm. 1.2], $\text{Cl}(A)$ is a quasi-attractor if and only if $\text{Cl}(A) \subset \mathcal{B}_p^*(\text{Cl}(A))$, where $\mathcal{B}_p^*(\text{Cl}(A))$ is the pointwise basin of Ls-attraction of A for $\mathcal{L}$. But this fact is followed by [22, Thm. 1]. Moreover, in this case, $\text{Cl}(A) = \text{Ls}\mathcal{L}^n(K)$ for all nonempty compact sets $K \subset \text{Cl}(A)$, where $\text{Ls}\mathcal{L}^n(K)$ given by (3). In particular, $\lim_{n \rightarrow \infty} \mathcal{L}^n(K) = \text{Cl}(A)$. Thus, $\text{Cl}(A)$ is a forward minimal self-similar compact set and hence a quasi-attractor.

We claim that the quasi-attractor $\text{Cl}(A)$ is well-fibred. For simplicity, we take $B = \text{Cl}(A)$. It is not difficult to see that, in the metric space case, a quasi-attractor B is well-fibred if and only if for every compact set K in B so that $K \neq B$, there is a sequence $h_n \in \langle \mathcal{F} \rangle^+$ such that the diameter $\text{diam}(h_n(K))$ converges to zero as $n \rightarrow \infty$. On the other hand, for each $\omega \in \mathcal{S}$, $\lim_{n \rightarrow +\infty} \text{diam}(f_{\omega_0} \circ f_{\omega_1} \circ \dots \circ f_{\omega_n}(X)) = 0$. By this facts, the subset $\text{Cl}(A)$ is a well-fibred quasi-attractor. Then the proof is completed. $\square$

By [9, Thm. 1.6] every well-fibred quasi-attractor A of an IFS of continuous maps of a Hausdorff topological space satisfies the deterministic chaos game. Hence we get the next result.

Corollary 3.2. *Assume $\text{IFS}(X; \mathcal{F})$ has the weak average contraction condition. Then $\text{IFS}(X; \mathcal{F})$ admits a well-fibred quasi-attractor satisfying the deterministic chaos game.*

Now Theorem A can be followed by the previous corollary.

Remark 3.3. We recall that a fixed point A of the Hutchinson operator is stable if for every open neighbourhood V of A there is an open neighbourhood V_0 of A such that $\mathcal{L}^n(V_0) \subset V$ for all $n \geq 0$. Let $\text{IFS}(X; \mathcal{F})$ be an iterated function system with the target set A . According to [21, Thm. 3], if $\text{Cl}(A)$ is a stable fixed point of the Hutchinson operator then it satisfies the deterministic chaos game. Under our mild assumptions on $\text{IFS}(X; \mathcal{F})$ (the weak average contraction condition), it is unknown the stability of the quasi-attractor $\text{Cl}(A)$ and thus our main result can not be followed by [21, Thm. 3].

We say that $\text{IFS}(X; \mathcal{G})$ with $\mathcal{G} = \{g_1, \dots, g_k\}$ is ε -close to $\text{IFS}(X; \mathcal{F})$ if for each $i = 1, \dots, k$, $\rho(f_i, g_i) < \varepsilon$, where ρ is the metric defined in (6). Then $\text{IFS}(X; \mathcal{G})$ is a small perturbation of $\text{IFS}(X; \mathcal{F})$ whenever it is ε -close to $\text{IFS}(X; \mathcal{F})$ for sufficiently small $\varepsilon > 0$.

Corollary 3.4. Assume $\text{IFS}(X; \mathcal{F})$ has the weak average contraction condition. Then each small perturbation $\text{IFS}(X; \mathcal{G})$ of the iterated function system $\text{IFS}(X; \mathcal{F})$ admits a well-fibred quasi-attractor satisfying the deterministic chaos game.

