

## DYNAMICAL SYMMETRIES FOR GRADED VECTOR FIELDS

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(Received: 6 September 2018 , Accepted: 7 February 2019)

**ABSTRACT.** Suppose that  $\mathcal{M} = (M, \mathcal{A}_M)$  is a graded manifold and consider a direct subsheaf  $\mathcal{D}$  of  $Der\mathcal{A}_M$  and a graded vector field  $\Gamma$  on  $\mathcal{M}$ ; both satisfying certain conditions. We associate to the graded vector field  $\Gamma \in Der\mathcal{A}_M$ , a set of 1-forms  $(Der\mathcal{A}_M)^*_{\Gamma}$  and show that if  $\phi \in (Der\mathcal{A}_M)^*_{\Gamma}$  is a non-degenerate graded 1-form and  $X \in Der\mathcal{A}_M$  such that for some superfunction  $f$  on  $\mathcal{M}$ ,  $\mathcal{L}_X(J(\phi)) = df$ , and  $i_X(\phi - (-1)^{|\Gamma|(|\phi|+1)}d(\phi(\Delta))) = 0$ , then the superfunction  $F = f - J(\phi)(X)$  satisfies  $\mathcal{L}_{\Gamma}F = \Gamma(F) = 0$ . This result, generalizes the conditions under which there exist a solution for the inverse problem.

**AMS Classification:** 58A50, 58A30, 58C50, 70H33

**Keywords:** supermanifold; involutive distribution; second-order differential equation field; Lagrangian systems; inverse problem

### 1. INTRODUCTION

Over the past few years, many studies have been devoted to the theory of symmetry for systems of second-order differential equations  $\Gamma$ . One of the issues which

has attracted attention in these studies is the generalization of properties of Lagrangian systems. In the developments in these directions, there have been many important fundamental contributions, such as [12], [7], [15], [9], [14]. In some papers, the geometric characteristics of these systems are calculated. For example, a characterization of symmetries of  $\Gamma$  in terms of the transformations properties of the Lagrangian  $L$  is considered in [7], also, geometric conditions for this vector field,  $\Gamma$ , to be a solution of the dynamics defined by a Lagrangian function, are found in [8]. In some other papers, a version of Noether's theorem is described: Sarlet et al. [15] associate to  $\Gamma$ , a particular subset of 1-forms on  $TM$  such that Lagrangian systems are characterized by the property that this subset contains an element which is non-degenerate and exact.

In this paper, we will show how it is possible to enlarge some of these results to cover the case of a graded Lagrangian dynamic. Of course, it should be noted that good efforts have been made in this regard, see [4], [11], [16, 17].

We remark that the Lagrangian formalism of supermechanics takes place on the tangent supermanifold of a graded manifold of dimension  $(m, n)$ . The work of Ibort et al [11], is an interesting study of the inverse problem of Lagrangian supermechanics. Although the tangent supermanifold is not a bundle but from the point of view of physicist, the arena to develop Lagrangian supermechanics is the tangent supermanifold. On the other hand, the tangent superbundle will be a suitable generalization of the tangent manifold from the theoretical point of view, but its dimension is too big which is  $(2m + n, 2n + m)$  while the dimension of the tangent supermanifold is  $(2m, 2n)$ . Related references are [3, 6]. We use the tangent supermanifold in this paper.

In this paper we start with a graded vector field  $\Gamma$  on a graded manifold  $\mathcal{M} = (M, \mathcal{A}_M)$  of dimension  $(m, n)$  and a direct subsheaf  $\mathcal{D}$  of  $Der\mathcal{A}_M$  of rank  $(r, s)$ , both satisfying certain conditions. By using the Lie bracket of  $\Gamma$  and graded vector fields of  $\mathcal{D}$ , we construct another direct subsheaf  $\mathcal{C} := \mathcal{D} + [\Gamma, \mathcal{D}]$  of  $Der\mathcal{A}_M$  such that  $rank_p(\mathcal{C}) = (r + rank[\Gamma, \mathcal{D}]_0, s + rank[\Gamma, \mathcal{D}]_1)$ . As shown in [1], if  $|\Gamma| = 0$ ,  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have maximal ranks respectively  $r$  and  $s$ , and if  $|\Gamma| = 1$ , depending on the dimension of the graded manifold  $\mathcal{M}$ , there are several cases for introducing the maximal rank of  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$ , which in this article, we only consider one of

them. In the case where  $\mathcal{M}$  is a tangent supermanifold  $(TM', \mathcal{A}_{TM'})$ , we see that the local expression for an even graded vector field  $\Gamma$  is the same as Ibort's one [11].

We associate to the graded vector field  $\Gamma \in \text{Der}\mathcal{A}_M$ , a set of 1-forms  $(\text{Der}\mathcal{A}_M)_\Gamma^*$  defined by,

$$(1) \quad (\text{Der}\mathcal{A}_M)_\Gamma^* = \{\phi \in (\text{Der}\mathcal{A}_M)^* \mid \mathcal{L}_\Gamma(J(\phi)) = (-1)^{|\Gamma|} \phi\}.$$

We show that if  $L$  is a pseudo-Lagrangian function for  $\Gamma$ , i.e.  $dL \in (\text{Der}\mathcal{A}_M)_\Gamma^*$ , then for any closed 1-form  $\alpha$  on  $\mathcal{M}$ , such that  $L = i_\Gamma \alpha$ , we have  $\mathcal{L}_\Gamma(\alpha - (-1)^{|\Gamma|} J(\mathcal{L}_\Gamma \alpha)) = 0$ . Also, if  $\phi \in (\text{Der}\mathcal{A}_M)_\Gamma^*$  is a non-degenerate graded 1-form and  $X \in \text{Der}\mathcal{A}_M$  such that for some superfunction  $f$  on  $\mathcal{M}$ ,  $i_X(\phi - (-1)^{|\Gamma|(|\phi|+1)} d(\phi(\Delta))) = 0$  and  $\mathcal{L}_X(J(\phi)) = df$ , then the superfunction  $F = f - J(\phi)(X)$  satisfies  $\mathcal{L}_\Gamma F = \Gamma(F) = 0$ . We can apply these results to find another Lagrangian as a solution of the inverse problem for  $\Gamma$ . In other words, these results generalize the conditions under which there exist a solution for the inverse problem.

## 2. PRELIMINARIES

Let  $\mathcal{M} = (M, \mathcal{A}_M)$  be an  $(m, n)$ -dimensional graded manifold, the sheaf of left  $\mathcal{A}_M$ -modules of derivations of a graded manifold  $\mathcal{M}$  is the subsheaf of  $\text{End}\mathcal{A}_M$  whose sections are linear graded derivations and denoted by  $\text{Der}\mathcal{A}_M$  the sheaf of graded derivations of  $\mathcal{A}_M$ . Let  $\mathcal{D}$  be a locally free sheaf of  $\mathcal{A}_M$ -modules.  $\mathcal{D}$  is a direct subsheaf of  $\text{Der}\mathcal{A}_M$  of rank  $(r, s)$ , if for each point  $p \in M$  there is an open subset  $U$  over which any set of generators  $\{D_i, D_\mu \mid 1 \leq i \leq r, 1 \leq \mu \leq s\}$  of the module  $\mathcal{D}(U)$  can be enlarged to a set

$$\left\{ C_u, D_i, D_\mu, C_\alpha \mid \begin{array}{l} 1 \leq i \leq r \\ r+1 \leq u \leq m \end{array} \text{ and } \begin{array}{l} 1 \leq \mu \leq s \\ s+1 \leq \alpha \leq n \end{array} \mid \begin{array}{l} C_u = 0 \\ D_i = 0 \end{array} \text{ and } \begin{array}{l} D_\mu = 1 \\ C_\alpha = 1 \end{array} \right\}$$

of free generators of  $\text{Der}\mathcal{A}_M$ . A direct subsheaf  $\mathcal{D}$  of  $\text{Der}\mathcal{A}_M$  is involutive if  $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$  (see [13]).

