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COSMOLOGICAL APPLICATION IN KALUZA-KLEIN THEORY: GENERALIZED SUDDEN SINGULARITY

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Abstract. In this paper, we considered the cosmological application by taking G and Λ to be a function of t in Kaluza-Klein cosmology. We use the Taylor's expansion of cosmological function $\Lambda(t)$, up to the first order of time t and evaluated the cosmological parameters by using the modified equation of state of the form $p = -\rho - f(\rho)$, where $f(\rho) = \alpha\rho$. The analytical properties of $R(t), \rho(t)$ and $H(t)$ are investigated and it is observed that from the solutions of the field equations, the generalized sudden singularity occurs at a finite time in the framework of Kaluza-Klein theory of gravitation.

AMS Classification: 54H20

Keywords: Kaluza-Klein Theory; Taylor's Expansion of cosmological function; Generalized Sudden Singularity

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1. Introduction

The most existing astrophysical work on the cosmological term, operate on the assumption that Λ is a constant. However, quantum field theorists and others treat the cosmological term as a dynamical quantity [[3], [6], [9], [11], [16], [20], [21], [24]]. Chen and Wu [7] argue in favour of R^{-2} dependence on Λ, based on a dimensional argument in line with quantum cosmology. In the spirit of quantum cosmology they also argued that one can always write the vacuum energy density as M_{pl}^4 times a dimensionless quantity, where $M_{pl} = (hc/G)^{1/2}$ is the Plank mass. They therefore, considered

$$\Lambda \propto \frac{1}{l_{pl}^2}\left[\frac{l_{pl}}{R}\right]^n, \tag{1}$$

where $l_{pl} = (Gh/c^3)^{1/2}$ is the Plank length. One may for example, also write $\Lambda \propto \frac{1}{l_{pl}^2}\left[\frac{t_{pl}}{t_H}\right]^n$, where t_{pl} and $t_H = H^{-1}$ are the Plank and Hubble times respectively. Some of the discussions on the cosmological constant problem and consequences on the cosmology with a time-varying cosmological constant have been discussed by [[4], [5], [8], [15], [18], [19], [22]]. Al-Rawaf and Overduin [[1],[17]] have proposed a cosmological model with a cosmological constant of the form $\Lambda \propto \frac{\ddot{R}}{R}$.

The concept of extra dimension is relevant in cosmology particularly at the early universe-theoretically the present four dimension stage of the universe might have been preceded by a higher dimensional phase. One of the classical example of this theory is the Kaluza-Klein theory of gravitation [[10], [12], [13]]. It is a five dimensional general relativity in which extra dimension is used to couple to gravity and electromagenetism [[2], [14], [16]]. Due to importance of the higher dimension theory at the early epoch of the universe, we have solve in this work the gravitational field equations of Kaluza-Klein theory in the domain of cosmology considering the gravitational and cosmological constant as a function of t.

Wanas et al. [25] solved the gravitation field equations of Kaluza-Klein theory in the domain of cosmology considering the gravitational and cosmological constant G and Λ to be the function of time t. They used Taylor's expansion of cosmological function, $\Lambda(t)$, up to the first order of time t and calculated the cosmological parameters by assuming equation of state of the form $p = (\omega - 1)\rho$.

In this paper, we generalized the work of Wanas et al. [25] by assuming the modified equation of state of the form $p = -\rho - f(\rho)$ in the domain of cosmology

and solve the gravitational field equations of Kaluza-Klein theory by considering the gravitational and cosmological constant as a function of t.

The paper is organize as follows: in section 2, Einstein's field equations with cosmological constant of the form $\Lambda \approx \epsilon_0 + \epsilon_1 t + ...$ and gravitational constant $G \propto \epsilon_0 + \epsilon_1 t$ are applied to the five dimensional Kaluza-Klein type metric by using the modified equation of state of the form $p = -\rho - f(\rho)$, where $f(\rho) = \alpha\rho$. In section 3, we obtain the solution of the field equations for the flat universe in the domain of cosmology by considering Taylor's expansion of cosmological constant Λ up to the first order of time t. We calculate the corresponding cosmological parameters for dust, radiation and stiff matter case. It is also observed that from the solutions of the field equations the generalized sudden singularity at finite times occurs in the framework of Kaluza-Klein theory of gravitation. The physical properties of the model are discuss in section 4. Concluding remarks are given in section 5.