Proof. Assume $\mathcal{F} = \{f_1, \dots, f_k\}$ and consider $\text{IFS}(X; \mathcal{F})$ the iterated function system generated by $\mathcal{F}$ so that it satisfies the weak average contraction condition. By this condition and ergodicity of Bernoulli measure, $\text{IFS}(X; \mathcal{F})$ admits a negative maximal Lyapunov exponent, say λ . This means that for $\mathbb{P}$ -almost all $\omega \in \Omega$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|f_\omega^n\| = \lambda < 0.$$

Take a small perturbation $\text{IFS}(X; \mathcal{G})$ of the iterated function system $\text{IFS}(X; \mathcal{F})$, with generators $\mathcal{G} = \{g_1, \dots, g_k\}$. Let $g(\omega, x) = g_{\omega_0}(x)$ for each $\omega \in \Omega$ and $x \in X$. Therefore, $g^n(\omega, \cdot) = g_{\omega_{n-1}} \circ \dots \circ g_{\omega_0}$. By Theorem 5.4 of [20], one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|f_\omega^n\| = \inf \frac{1}{n} \log \|f_\omega^n\|.$$

Thus given a sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that if $\rho(f_i, g_i) < \delta$, for each $i = 1, \dots, k$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|g^n(\omega, \cdot)\| = \lambda + \epsilon < 0, \text{ for } \mathbb{P} \text{ a.e. } \omega \in \Omega.$$

This means that $\text{IFS}(X; \mathcal{G})$ has a negative maximal Lyapunov exponent. Now we apply Theorem 3.1 and Corollary 3.2 for $\text{IFS}(X; \mathcal{G})$, we conclude that $\text{IFS}(X; \mathcal{G})$

admits a well-fibred quasi-attractor satisfying the deterministic chaos game; thus the proof is completed. $\square$

Theorem B can be followed by the previous corollary. By [9, Thm. 1.4], the compact set A is a quasi-attractor, if and only if there is $\omega \in \Omega$ such that

$$A \subset \text{Cl}(\mathcal{O}_\omega^+(z)) \text{ for all } z \in \mathcal{B}_p^*(A).$$

Therefore, the attractor A is fiberwise minimal. Hence, Corollary C is verified.

4. EXAMPLE

We assume that X is a metric space and $f : X \rightarrow X$ is a continuous map.

The map f is *weak contractive* whenever for each $x, y \in X$ with $x \neq y$, $d(f(x), f(y)) < d(x, y)$.

It is a well-known fact [18, Coro. 3] that if f is weak contractive and X is compact then there exists a unique fixed point $x \in X$ of the map f . Furthermore, for every $y \in X$, $\lim_{k \rightarrow \infty} f^k(y) = x$ uniformly. Then we say that x is a *weak attracting fixed point*.

Clearly if f is a weak contractive map then

$$d(f^n(y), f^n(z)) \rightarrow 0, \text{ as } n \rightarrow \infty,$$

for each $y, z \in X$.

Now, we illustrate one example of an iterated function system which fits in our situation.

Take two C^r -diffeomorphisms ($r \geq 2$) $f_i, i = 0, 1$, so that they fulfill the following conditions:

- 1) $f_i, i = 0, 1$, is an orientation preserving weak contractive map takes the interval $I = [-1, 1]$ strictly into itself and it is C^2 close to the identity.
- 2) For each $i = 0, 1$, f_i has a unique weak attracting fixed point p_i , i.e. $Df_i(p_i) = 1$. Moreover, $\text{diam}(f_i^n(I)) \rightarrow 0$ as $n \rightarrow +\infty$.
- 3) The weak attracting fixed points $p_i, i = 0, 1$, are pairwise disjoint. Moreover, $p_i \neq 0, 1$ and $f_i(p_j) \neq p_i$ for each $j \neq i$.

It is easily seen that the iterated function system $\text{IFS}(X; \mathcal{F})$ generated by $\mathcal{F} = \{f_0, f_1\}$ admits a negative Lyapunov exponent $\lambda < 0$ and hence satisfies the weak average contraction condition.

Consider the weak contractive map f_0 given by (1) with a weak attracting fixed point p_0 and take a map g such that g is C^2 -close to f_0 and $g = id$ on a sufficiently small neighborhood U of the point p_0 . Then the iterated function system $\text{IFS}(X; \mathcal{G})$ generated by $\mathcal{G} = \{g, f_1\}$ is a small perturbation of $\text{IFS}(X; \mathcal{F})$ and by Corollary 3.4 it admits a negative maximal Lyapunov and hence it enjoys the weak average contraction condition. In particular, it satisfies the assumptions of Theorem A, Theorem B and Corollary C.

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