Let  $\mathcal{D}$  be an involutive direct subsheaf of  $\text{Der}\mathcal{A}_M$  of rank  $(r, s)$  such that  $2r \leq m$ ,  $2s \leq n$  and  $\Gamma$  a homogeneous graded vector field on  $\mathcal{M}$  such that  $\Gamma \notin \mathcal{D}$ . Set

$$\mathcal{C} := \mathcal{D} + [\Gamma, \mathcal{D}] = \{D_1 + [\Gamma, D_2] : D_1, D_2 \in \mathcal{D}\}.$$

Here a brief description of the geometry of  $\Gamma$  and  $\mathcal{D}$  is given, for more details, see [1].

**Theorem 2.1.** *Suppose that the graded vector field  $\Gamma$  on  $\mathcal{M}$  is such that  $[\Gamma, \mathcal{D}] \cap \mathcal{D} = \{0\}$ .*

(1) *If  $|\Gamma| = 0$ ,  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have maximal ranks respectively  $r$  and  $s$ . Then  $\text{rank}_p(\mathcal{C}) = (2r, 2s)$ .*

(2) *If  $|\Gamma| = 1$ , then*

- *for  $m = 2r, n = 2s$ ,*
  - *if  $r \leq s$ , both  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have the same maximal rank  $r$ ,*
  - *if  $r > s$  both  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have the same maximal rank  $s$ ,*
- *for  $m = 2r + 1, n = 2s$ ,*
  - *if  $r < s$ ,  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have maximal ranks respectively  $r + 1$  and  $r$ ,*
  - *if  $r \geq s$  both  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have the same maximal rank  $s$ ,*
- *for  $m = 2r, n = 2s + 1$ ,*
  - *if  $r \leq s$ , both  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have the same maximal rank  $r$ ,*
  - *if  $r > s$ ,  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have maximal ranks respectively  $s$  and  $s + 1$ ,*
- *for  $m = 2r + 1, n = 2s + 1$ ,*
  - *if  $r = s$ , both  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have the same maximal rank  $r$ ,*
  - *if  $r < s$ ,  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have maximal ranks respectively  $r + 1$  and  $r$ ,*
  - *if  $r > s$ ,  $[\Gamma, \mathcal{D}]_0$  and  $[\Gamma, \mathcal{D}]_1$  have maximal ranks respectively  $s$  and  $s + 1$ .*

*In each of these cases,  $\text{rank}_p(\mathcal{C}) = (r + \text{rank}[\Gamma, \mathcal{D}]_0, s + \text{rank}[\Gamma, \mathcal{D}]_1)$ .*

All the details of the proof of this theorem can be found in [1].

Hereafter, unless otherwise stated, we will assume that the graded vector field  $\Gamma$  satisfies the conditions of Theorem 2.1, and  $[\mathcal{C}, \mathcal{C}] \subset \mathcal{C}$ . From the above theorem, we see that if  $\{D_i, D_\mu\}$  is a local basis of  $\mathcal{D}$  and if we set  $C_a = [\Gamma, D_a]$  (for  $|D_a| = 0$ ) and  $C_b = [\Gamma, D_b]$  (for  $|D_b| = 1$ ), then  $\{C_a, D_i, D_\mu, C_b\}$  is a local basis for  $\mathcal{C}$ , where  $\{C_a, C_b\}$  are generators of  $[\Gamma, \mathcal{D}]$ . Thus  $\mathcal{C}$  is a direct subsheaf of  $\text{Der}\mathcal{A}_M$ . Moreover,  $C_a = [\Gamma, D_a]$  and  $C_b = [\Gamma, D_b]$  are respectively odd and even graded vector fields whenever  $\Gamma$  is odd and  $|D_a| = 0, |D_b| = 1$ .

Now we consider a graded tensor field on  $\mathcal{C}$  which can be extended to a tensor field on  $\mathcal{M}$  as a nonlinear connection similar to the classical case. Consider the

tensor field  $J : \mathcal{C} \rightarrow \mathcal{C}$  of type (1,1) by

$$(2) \quad J(X_a) = 0 \quad \text{and} \quad J(Y_a) = -X_a, \quad X_a \in \{D_i, D_\mu\}, Y_a = [\Gamma, X_a].$$

It is called pseudo almost tangent structure (see also [5, 11]). Clearly  $J^2 = 0$  and  $|J| = |\Gamma|$ . If  $|\Gamma| = 0$  then  $\text{Im } J = \text{Ker } J = \mathcal{D}$ .

From now on, we suppose that  $m = n = 2r$  and  $r = s$ . The reason for choosing this particular case is that all the results are valid for the case in which  $\Gamma$  is an even graded vector field,  $r$  and  $s$  are arbitrary and  $m = 2r, n = 2s$ , but some of the theorems can not be concluded for the odd graded vector field  $\Gamma$ .

**Theorem 2.2.** *Let  $\mathcal{D}$  and  $\mathcal{C}$  be involutive and  $\Gamma \in \mathcal{C}$ . Assume that the set  $N = \{z \in M : \Gamma(z) \in \mathcal{D}(z)\} \subset M$ , is nonempty.*

(1) *If  $|\Gamma| = 0$ , then we may choose supercoordinates  $(x_i, y_i; \eta_\mu, \zeta_\mu), i = 1, \dots, r, \mu = 1, \dots, s$ , with respect to which*

$$\Gamma = y_i \frac{\partial}{\partial x_i} + \Gamma_i(x, y; \eta, \zeta) \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + \Gamma'_\mu(x, y; \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}.$$

(2) *If  $|\Gamma| = 1$ , then we may choose supercoordinates  $(x_i, y_i; \eta_\mu, \zeta_\mu), i, \mu, a, b = 1, \dots, r$ , with respect to which*

$$\Gamma = \zeta_a \frac{\partial}{\partial x_a} + \Gamma_i(x, y; \eta, \zeta) \frac{\partial}{\partial y_i} + y_b \frac{\partial}{\partial \eta_b} + \Gamma'_\mu(x, y; \eta, \zeta) \frac{\partial}{\partial \zeta_\mu}.$$

See [1], Theorem 4.3., for details.

### 3. DYNAMICAL SYMMETRY OF SUPER SODE

First, we recall some basic relations from graded tensor calculus necessary to this paper.