2. Field equations

Let us consider the Kaluza-Klein type metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + sin^2\theta d\phi^2) + (1-kr^2)d\psi^2\right], \tag{2}$$

where $R(t)$ is the scale factor and $k = 0. - 1, or + 1$ is the curvature parameter for flat, open and closed universe, respectively.

The Einstein field equations with time-dependent cosmological term Λ is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} + \Lambda(t)g_{\mu\nu}, \tag{3}$$

where $R_{\mu\nu}$ is the Ricci tensor, $\Lambda = \Lambda(t)$ is the variable cosmological constants and $G = G(t)$ be the gravitational constant.

Assuming that matter filling the universe is in the form of a perfect fluid, then

$$T_0^0 = \rho + \frac{\Lambda}{8\pi G},$$

$$T_1^1 = T_2^2 = T_3^3 = T_4^4 = -p + \frac{\Lambda}{8\pi G}, \tag{4}$$

where $\rho = \rho(t)$ is the energy density and $p = p(t)$ is the pressure.

Applying Eq. (3) to metric given by Eq. (2) and using Eq. (4) we obtain the

following set of differential equations

$$8\pi G + \Lambda = 6\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}\right], \tag{5}$$

$$8\pi G - \Lambda = -3\left[\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 - \frac{k}{R^2}\right]. \tag{6}$$

Now we have two differential equations in five unknown $R(t)$, $\rho(t)$, $p(t)$, $\Lambda(t)$ and $G(t)$. Therefore, we need some extra conditions to be able to solve Eqs. (5) and (6), for each value of k. One of these conditions is usually provided by an equation of state characterizing the fluid filling the model.

Here, we take the modified equation of state of the form

$$p = -\rho - f(\rho), \tag{7}$$

where $f(\rho) = \alpha\rho$.

The energy conservation equation has the form $T^{\mu\nu}_{;\nu} = 0$, gives the relation,

$$\dot{\rho} + 4\frac{\dot{R(t)}}{R(t)}(\rho + p) + \frac{\Lambda}{8\pi G}\left(\frac{\dot{\Lambda}}{\Lambda} - \frac{\dot{G}}{G}\right) = 0. \tag{8}$$

Eqs. (5), (6) and (7) are three equations in the five unknown functions. Therefore, we split Eq. (8) for simplicity into the two equations

$$\dot{\rho} + 4\frac{\dot{R(t)}}{R(t)}(\rho + p) = 0, \tag{9}$$

and

$$\frac{\dot{\Lambda}}{\Lambda} - \frac{\dot{G}}{G} = 0, \tag{10}$$

in order to impose two further additional conditions on the unknown functions. Now, we have five Eqs. (5), (6), (7), (9) and (10) in five unknown functions $R(t), \rho(t), p(t), \Lambda(t)$ and $G(t)$.

Eq. (10) shows that Λ is a constant whenever G is constant and vice versa.

To solve the conservation Eq. (9), by using Eq. (7) with $f(\rho) = \alpha\rho$, we get

$$\rho = C_1 R^{4\alpha}, \tag{11}$$

where $R(t)$ is a function of cosmic time and C_1 is a constant of integration.