**Lemma 3.1.** [2] *For each  $X, Y \in \text{Der } \mathcal{A}_M$ ,  $\psi \in (\text{Der } \mathcal{A}_M)^*$  and (1, 1) tensor filed  $T = Z \otimes \omega$ , we have*

- (1)  $\psi(T(X)) = (-1)^{|T||\psi|} (T(\psi))(X),$
- (2)  $\mathcal{L}_X(\psi(Y)) = (\mathcal{L}_X \psi)(Y) + (-1)^{|\psi||X|} \psi(\mathcal{L}_X Y),$
- (3)  $\mathcal{L}_X(T(Y)) = (\mathcal{L}_X T)(Y) + (-1)^{|X||T|} T(\mathcal{L}_X Y),$
- (4)  $\mathcal{L}_{[X, Y]} = [\mathcal{L}_X, \mathcal{L}_Y].$

With the help of the theorem 2.2, we can write the local representation of  $J$ . The behavior of the tensor  $J$  is similar to the vertical tensor in classical mod. Let  $|\Gamma| = 0$ , and let  $(x, y, \eta, \zeta)$  be the corresponding coordinate (as defined in Theorem 2.2). Then

$$J = \frac{\partial}{\partial y_i} \otimes dx_i - \frac{\partial}{\partial \zeta_\mu} \otimes d\eta_\mu.$$

In case  $|\Gamma| = 1$ ,  $r = s$  and  $(x, y, \eta, \zeta)$  is the corresponding coordinate (as defined in Theorem 2.2), we have

$$J = -\frac{\partial}{\partial \zeta_i} \otimes dx_i - \frac{\partial}{\partial y_\mu} \otimes d\eta_\mu.$$

In both cases, we have  $J^2 = 0$  and  $J(\Gamma) = \Delta$ , where  $\Delta = y_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \zeta_\mu}$ , similar to the one in classical case.

Let  $X \in \text{Der}\mathcal{A}_M$ , we have

$$(3) \quad \mathcal{L}_{J(X)}J = J \circ \mathcal{L}_X J = -(-1)^{|J||X|} \mathcal{L}_X J \circ J.$$

Thus, if  $X \in \text{Der}\mathcal{A}_M$  satisfies  $J(X) = 0$ , then  $J \circ \mathcal{L}_X J = 0$ . In particular for each  $Y \in \text{Der}\mathcal{A}_M$  and  $\psi \in (\text{Der}\mathcal{A}_M)^*$ , we have

$$(4) \quad (\mathcal{L}_\Gamma J) \circ J(Y) = -(-1)^{|\Gamma|} J \circ (\mathcal{L}_\Gamma J)(Y) = (-1)^{|\Gamma|} J(Y)$$

$$(5) \quad -(-1)^{|\Gamma|} (\mathcal{L}_\Gamma J) \circ J(\psi) = J \circ (\mathcal{L}_\Gamma J)(\psi) = (-1)^{|\Gamma|} J\psi.$$

It is also straightforward to obtain

$$(6) \quad (\mathcal{L}_\Gamma J)^2 = I,$$

for  $|\Gamma| = 0$ , or

$$(7) \quad (\mathcal{L}_\Gamma J)^2 = I - 2\left(\frac{\partial}{\partial y_i} \otimes J(d\Gamma_i) - \frac{\partial}{\partial \zeta_\mu} \otimes J(d\Gamma'_\mu)\right)$$

respectively, for  $|\Gamma| = 1$ .

We associate to any homogeneous graded vector field  $\Gamma \in \text{Der}\mathcal{A}_M$ , a set of 1-forms  $(\text{Der}\mathcal{A}_M)_\Gamma^*$  defined by,

$$(8) \quad (\text{Der}\mathcal{A}_M)_\Gamma^* = \{\phi \in (\text{Der}\mathcal{A}_M)^* \mid \mathcal{L}_\Gamma(J(\phi)) = (-1)^{|\Gamma|} \phi\}.$$

From Theorem 2.2, the local expression for  $\phi$  is

$$(9) \quad \phi = A^i dy_i + \Gamma(A^i) dx_i + B^\mu d\zeta_\mu + \Gamma(B^\mu) d\eta_\mu,$$

for  $|\Gamma| = 0$ , or

$$(10) \quad \phi = A^i dy_i + B^\mu d\zeta_\mu + (-1)^{|B^i|} \Gamma(B^i) dx_i + (-1)^{|A^\mu|} \Gamma(A^\mu) d\eta_\mu,$$

respectively, for  $|\Gamma| = 1$ , where,  $A^i$  and  $B^\mu$  are smooth superfunctions on  $\mathcal{M}$ . In both cases, we have

$$\mathcal{L}_\Gamma(J(f\phi)) = (-1)^{|\Gamma|} f\phi + (-1)^{|\Gamma||f|} \Gamma(f)J(\phi).$$

This shows that if  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$  and if  $f \in \mathcal{A}_M$  is a superfunction satisfying  $\Gamma(f) = 0$ , then  $f\phi \in (Der\mathcal{A}_M)_\Gamma^*$ .

Now let  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$  and  $f \in \mathcal{A}_M$ , since  $J^2 = 0$  and  $\mathcal{L}_\Gamma(J(\phi)) = (-1)^{|\Gamma|}\phi$ , we have

$$\begin{aligned} \mathcal{L}_\Gamma(J(f\phi) + (-1)^{|\Gamma|(1+|f|)} \Gamma(f)J(\phi)) \\ &= \mathcal{L}_\Gamma((-1)^{|\Gamma||f|} fJ(\phi) + (-1)^{|\Gamma|(1+|f|)+|\Gamma||\Gamma(f)|} \Gamma(f)J^2(\phi)) \\ &= (-1)^{|\Gamma||f|} (\Gamma(f)J(\phi) + (-1)^{|\Gamma|(1+|f|)} f\phi) \\ &= (-1)^{|\Gamma|} (f\phi + (-1)^{|\Gamma|(1+|f|)} \Gamma(f)J(\phi)). \end{aligned}$$

This shows that  $f\phi \in (Der\mathcal{A}_M)_\Gamma^*$ . Thus  $(Der\mathcal{A}_M)_\Gamma^*$  can be endowed with a  $\mathcal{A}_M$ -module structure by means of the product

$$f \star \phi = f\phi + (-1)^{|\Gamma|(1+|f|)} \Gamma(f)J(\phi), \quad f \in \mathcal{A}_M, \quad \phi \in (Der\mathcal{A}_M)_\Gamma^*.$$

**Proposition 3.2.** *If  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$ , then we have*

$$(11) \quad i_\Gamma dJ(\phi) = (-1)^{|\Gamma|} \phi - (-1)^{|\Gamma||\phi|} d\phi(\Delta)$$

and, for  $|\Gamma| = 0$ ,

$$(12) \quad i_\Gamma(\phi - d\phi(\Delta)) = 0.$$

*Proof.* Since  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$ , from the definition we have  $\mathcal{L}_\Gamma(J(\phi)) = (-1)^{|\Gamma|}\phi$  and then

$$\begin{aligned} i_\Gamma dJ(\phi) &= (-1)^{|\Gamma|} \phi - d(J(\phi)(\Gamma)) \\ &= (-1)^{|\Gamma|} \phi - (-1)^{|\Gamma||\phi|} d(\phi(J(\Gamma))). \end{aligned}$$

Since  $J(\Gamma) = \Delta$ , thus  $i_\Gamma dJ(\phi) = (-1)^{|\Gamma|}\phi - (-1)^{|\Gamma||\phi|}d\phi(\Delta)$ . The second part of the theorem can be easily obtained from  $i_\Gamma(i_\Gamma dJ(\phi)) = 0$  for  $|\Gamma| = 0$ .  $\square$

**Remark 3.3.** *Let us recall that in the non graded case, once we have a second-order differential equation  $\Gamma$ , we can associate to it a 1-form  $\phi = dL$  which satisfies  $\mathcal{L}_\Gamma(J\phi) = \phi$ , where  $J$  is the almost tangent structure on the double tangent manifold and  $L$  is a regular function on the tangent manifold. Regularity of  $L$  is used to define the notion of regularity for  $\phi$ . The analogous projection process can also be done in the graded case. Then, suppose that  $\phi \in (\text{Der}\mathcal{A}_M)_\Gamma^*$ . If  $|\Gamma| = 0$ , then from Theorem 2.2, we have  $J(\phi) = A^i dx_i + B^\mu d\eta_\mu$ . Let*

$$(13) \quad \begin{bmatrix} \frac{\partial A^j}{\partial y_i} & \frac{\partial B^\nu}{\partial y_i} \\ \frac{\partial A^j}{\partial \zeta_\mu} & \frac{\partial B^\nu}{\partial \zeta_\mu} \end{bmatrix}$$

denotes the matrix obtained from  $dJ(\phi)$  such that whose entries,  $\frac{\partial A^j}{\partial y_i}$ ,  $\frac{\partial B^\nu}{\partial y_i}$ ,  $\frac{\partial A^j}{\partial \zeta_\mu}$  and  $\frac{\partial B^\nu}{\partial \zeta_\mu}$  are the coefficients of  $dy_i \wedge dx_j$ ,  $dy_i \wedge d\eta_\nu$ ,  $d\zeta_\mu \wedge dx_j$  and  $d\zeta_\mu \wedge d\eta_\nu$  respectively. Then the regularity condition for  $\phi$  is that the above matrix (13) is non-singular. Similarly, for  $|\Gamma| = 1$  and  $r = s$ , and from Theorem 2.2, we have  $J(\phi) = (-1)^{|\phi|}(-A^\mu d\eta_\mu + B^i dx_i)$ , and the regularity condition for  $\phi$  (in this case) is that the matrix

$$(14) \quad \begin{bmatrix} \frac{\partial B^j}{\partial y_i} & -\frac{\partial A^\nu}{\partial y_i} \\ \frac{\partial B^j}{\partial \zeta_\mu} & -\frac{\partial A^\nu}{\partial \zeta_\mu} \end{bmatrix}$$

obtained from  $dJ(\phi)$  is non-singular, where  $\frac{\partial B^\nu}{\partial y_i}$ ,  $-\frac{\partial A^j}{\partial y_i}$ ,  $\frac{\partial B^\nu}{\partial \zeta_\mu}$  and  $-\frac{\partial A^j}{\partial \zeta_\mu}$  are the coefficients of  $dy_i \wedge dx_j$ ,  $dy_i \wedge d\eta_\nu$ ,  $d\zeta_\mu \wedge dx_j$  and  $d\zeta_\mu \wedge d\eta_\nu$  respectively. Then  $\phi$  is called non-degenerate if the above matrix (14) is non-singular.

**Definition 3.4.** *Let  $\Gamma$  be even (respectively odd),  $\phi \in (\text{Der}\mathcal{A}_M)_\Gamma^*$  is called regular or non-degenerate if the above matrix (13) (respectively (14)), obtained from  $dJ(\phi)$  is non-degenerate.*

**Definition 3.5.** *A graded vector field  $\Gamma$  is called a pseudo-Lagrangian vector field if there exists a non-degenerate  $\phi \in (\text{Der}\mathcal{A}_M)_\Gamma^*$  which is exact:  $\phi = dL$  for some  $L \in \mathcal{A}_M$ .  $L$  is called a pseudo-Lagrangian function for  $\Gamma$ .*

It should be remembered that if  $\phi = dL \in (Der\mathcal{A}_M)_\Gamma^*$ , for some  $L \in \mathcal{A}_M$ , then from the condition  $\mathcal{L}_\Gamma(J(\phi)) = (-1)^{|\Gamma|}\phi$  we can obtain two equations. Using (9) and (10), it is easy to check that these equations, in terms of the supercoordinates we have been using, are

$$(15) \quad \Gamma\left(\frac{\partial L}{\partial y_i}\right) = y_j \frac{\partial^2 L}{\partial y_i \partial x_j} + \Gamma_j \frac{\partial^2 L}{\partial y_i \partial y_j} + \zeta_\nu \frac{\partial^2 L}{\partial y_i \partial \eta_\nu} + \Gamma'_\nu \frac{\partial^2 L}{\partial y_i \partial \zeta_\nu} = \frac{\partial L}{\partial x_i},$$

$$(16) \quad \Gamma\left(\frac{\partial L}{\partial \zeta_\mu}\right) = y_i \frac{\partial^2 L}{\partial \zeta_\mu \partial x_i} + \Gamma_j \frac{\partial^2 L}{\partial \zeta_\mu \partial y_j} + \zeta_\nu \frac{\partial^2 L}{\partial \zeta_\mu \partial \eta_\nu} + \Gamma'_\nu \frac{\partial^2 L}{\partial \zeta_\mu \partial \zeta_\nu} = \frac{\partial L}{\partial \eta_\mu}$$

for  $|\Gamma| = 0$ , or

$$(17) \quad \Gamma\left(\frac{\partial L}{\partial y_i}\right) = \zeta_j \frac{\partial^2 L}{\partial y_i \partial x_j} + \Gamma_j \frac{\partial^2 L}{\partial y_i \partial y_j} + y_\nu \frac{\partial^2 L}{\partial y_i \partial \eta_\nu} + \Gamma'_\nu \frac{\partial^2 L}{\partial y_i \partial \zeta_\nu} = -\frac{\partial L}{\partial \eta_i},$$

$$(18) \quad \Gamma\left(\frac{\partial L}{\partial \zeta_\mu}\right) = \zeta_i \frac{\partial^2 L}{\partial \zeta_\mu \partial x_i} + \Gamma_j \frac{\partial^2 L}{\partial \zeta_\mu \partial y_j} + y_\nu \frac{\partial^2 L}{\partial \zeta_\mu \partial \eta_\nu} + \Gamma'_\nu \frac{\partial^2 L}{\partial \zeta_\mu \partial \zeta_\nu} = \frac{\partial L}{\partial x_\mu},$$

respectively, for  $|\Gamma| = 1$ .