3. Solution for $k = 0$

We solve the field Eqs. (5) and (6) by using Taylor's expansion of cosmological constant $\Lambda(t)$ up to the first order of time t in the form

$$\Lambda(t) \approx \epsilon_0 + \epsilon_1 t + ..., \tag{12}$$

where ϵ_0 and ϵ_1... are arbitrary constants.
From Eq. (10) we obtain

$$G(t) = C_2(\epsilon_0 + \epsilon_1 t), \tag{13}$$

where C_2 is another constant of integration.
Eqs. (5) and (6), for the case $k = 0$, takes the form

$$8\pi G\rho + \Lambda = 6\left(\frac{\dot{R}}{R}\right)^2, \tag{14}$$

$$8\pi Gp - \Lambda = -3\left[\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 - \Lambda\right]. \tag{15}$$

From Eqs. (14), (15) and equation of state (7), we obtain

$$3\frac{\ddot{R}}{R} - 3\left(\frac{\dot{R}(t)}{R(t)}\right)^2 + \alpha\left[6\frac{\dot{R^2}(t)}{R^2(t)} - \Lambda\right] = 0,$$

$$\frac{d}{dt}\left(\frac{\dot{R}}{R}\right) - \alpha\left[2\left(\frac{\dot{R}}{R}\right)^2 - \frac{\Lambda}{3}\right] = 0.$$

By using the value of Λ from Eq. (12) in above equation, we get

$$\dot{H(t)} - \alpha\left[2H^2(t) - \frac{\epsilon_0}{3} - \frac{\epsilon_1 t}{3}\right] = 0. \tag{16}$$

Eq. (16) is a differential equation with two variables so we are going to study the following three cases: (i) dust, (ii) radiation and (iii) stiff matter.

3.1. **Dust case ($\alpha = -1$).** From the Eq. (16) with $\alpha = -1$, i.e., $p = 0$ from Eq. (7), we get

$$H\dot{(}t) + 2H^2(t) - \frac{\epsilon_0}{3} - \frac{\epsilon_1 t}{3} = 0. \tag{17}$$

After integrating above Eq. (17), we get [1]

$$H(t) = \left(\frac{\epsilon_1}{12}\right)^{\frac{1}{3}} \left[\frac{A^r_{BiP}(x) + A^r_{AiP}(x)C_3}{A^r_{Bi}(x) + A^r_{Ai}(x)C_3}\right], \tag{18}$$

where C_3 is constant of integration and $x = \left(\left[\frac{18}{\epsilon_1^2}\right]^{\frac{1}{3}} (\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3})\right)$.
Differentiate Eq. (18), we get

$$\dot{H}(t) = \left(\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3}\right) - \left(\frac{\epsilon_1^2}{18}\right)^{\frac{1}{3}} \left(\frac{\left[A^r_{BiP}(x) + A^r_{AiP}(x)C_3\right]^2}{\left[A^r_{Bi}(x) + A^r_{Ai}(x)C_3\right]^2}\right). \tag{19}$$

For the scale factor R, from Eq. (18) we get

$$\frac{\dot{R}}{R} = \left(\frac{\epsilon_1}{12}\right)^{\frac{1}{3}} \left[\frac{A^r_{BiP}(x) + A^r_{AiP}(x)C_3}{A^r_{Bi}(x) + A^r_{Ai}(x)C_3}\right],$$

after integrating, we obtain the scale factor in the form

$$R(t) = C_4\sqrt{A^r_{Bi}(x) + A^r_{Ai}(x)C_3}\,, \tag{20}$$

where C_4 is constant of integration.
Similarly from Eqs. (11) and (20), we get energy density in the form

$$\rho(t) = \left[\frac{C_1}{(A^r_{Bi}(x) + A^r_{Ai}(x)C_3)^2 C_4^4}\right]. \tag{21}$$

[1] $A^r_{Bi}(x), A^r_{Ai}(x)$ are the $AiryBi(x)$, $AiryAi(x)$ be the wave functions and $A^r_{BiP}(x)$, $A^r_{AiP}(x)$ are the $AiryBiPrime(x), AiryAiPrime(x)$ functions. The Airy functions $AiryAi(x)$ and $AiryBi(x)$ are the two independent solutions $Ai(x)$ and $Bi(x)$ to the differential equation $y'' - xy = 0$. $Ai(x)$ tends to zero for large positive x, while $Bi(x)$ increases unboundedly. The Airy functions are related to modified Bessel functions with on-third-integer orders. The Airy functions often appear as the solutions to boundary value problems in electromagnetic theory and quantum mechanics.

3.2. **Radiation case ($\alpha = -\frac{5}{4}$).** From the Eq. (16) with $\alpha = -\frac{5}{4}$, i.e., $p = \rho/4$ from Eq. (7), we get

$$H\dot{(t)} + \frac{5}{2}H^2(t) - \frac{\epsilon_0}{3} - \frac{\epsilon_1 t}{3} = 0. \tag{22}$$

After integrating above equation, we get

$$H(t) = \left(\frac{\epsilon}{15}\right)^{\frac{1}{3}} \left[\frac{A^r_{BiP}(y) + A^r_{AiP}(y)C_5}{A^r_{Bi}(y) + A^r_{A}(y)C_5}\right], \tag{23}$$

where C_5 is constant of integration and $y = \left(\left[\frac{225}{8\epsilon_1^2}\right]^{\frac{1}{3}} (\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3})\right)$.
Differentiate Eq. (23), we get

$$\dot{H}(t) = \frac{5}{4}\left(\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3}\right) - \left(\frac{5\epsilon_1^2}{72}\right)^{\frac{1}{3}} \left(\frac{\left[A^r_{BiP}(y) + A^r_{AiP}(y)C_5\right]^2}{\left[A^r_{Bi}(y) + A^r_{Ai}(y)C_5\right]^2}\right). \tag{24}$$

From Eq. (23) we obtain the scale factor in the form

$$R(t) = C_6 \left[A^r_{Bi}(y) + A^r_{Ai}(y)C_5\right]^{\frac{2}{5}}, \tag{25}$$

where C_6 is constant of integration.
Similarly from Eqs. (11) and (25), we get energy density in the form

$$\rho(t) = \left[\frac{C_1}{\left(A^r_{Bi}(y) + A^r_{Ai}(y)C_5\right)^2 C_6^5}\right]. \tag{26}$$

3.3. **Stiff matter case ($\alpha = -2$).** From the Eq. (16) with $\alpha = -2$, i.e., $p = \rho$ from Eq. (7), we get

$$H\dot{(t)} + 4H^2(t) - \frac{\epsilon_0}{3} - \frac{\epsilon_1 t}{3} = 0. \tag{27}$$

After integrating above equation, we get

$$H(t) = \left(\frac{\epsilon_1}{24}\right)^{\frac{1}{3}} \left[\frac{A^r_{BiP}(z) + A^r_{AiP}(z)C_7}{A^r_{Bi}(z) + A^r_{A}(z)C_7}\right], \tag{28}$$

where C_7 is constant of integration and $z = \left(\left[\frac{72}{\epsilon_1^2}\right]^{\frac{1}{3}} (\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3})\right)$.
Differentiate Eq. (28), we get

$$\dot{H}(t) = 2\left(\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3}\right) - \left(\frac{2\epsilon_1^2}{18}\right)^{\frac{1}{3}} \left(\frac{\left[A^r_{BiP}(z) + A^r_{AiP}(z)C_7\right]^2}{\left[A^r_{Bi}(z) + A^r_{Ai}(z)C_7\right]^2}\right). \tag{29}$$

From Eq. (28) we obtain the scale factor in the form

$$R(t) = C_8 \left[A^r_{Bi}(z) + A^r_{Ai}(z)C_7\right]^{\frac{1}{4}}, \tag{30}$$

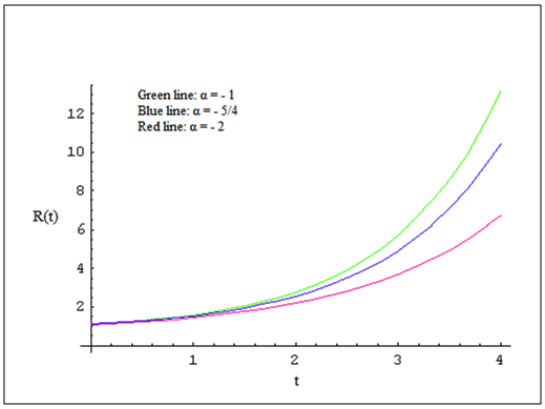


FIGURE 1. Relation between the scale factor $R(t)$ and cosmic time t of different cases for $\epsilon_0 = 1, \epsilon_1 = 1, C_3 = 1, C_4 = 1$

where C_8 is constant of integration.