**Theorem 3.6.** *Let  $\alpha$  be a closed 1-form such that  $\mathcal{L}_\Gamma(\alpha - (-1)^{|\Gamma|}J(\mathcal{L}_\Gamma\alpha)) = 0$ , if  $L = i_\Gamma\alpha$ , then  $\Gamma$  is a pseudo-Lagrangian graded vector field. Conversely, if  $L$  is a pseudo-Lagrangian function for  $\Gamma$ , then for any closed 1-form  $\alpha$  such that  $L = i_\Gamma\alpha$ , we have  $\mathcal{L}_\Gamma(\alpha - (-1)^{|\Gamma|}J(\mathcal{L}_\Gamma\alpha)) = 0$ .*

*Proof.* Given a closed 1-form  $\alpha$  satisfying  $\mathcal{L}_\Gamma(\alpha - (-1)^{|\Gamma|}J(\mathcal{L}_\Gamma\alpha)) = 0$ . If  $L = i_\Gamma\alpha$  then  $dL = \mathcal{L}_\Gamma\alpha$ , and

$$\begin{aligned} \mathcal{L}_\Gamma(J(dL)) - (-1)^{|\Gamma|}dL &= \mathcal{L}_\Gamma(J(\mathcal{L}_\Gamma\alpha)) - (-1)^{|\Gamma|}\mathcal{L}_\Gamma\alpha \\ &= -(-1)^{|\Gamma|}\mathcal{L}_\Gamma(\alpha - (-1)^{|\Gamma|}J(\mathcal{L}_\Gamma\alpha)) = 0. \end{aligned}$$

Conversely, let  $\alpha$  be a closed 1-form  $i_\Gamma\alpha = L$ . Then  $\mathcal{L}_\Gamma\alpha = dL$  and

$$\begin{aligned} \mathcal{L}_\Gamma(\alpha - (-1)^{|\Gamma|}J(\mathcal{L}_\Gamma\alpha)) &= \mathcal{L}_\Gamma\alpha - (-1)^{|\Gamma|}\mathcal{L}_\Gamma(J(\mathcal{L}_\Gamma\alpha)) \\ &= dL - (-1)^{|\Gamma|}\mathcal{L}_\Gamma(J(dL)). \end{aligned}$$

Since  $L$  is a pseudo-Lagrangian function for  $\Gamma$ , we have  $\mathcal{L}_\Gamma(\alpha - (-1)^{|\Gamma|}J(\mathcal{L}_\Gamma\alpha)) = 0$ . □

We now introduce tensor fields which preserve  $(Der\mathcal{A}_M)_\Gamma$  and  $(Der\mathcal{A}_M)_\Gamma^*$ .

**Definition 3.7.** Suppose that  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are two  $(1, 1)$  tensor fields. We say that  $\mathcal{B}_1$  commutes with  $\mathcal{B}_2$ , if we have

$$[\mathcal{B}_1, \mathcal{B}_2] = \mathcal{B}_1\mathcal{B}_2 - (-1)^{|\mathcal{B}_1||\mathcal{B}_2|}\mathcal{B}_2\mathcal{B}_1 = 0.$$

Let  $\mathcal{B}$  be a  $(1, 1)$  tensor field such that  $[J, \mathcal{B}] = 0$  and  $\mathcal{L}_\Gamma\mathcal{B} \circ J = 0$ . If  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$ , then from Lemma (3.1), we have

$$\begin{aligned} \mathcal{L}_\Gamma(J(\mathcal{B}(\phi))) - (-1)^{|\Gamma|}|\mathcal{B}(\phi)|\mathcal{B}(\phi) &= \mathcal{L}_\Gamma(J(\mathcal{B}(\phi))) - \mathcal{B}(\mathcal{L}_\Gamma(J(\phi))) \\ &= \mathcal{L}_\Gamma(J(\mathcal{B}(\phi))) - (-1)^{|\Gamma||\mathcal{B}|}(\mathcal{L}_\Gamma(\mathcal{B}(J(\phi)))) - (\mathcal{L}_\Gamma\mathcal{B})(J(\phi)) \\ (19) \quad &= \mathcal{L}_\Gamma([J, \mathcal{B}](\phi)) + (-1)^{|\mathcal{B}||\Gamma|}(\mathcal{L}_\Gamma\mathcal{B} \circ J)(\phi) = 0. \end{aligned}$$

Therefore  $\mathcal{B}((Der\mathcal{A}_M)_\Gamma^*) \subset (Der\mathcal{A}_M)_\Gamma^*$ , i.e.  $\mathcal{B}$  preserves  $(Der\mathcal{A}_M)_\Gamma^*$ . From Theorem 2.2, the local expression for  $\mathcal{B}$  is

$$\begin{aligned} \mathcal{B} = & a_j^k \left\{ \frac{\partial}{\partial x_j} \otimes dx_k + \frac{\partial}{\partial y_j} \otimes dy_k \right\} + \Gamma(a_j^k) \left( \frac{\partial}{\partial y_j} \otimes dx_k \right) \\ & + b_\mu^\nu \left\{ \frac{\partial}{\partial \eta_\mu} \otimes d\eta_\nu + \frac{\partial}{\partial \zeta_\mu} \otimes d\zeta_\nu \right\} + \Gamma(b_\mu^\nu) \left( \frac{\partial}{\partial \zeta_\mu} \otimes d\eta_\nu \right) \\ & + c_j^\mu \left\{ \frac{\partial}{\partial x_j} \otimes d\eta_\mu + \frac{\partial}{\partial y_j} \otimes d\zeta_\mu \right\} + \Gamma(c_j^\mu) \left( \frac{\partial}{\partial y_j} \otimes d\eta_\mu \right) \\ (20) \quad & + d_\mu^j \left\{ \frac{\partial}{\partial \eta_\mu} \otimes dx_j + \frac{\partial}{\partial \zeta_\mu} \otimes dy_j \right\} + \Gamma(d_\mu^j) \left( \frac{\partial}{\partial \zeta_\mu} \otimes dx_j \right). \end{aligned}$$

for  $|\Gamma| = 0$ , or

$$\begin{aligned} \mathcal{B} = & a_j^k \left\{ \frac{\partial}{\partial x_j} \otimes dx_k - \frac{\partial}{\partial \zeta_j} \otimes d\zeta_k \right\} + (-1)^{|\mathcal{B}|} \Gamma(a_j^k) \left( \frac{\partial}{\partial \zeta_j} \otimes dx_k \right) \\ & + b_\mu^\nu \left\{ \frac{\partial}{\partial \eta_\mu} \otimes d\eta_\nu - \frac{\partial}{\partial y_\mu} \otimes dy_\nu \right\} - (-1)^{|\mathcal{B}|} \Gamma(b_\mu^\nu) \left( \frac{\partial}{\partial y_\mu} \otimes d\eta_\nu \right) \\ & + c_j^\mu \left\{ \frac{\partial}{\partial x_j} \otimes d\eta_\mu - \frac{\partial}{\partial \zeta_j} \otimes dy_\mu \right\} - (-1)^{|\mathcal{B}|} \Gamma(c_j^\mu) \left( \frac{\partial}{\partial \zeta_j} \otimes d\eta_\mu \right) \\ (21) \quad & + d_\mu^j \left\{ \frac{\partial}{\partial \eta_\mu} \otimes dx_j - \frac{\partial}{\partial y_\mu} \otimes d\zeta_j \right\} + (-1)^{|\mathcal{B}|} \Gamma(d_\mu^j) \left( \frac{\partial}{\partial y_\mu} \otimes dx_j \right), \end{aligned}$$

respectively, for  $|\Gamma| = 1$ , where  $a_j^k, b_\mu^\nu, c_j^\mu$  and  $d_\mu^j$  are smooth superfunctions on  $\mathcal{M}$ .