Similarly, from Eqs. (11) and (30), we get energy density in the form

$$\rho(t) = \left[\frac{C_1}{\left(A^r_{Bi}(z) + A^r_{Ai}(z)C_7\right)^2 C_8^8}\right]. \tag{31}$$

Table 1. Comparison between the results of scale factor $R(t)$, Hubble parameter $H(t)$, Density $\rho(t)$ for different value of α

For α	$R(t)$	$H(t)$	$\rho(t)$
-1	$C_4\sqrt{A^r_{Bi}(x) + A^r_{Ai}(x)C_3}$	$\left(\frac{\epsilon_1}{12}\right)^{\frac{1}{3}} \frac{A^r_{BiP}(x)+A^r_{AiP}(x)C_3}{A^r_{Bi}(x)+A^r_{Ai}(x)C_3}$	$\frac{C_1}{[A^r_{Bi}(x)+A^r_{Ai}(x)C_3]^2 C_4^4}$
$-\frac{5}{4}$	$C_4\left[A^r_{Bi}(y) + A^r_{Ai}(y)C_5\right]^{\frac{2}{5}}$	$\left(\frac{\epsilon}{15}\right)^{\frac{1}{3}} \frac{A^r_{BiP}(y)+A^r_{AiP}(y)C_5}{A^r_{Bi}(y)+A^r_{A}(y)C_5}$	$\frac{C_1}{\left[A^r_{Bi}(y)+A^r_{Ai}(y)C_5\right]^2 C_6^5}$
-2	$C_4\left[A^r_{Bi}(z) + A^r_{Ai}(z)C_7\right]^{\frac{1}{4}}$	$\left(\frac{\epsilon_1}{24}\right)^{\frac{1}{3}} \frac{A^r_{BiP}(z)+A^r_{AiP}(z)C_7}{A^r_{Bi}(z)+A^r_{A}(z)C_7}$	$\frac{C_1}{\left[A^r_{Bi}(z)+A^r_{Ai}(z)C_7\right]^2 C_8^8}$

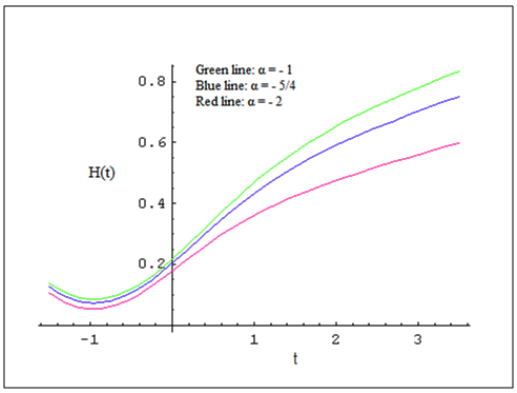


FIGURE 2. Relation between the Hubble parameter $H(t)$ and cosmic time t of different cases for $\epsilon_0 = 1, \epsilon_1 = 1, C_3 = 1, C_4 = 1$

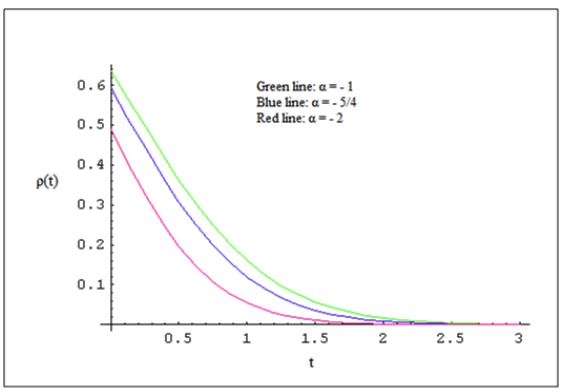


FIGURE 3. Relation between the density $\rho(t)$ and cosmic time t of different cases for $\epsilon_0 = 1, \epsilon_1 = 1, C_3 = 1, C_4 = 1$

Table 2. Comparison between the results of $\dot{H}(t)$, deceleration parameter $q(t)$ for different value of α

(I)	$\alpha = -1$
$\dot{H(t)}$	$\left(\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3}\right) - \left(\frac{\epsilon_1^2}{18}\right)^{\frac{1}{3}} \frac{[A^r_{BiP}(x)+A^r_{AiP}(x)C_3]^2}{\left[A^r_{Bi}(x)+A^r_{Ai}(x)C_3\right]^2}$
$q(t)$	$1 - \frac{\left[\left(\frac{\epsilon_1}{3}t+\frac{\epsilon_0}{3}\right)-\left(\frac{\epsilon_1^2}{18}\right)^{\frac{1}{3}} \frac{\left(A^r_{BiP}(x)+A^r_{AiP}(x)C_3\right)^2}{\left(A^r_{Bi}(x)+A^r_{Ai}(x)C_3\right)^2}\right]}{\left[\left(\frac{\epsilon_1}{12}\right)^{\frac{1}{3}} \frac{A^r_{BiP}(x)+A^r_{AiP}(x)C_3}{A^r_{Bi}(x)+A^r_{Ai}(x)C_3}\right]^2}$

(II)	$\alpha = -\frac{5}{4}$
$\dot{H(t)}$	$\frac{5}{4}\left(\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3}\right) - \left(\frac{5\epsilon_1^2}{72}\right)^{\frac{1}{3}} \frac{[A^r_{BiP}(y)+A^r_{AiP}(y)C_5]^2}{\left[A^r_{Bi}(y)+A^r_{Ai}(y)C_5\right]^2}$
$q(t)$	$1 - \frac{\left[\frac{5}{4}\left(\frac{\epsilon_1}{3}t+\frac{\epsilon_0}{3}\right)-\left(\frac{5\epsilon_1^2}{72}\right)^{\frac{1}{3}} \frac{\left(A^r_{BiP}(y)+A^r_{AiP}(y)C_5\right)^2}{\left(A^r_{Bi}(y)+A^r_{Ai}(y)C_5\right)^2}\right]}{\left[\left(\frac{4\epsilon}{75}\right)^{\frac{1}{3}} \frac{A^r_{BiP}(y)+A^r_{AiP}(y)C_5}{A^r_{Bi}(y)+A^r_{A}(y)C_5}\right]^2}$

(III)	$\alpha = -2$
$\dot{H(t)}$	$2\left(\frac{\epsilon_1}{3}t + \frac{\epsilon_0}{3}\right) - \left(\frac{2\epsilon_1^2}{18}\right)^{\frac{1}{3}} \frac{[A^r_{BiP}(z)+A^r_{AiP}(z)C_7]^2}{\left[A^r_{Bi}(z)+A^r_{Ai}(z)C_7\right]^2}$
$q(t)$	$1 - \frac{\left[2\left(\frac{\epsilon_1}{3}t+\frac{\epsilon_0}{3}\right)-\left(\frac{2\epsilon_1^2}{18}\right)^{\frac{1}{3}} \frac{\left(A^r_{BiP}(z)+A^r_{AiP}(z)C_7\right)^2}{\left(A^r_{Bi}(z)+A^r_{Ai}(z)C_7\right)^2}\right]}{\left[\left(\frac{\epsilon_1}{24}\right)^{\frac{1}{3}} \frac{A^r_{BiP}(z)+A^r_{AiP}(z)C_7}{A^r_{Bi}(z)+A^r_{A}(z)C_7}\right]^2}$