**Proposition 3.8.** Let  $Y$  be a graded vector field. With respect to  $\Gamma$ , there exists a unique type  $(1, 1)$  tensor field  $\mathcal{B}_Y$  ( $|\mathcal{B}_Y| = |Y|$ ) on  $\mathcal{M}$ , satisfying the conditions  $[J, \mathcal{B}_Y] = 0$ ,  $\mathcal{L}_\Gamma\mathcal{B}_Y \circ J = 0$  and for each  $\psi \in (Der\mathcal{A}_M)^*$ , we have

$$(22) \quad J \circ (\mathcal{B}_Y - (-1)^{|\Gamma||\psi|}\mathcal{L}_Y J)(\psi) = 0.$$

*Proof.* To find  $\mathcal{B}_Y$ , there are two cases related to  $\Gamma$ , as described above. So given  $\Gamma$  as in Theorem 2.2 and in the first case, assume that  $|\Gamma| = 0$ . Thus  $Y$  can be written in the local representation as

$$Y = a^i \frac{\partial}{\partial x_i} + b^i \frac{\partial}{\partial y_i} + c^\mu \frac{\partial}{\partial \eta_\mu} + d^\mu \frac{\partial}{\partial \zeta_\mu}.$$

By using (20) and (22) we obtain

$$\begin{aligned} a_j^k &= \frac{\partial a^j}{\partial y_k}, & b_\mu^\nu &= -(-1)^{|Y|} \frac{\partial c^\mu}{\partial \zeta_\nu}, \\ c_j^\mu &= -(-1)^{|Y|} \frac{\partial a^j}{\partial \zeta_\mu}, & d_\mu^j &= \frac{\partial c^\mu}{\partial y_j}. \end{aligned}$$

So,

$$\begin{aligned} \mathcal{B}_Y &= \frac{\partial a^j}{\partial y_k} \left\{ \frac{\partial}{\partial x_j} \otimes dx_k + \frac{\partial}{\partial y_j} \otimes dy_k \right\} + \Gamma \left( \frac{\partial a^j}{\partial y_k} \right) \left( \frac{\partial}{\partial y_j} \otimes dx_k \right) \\ &\quad - (-1)^{|Y|} \frac{\partial c^\mu}{\partial \zeta_\nu} \left\{ \frac{\partial}{\partial \eta_\mu} \otimes d\eta_\nu + \frac{\partial}{\partial \zeta_\mu} \otimes d\zeta_\nu \right\} - (-1)^{|Y|} \Gamma \left( \frac{\partial c^\mu}{\partial \zeta_\nu} \right) \left( \frac{\partial}{\partial \zeta_\mu} \otimes d\eta_\nu \right) \\ &\quad - (-1)^{|Y|} \frac{\partial a^j}{\partial \zeta_\mu} \left\{ \frac{\partial}{\partial x_j} \otimes d\eta_\mu + \frac{\partial}{\partial y_j} \otimes d\zeta_\mu \right\} - (-1)^{|Y|} \Gamma \left( \frac{\partial a^j}{\partial \zeta_\mu} \right) \left( \frac{\partial}{\partial y_j} \otimes d\eta_\mu \right) \\ &\quad + \frac{\partial c^\mu}{\partial y_j} \left\{ \frac{\partial}{\partial \eta_\mu} \otimes dx_j + \frac{\partial}{\partial \zeta_\mu} \otimes dy_j \right\} + \Gamma \left( \frac{\partial c^\mu}{\partial y_j} \right) \left( \frac{\partial}{\partial \zeta_\mu} \otimes dx_j \right). \end{aligned}$$

Similarly, if  $|\Gamma| = 1$ , then with respect to the coordinates related to this case, we have  $Y = a^i \frac{\partial}{\partial x_i} + b^i \frac{\partial}{\partial y_i} + c^\mu \frac{\partial}{\partial \eta_\mu} + d^\mu \frac{\partial}{\partial \zeta_\mu}$ , and by using (21) and (22) we obtain

$$\begin{aligned} a_j^k &= (-1)^{|Y|} \frac{\partial a^j}{\partial \zeta_k}, & b_\mu^\nu &= -\frac{\partial c^\mu}{\partial y_\nu}, \\ c_j^\mu &= \frac{\partial a^j}{\partial y_\mu}, & d_\mu^j &= -(-1)^{|Y|} \frac{\partial c^\mu}{\partial \zeta_j}. \end{aligned}$$

So,

$$\begin{aligned} \mathcal{B}_Y &= (-1)^Y \frac{\partial a^j}{\partial \zeta_k} \left\{ \frac{\partial}{\partial x_j} \otimes dx_k - \frac{\partial}{\partial \zeta_j} \otimes d\zeta_k \right\} + \Gamma \left( \frac{\partial a^j}{\partial \zeta_k} \right) \left( \frac{\partial}{\partial \zeta_j} \otimes dx_k \right) \\ &\quad - \frac{\partial c^\mu}{\partial y_\nu} \left\{ \frac{\partial}{\partial \eta_\mu} \otimes d\eta_\nu - \frac{\partial}{\partial y_\mu} \otimes dy_\nu \right\} + (-1)^{|Y|} \Gamma \left( \frac{\partial c^\mu}{\partial y_\nu} \right) \left( \frac{\partial}{\partial y_\mu} \otimes d\eta_\nu \right) \\ &\quad + \frac{\partial a^j}{\partial y_\mu} \left\{ \frac{\partial}{\partial x_j} \otimes d\eta_\mu - \frac{\partial}{\partial \zeta_j} \otimes dy_\mu \right\} - (-1)^{|Y|} \Gamma \left( \frac{\partial a^j}{\partial y_\mu} \right) \left( \frac{\partial}{\partial \zeta_j} \otimes d\eta_\mu \right) \\ &\quad - (-1)^Y \frac{\partial c^\mu}{\partial \zeta_j} \left\{ \frac{\partial}{\partial \eta_\mu} \otimes dx_j - \frac{\partial}{\partial y_\mu} \otimes d\zeta_j \right\} - \Gamma \left( \frac{\partial c^\mu}{\partial \zeta_j} \right) \left( \frac{\partial}{\partial y_\mu} \otimes dx_j \right). \end{aligned}$$

□

Consider the set

$$(23) \quad (Der\mathcal{A}_M)_\Gamma = \{Y \in Der\mathcal{A}_M \mid J([\Gamma, Y]) = 0\}.$$