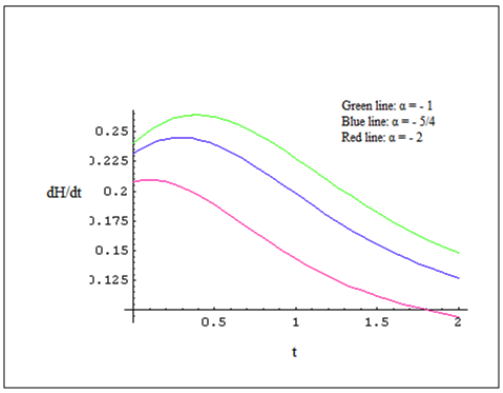


FIGURE 4. Relation between the $\dot{H}(t)$ and cosmic time t of different cases for $\epsilon_0 = 1, \epsilon_1 = 1, C_3 = 1, C_4 = 1$

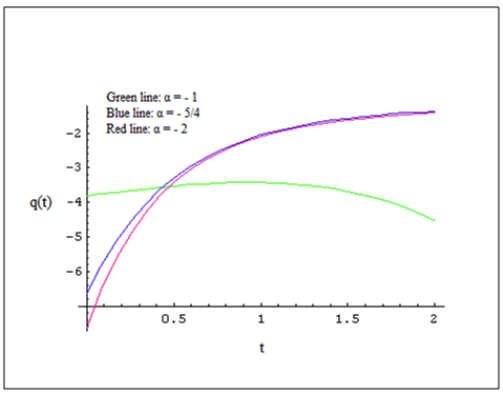


FIGURE 5. Relation between the deceleration parameter $q(t)$ and cosmic time t of different cases for $\epsilon_0 = 1, \epsilon_1 = 1, C_3 = 1, C_4 = 1$

3.4. **Generalized sudden singularity.** We note that all the physical quantities from Eqs. (18 - 21), (23 - 26) and (28 - 31), the energy density (ρ), pressure (p), scale factor (R), Hubble parameter (H) and its derivative in all cases are all constant when $t \to t_s(= -\frac{\epsilon_0}{\epsilon_1}) < \infty$, hence we approach to generalized sudden singularities i.e., $R \to R_s, \rho \to \rho_s, H \to H_s, |p| \to p_s, \dot{H} \to H_s$ when $t \to t_s$ ($t_s < \infty$), where t_s, ρ_s, H_s, p_s are all finite constants. It is noted that the higher derivative of H diverges when $t \to t_s$ and hence the values of $\Lambda(t)$ and $R(t)$ are finite when $t \to 0$

and value of $\Lambda(t)$ and $R(t)$ are unbounded when $t \to \infty$. Therefore, in this model the generalized sudden singularity occurs at a finite time $t \to t_s(= -\frac{\epsilon_0}{\epsilon_1})$.

4. Physical properties of the models

4.1. **Red Shift.** Consider an observer O, located at the origin of space coordinate and a galaxy S at an arbitrary point with spatial coordinates (r, θ, ϕ, ψ). According to Eq. (2), the velocity of a radial ray of light ($ds = d\theta = d\phi = d\psi = 0$) is given by

$$\frac{dr}{dt} = \pm\frac{\sqrt{1-kr^2}}{R(t)}, \tag{32}$$

hence if a light pulse is emitted by the galaxy S at the instant t_1, is received by O at the instant t_0, $(t_0 > t_1)$ we obtain from Eq. (32)

$$f(r_1) = -\int_{t_1}^{t_0} \frac{dt}{R(t)} = \begin{cases} sin^{-1}r_1, & k = +1 \\ r_1, & k = 0 \\ sinh^{-1}r_1, & k = -1. \end{cases} \tag{33}$$