From Theorem 2.2, the local expression for  $Y \in (Der\mathcal{A}_M)_\Gamma$  is

$$(24) \quad Y = a^i \frac{\partial}{\partial x_i} + \Gamma(a^i) \frac{\partial}{\partial y_i} + c^\mu \frac{\partial}{\partial \eta_\mu} + \Gamma(c^\mu) \frac{\partial}{\partial \zeta_\mu},$$

for  $|\Gamma| = 0$ , or

$$(25) \quad Y = a^i \frac{\partial}{\partial x_i} + c^\mu \frac{\partial}{\partial \eta_\mu} - (-1)^{|c^i|} \Gamma(c^i) \frac{\partial}{\partial y_i} + (-1)^{|a^\mu|} \Gamma(a^\mu) \frac{\partial}{\partial \zeta_\mu},$$

respectively, for  $|\Gamma| = 1$ , where,  $a^i$  and  $c^\mu$  are smooth superfunctions on  $\mathcal{M}$ . Given the local form (24) for  $Y$ , then

$$\begin{aligned} [\Gamma, Y] = & \Gamma(a^i) \frac{\partial}{\partial x_i} + \Gamma\Gamma(a^i) \frac{\partial}{\partial y_i} + \Gamma(c^\mu) \frac{\partial}{\partial \eta_\mu} + \Gamma\Gamma(c^\mu) \frac{\partial}{\partial \zeta_\mu} \\ & - \Gamma(a^i) \frac{\partial}{\partial x_i} - Y(\Gamma_i) \frac{\partial}{\partial y_i} - \Gamma(c^\mu) \frac{\partial}{\partial \eta_\mu} - Y(\Gamma'_\mu) \frac{\partial}{\partial \zeta_\mu}. \end{aligned}$$

Also for (25), we have

$$\begin{aligned} [\Gamma, Y] = & \Gamma(a^i) \frac{\partial}{\partial x_i} + (-1)^{|Y|} \Gamma\Gamma(c^i) \frac{\partial}{\partial y_i} + \Gamma(c^\mu) \frac{\partial}{\partial \eta_\mu} + (-1)^{|Y|} \Gamma\Gamma(a^\mu) \frac{\partial}{\partial \zeta_\mu} \\ & - (-1)^{|Y|} ((-1)^{|Y|} \Gamma(a^i) \frac{\partial}{\partial x_i} + Y(\Gamma_i) \frac{\partial}{\partial y_i} + (-1)^{|Y|} \Gamma(c^\mu) \frac{\partial}{\partial \eta_\mu} + Y(\Gamma'_\mu) \frac{\partial}{\partial \zeta_\mu}). \end{aligned}$$

**Definition 3.9.** A pseudo-dynamical symmetry of  $\Gamma$  is a graded vector field  $Y \in Der\mathcal{A}_M$  such that  $[\Gamma, Y] = 0$ .

From the definition above it is clear that if  $Y$  is a pseudo-dynamical symmetry of  $\Gamma$ , then  $Y \in (Der\mathcal{A}_M)_\Gamma$ . This yields that the coefficients of  $Y$  satisfy

$$(26) \quad \Gamma\Gamma(a^i) = Y(\Gamma_i), \text{ and } \Gamma\Gamma(c^\mu) = Y(\Gamma'_\mu)$$

for  $|\Gamma| = 0$ , or

$$(27) \quad \Gamma\Gamma(c^i) = Y(\Gamma_i), \text{ and } \Gamma\Gamma(a^\mu) = Y(\Gamma'_\mu)$$

respectively, for  $|\Gamma| = 1$ .

The following result is one useful way of formulating the general relation between

$(Der\mathcal{A}_M)_\Gamma$  and  $(Der\mathcal{A}_M)_\Gamma^*$ . Thus, if  $Y \in (Der\mathcal{A}_M)_\Gamma$  and  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$ , then from (8), (23) and Lemma 3.1, we have

$$\begin{aligned} \phi(Y) &= (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\phi))(Y) + \phi(J(\mathcal{L}_\Gamma Y)) \\ (28) \quad &= (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\phi))(Y) + (-1)^{|\Gamma||\phi|} (J(\phi))(\mathcal{L}_\Gamma Y) = (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\phi)(Y)), \end{aligned}$$

In case  $|\Gamma| = 0$  and From (9) and (24), we have

$$(29) \quad \phi(Y) = \Gamma(A^i a^i) + (-1)^{|Y|} \Gamma(B^\mu c^\mu).$$

Similarly, if  $|\Gamma| = 1$  From (10) and (25), we have

$$(30) \quad \phi(Y) = -(-1)^{|\phi|} \Gamma(B^i a^i) + (-1)^{|\phi|+|Y|} \Gamma(A^\mu c^\mu).$$

**Proposition 3.10.** *Let  $\mathcal{B}$  be a  $(1, 1)$  tensor field such that  $[J, \mathcal{B}] = 0$  and  $\mathcal{L}_\Gamma \mathcal{B} \circ J = 0$ . Then  $\mathcal{B}$  preserves  $(Der\mathcal{A}_M)_\Gamma$  and  $(Der\mathcal{A}_M)_\Gamma^*$ .*

*Proof.* In (19) we proved that  $\mathcal{B}$  preserves  $(Der\mathcal{A}_M)_\Gamma^*$ . For  $Y \in (Der\mathcal{A}_M)_\Gamma$  and  $(Der\mathcal{A}_M)_\Gamma^*$ , we have

$$\begin{aligned} \phi(\mathcal{B}(Y)) &= (-1)^{|R||\phi|} \mathcal{B}(\phi)(Y) = (-1)^{|R||\phi|+|\Gamma|} \mathcal{L}_\Gamma(J(\mathcal{B}(\phi))(Y)) \\ &= (-1)^{|R|(|\phi|+|\Gamma|)+|\Gamma|} \mathcal{L}_\Gamma(\mathcal{B}(J(\phi))(Y)) = (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\phi)(\mathcal{B}(Y))) \\ &= (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\phi))(\mathcal{B}(Y)) + (-1)^{|\Gamma||\phi|} J(\phi)(\mathcal{L}_\Gamma(\mathcal{B}(Y))) \\ &= \phi(\mathcal{B}(Y)) + \phi(J([\Gamma, \mathcal{B}(Y)])). \end{aligned}$$

Therefore  $\phi(J([\Gamma, \mathcal{B}(Y)])) = 0$ . Since  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$  is arbitrary, it follows that  $J([\Gamma, \mathcal{B}(Y)]) = 0$ , which implies that  $\mathcal{B}((Der\mathcal{A}_M)_\Gamma) \subset (Der\mathcal{A}_M)_\Gamma$ , i.e.  $\mathcal{B}$  preserves  $(Der\mathcal{A}_M)_\Gamma$ .  $\square$

**Proposition 3.11.** *Let  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$ . Assume that  $X$  is a symmetry of  $\Gamma$ , such that  $\mathcal{L}_X J = 0$ , then  $\mathcal{L}_X \phi \in (Der\mathcal{A}_M)_\Gamma^*$ .*

*Proof.* We show that  $\mathcal{L}_X \phi = (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\mathcal{L}_X \phi))$ . Since  $\phi = (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\phi))$ , it follows that

$$\begin{aligned} \mathcal{L}_X \phi &= (-1)^{|\Gamma|(|X|+1)} \mathcal{L}_\Gamma \mathcal{L}_X(J(\phi)) \\ &= (-1)^{|\Gamma|(|X|+1)} \mathcal{L}_\Gamma(\mathcal{L}_X J(\phi)) + (-1)^{|\Gamma|} (\mathcal{L}_\Gamma(J(\mathcal{L}_X \phi))) \\ &= (-1)^{|\Gamma|} \mathcal{L}_\Gamma(J(\mathcal{L}_X \phi)). \end{aligned}$$

$\square$

**Proposition 3.12.** *Given a non-degenerate graded 1-form  $\phi \in (Der\mathcal{A}_M)_\Gamma^*$ . Let  $X \in Der\mathcal{A}_M$  has the following two properties:*

$$(31) \quad (1) \mathcal{L}_X(J(\phi)) = df, \text{ for some superfunction } f \text{ on } \mathcal{M},$$

$$(32) \quad (2) i_X \left( \phi - (-1)^{|\Gamma|(|\phi|+1)} d(\phi(\Delta)) \right) = 0,$$

then the superfunction  $F = f - J(\phi)(X)$  satisfies  $\mathcal{L}_\Gamma F = \Gamma(F) = 0$ . Conversely, Suppose  $F$  is a superfunction on  $\mathcal{M}$  such that  $\mathcal{L}_\Gamma F = 0$ , there is a graded vector field  $X \in Der\mathcal{A}_M$  satisfying (31) and (32).

*Proof.* From (31) we get

$$i_X(dJ(\phi)) = \mathcal{L}_X(J(\phi)) - d(i_X(J(\phi))) = df - d(J(\phi)(X)) = dF.$$

This yields that

$$i_\Gamma i_X(dJ(\phi)) = \Gamma(F).$$

From (11) and (32) we have

$$\Gamma(F) = i_\Gamma i_X(dJ(\phi)) = -(-1)^{|X||\Gamma|} i_X i_\Gamma(dJ(\phi)) = 0.$$

Conversely, since  $\phi$  is non-degenerate, there exists a graded vector field  $X$ , such that it is the solution of the dynamical equation  $i_X dJ(\phi) = dF$ . From  $F = f - J(\phi)(X)$  we have

$$df = dF + d(J(\phi)(X)) = i_X dJ(\phi) + d(i_X(J(\phi))) = \mathcal{L}_X(J(\phi)),$$

also, from  $\Gamma(F) = 0$  we obtain

$$\begin{aligned} 0 = \Gamma(F) &= i_\Gamma(dF) = i_\Gamma(df - d(i_X(J(\phi)))) = i_\Gamma(df - \mathcal{L}_X(J(\phi)) + i_X \circ d(J(\phi))) \\ &= i_\Gamma \circ i_X(d(J(\phi))) = -(-1)^{|\Gamma||X|} i_X \circ i_\Gamma(d(J(\phi))) \\ &= -(-1)^{|\Gamma||X|} i_X((-1)^{|\Gamma|} \phi - (-1)^{|\Gamma||\phi|} d(\phi(\Delta))). \end{aligned}$$

□

#### 4. THE TANGENT SUPERMANIFOLD AND AN INVERSE PROBLEM

An example of a graded manifold  $\mathcal{M}$  that covers many of the concepts described in the previous sections is the tangent supermanifold  $(TM', \mathcal{A}_{TM'})$ , where  $(M', \mathcal{A}_{M'})$  is a graded manifold. So, here, we verify the results of the previous sections for the tangent supermanifold.

Let  $L$  be a pseudo-Lagrangian function for a graded vector field  $\Gamma$  on  $\mathcal{M}$  and  $\phi = dL$ , we can rewrite the condition  $\mathcal{L}_\Gamma(J(dL)) = (-1)^{|\Gamma|}dL$  in the form

$$\begin{aligned} i_\Gamma d(J(dL)) &= \mathcal{L}_\Gamma(J(dL)) - di_\Gamma(J(dL)) = (-1)^{|\Gamma|}dL - di_\Gamma(J(dL)) \\ &= (-1)^{|\Gamma|}dL - d((-1)^{|\Gamma|}\Delta L) = -(-1)^{|\Gamma|}d(\Delta L - L), \end{aligned}$$

where  $\Delta = y_i \frac{\partial}{\partial y_i} + \zeta_\mu \frac{\partial}{\partial \zeta_\mu}$ . Applying this to tangent supermanifolds, it is easy to see that the above equation is the Euler-Lagrange equation [11], (to achieve this one can choose  $L$  to be a regular Lagrangian on the tangent supermanifold and  $\Gamma$  is the even Euler-Lagrange graded vector field, as the solution of the dynamical equation  $i_\Gamma \Omega = dE$ , where  $\Omega = d(J(dL))$  and  $E = \Delta L - L$ ). The dynamical equations above are equivalent to equations (16)-(18). A Lagrangian function  $L \in \text{Der} \mathcal{A}_{TM'}$  such that is a solution of the Euler-Lagrange equations for  $L$ , will be called a solution of the inverse problem for  $\Gamma$ . A useful description of necessary and sufficient conditions for the existence of a Lagrangian for the super SODE  $\Gamma$  were given in [10]. Then some results from this reference such as generalized symmetries, generalized Noether's theorem are valid in this case. So using  $L$ , we can identify other members of the set  $(\text{Der} \mathcal{A}_M)_\Gamma^*$ .

**Proposition 4.1.** *Let  $f$  be a superfunction on a graded manifold  $(M', \mathcal{A}_{M'})$  and  $\phi \in (\text{Der} \mathcal{A}_{TM'})_\Gamma^*$  then  $\phi' = \phi + d\Gamma(f) \in (\text{Der} \mathcal{A}_{TM'})_\Gamma^*$ .*

*Proof.* A simple coordinate calculation shows that  $J(d\Gamma(f)) = (-1)^{|\Gamma|}df$ , then from (8) we have

$$\begin{aligned} \mathcal{L}_\Gamma(J(\phi + d\Gamma(f))) &= \mathcal{L}_\Gamma(J(\phi)) + \mathcal{L}_\Gamma(J(d\Gamma(f))) \\ &= (-1)^{|\Gamma|}\phi + (-1)^{|\Gamma|}\mathcal{L}_\Gamma(df) \\ &= (-1)^{|\Gamma|}(\phi + d\Gamma(f)). \end{aligned}$$

□

**Example 4.2.** Consider the real supersphere  $\mathbb{S}^{2|2}$  with coordinates  $(x_i, \eta_\mu)$  and let  $(x_i, \eta_\mu, y_i, \zeta_\mu)$  be local coordinates on the tangent supermanifold of  $\mathbb{S}^{2|2}$ . If we assume that  $L = \frac{1}{2}(y_1^2 + y_2^2 \sin^2 x_1) - \zeta_1 \zeta_2$  and put  $\phi = dL$ , then

$$\phi = (\sin x_1 \cos x_1)(y_2)^2 dx_1 + y_1 dy_1 + (\sin^2 x_1) y_2 dy_2 + \zeta_2 d\zeta_1 - \zeta_1 d\zeta_2.$$

Thus the vector field

$$\Gamma = y_i \frac{\partial}{\partial x_i} + \zeta_\mu \frac{\partial}{\partial \eta_\mu} + (\sin x_1 \cos x_1)(y_2)^2 \frac{\partial}{\partial y_1} - (\sin^{-1} x_1)^2 (2 \sin x_1 \cos x_1) y_1 y_2 \frac{\partial}{\partial y_2}$$

is a Lagrangian vector field.

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