The value of the red-shift Z in all above models can be proved to have the form

$$Z = \frac{\delta\lambda}{\lambda} = \frac{R_0}{R_1} - 1. \tag{34}$$

4.2. **Particle horizons.** Consider a light signal emitted by S at the instant t_1 and received by O at the instant t_0 and it moves along null geodesic then from Eq. (32) gives

$$\frac{dr}{dt} = \pm\frac{\sqrt{1-kr^2}}{R(t)} \Rightarrow \frac{dr}{\sqrt{1-kr^2}} = \frac{dt}{R(t)} = f(r_1). \tag{35}$$

We have to distinguish between two different distances, the radial coordinate distance defined by

$$r = \int_{t_1}^{t_0} \frac{dt}{R(t)}, \tag{36}$$

and the proper distance s defined by

$$s = R(t_0)r. \tag{37}$$

Recalling the red-shift as given by Eq. (34), one can always express the proper distance s in terms of the red-shift Z. Then if $s \to \infty$ as $Z \to \infty$, we can say that the model is free from particle horizons, i.e., isotropic observers can communicate by sending and receiving signals. But if s attains a finite value as $Z \to \infty$ we say

that the model contains particle horizons.

For example, we are going to explore the existence of particle horizons in one of the models discussed above.

For the dust model i.e. ($\alpha = -1$) with $\Lambda(t) = \epsilon_0 + \epsilon_1 t$, and the scale factor $R(t)$ is given by Eq. (20), we have

$$f(r_1) = r_1 = \int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_{t_1}^{t_0} \frac{dt}{C_4\sqrt{A^r_{Bi}(x) + A^r_{Ai}(x)C_3}}.$$

Taking Taylor's expansion of the above equation, up to the first order in t, we obtain

$$r_1 = \int_{t_1}^{t_0} \frac{dt}{C_4\left(\frac{1}{3^{\frac{1}{6}}\Gamma[\frac{2}{3}]} + \frac{C_3}{3^{\frac{2}{3}}\Gamma[\frac{2}{3}]}\right)^{\frac{1}{2}}\left[1 + \frac{1}{2}\frac{\left(\frac{3^{\frac{1}{6}}}{\Gamma[\frac{1}{3}]} - \frac{C_3}{3^{\frac{1}{3}}\Gamma[\frac{1}{3}]}\right)\frac{18\alpha^2}{\epsilon_1^2}\left(\frac{\epsilon_0}{3} + \frac{\epsilon_1}{3}t\right)}{\left(\frac{1}{3^{\frac{1}{6}}\Gamma[\frac{2}{3}]} + \frac{C_3}{3^{\frac{2}{3}}\Gamma[\frac{2}{3}]}\right)}\right]},$$

which can be integrated to give

$$r_1 = \left(\frac{2^{\frac{2}{3}}\sqrt{\sqrt{3} + C_3}\Gamma[\frac{1}{3}]}{(\frac{\epsilon_1}{3})^{\frac{1}{3}}(\sqrt{3} - C_3)C_4\sqrt{\Gamma[\frac{2}{3}]}}\right) \times$$

$$\log\left(\frac{\left[2\epsilon_1^{\frac{2}{3}}3^{(-\frac{1}{2})}(3 + \sqrt{3}C_3)\Gamma[\frac{1}{3}] + (2)^{\frac{1}{3}}(\sqrt{3} - C_3)\Gamma[\frac{2}{3}](\epsilon_1 t_0 + \epsilon_0)\right]}{\left[2\epsilon_1^{\frac{2}{3}}3^{(-\frac{1}{2})}(3 + \sqrt{3}C_3)\Gamma[\frac{1}{3}] + (2)^{\frac{1}{3}}(\sqrt{3} - C_3)\Gamma[\frac{2}{3}](\epsilon_1 t_1 + \epsilon_0)\right]}\right),$$

$$\Rightarrow r_1 = \left(\frac{2^{\frac{2}{3}}\sqrt{\sqrt{3} + C_3}\Gamma[\frac{1}{3}]}{(\frac{\epsilon_1}{3})^{\frac{1}{3}}(\sqrt{3} - C_3)C_4\sqrt{\Gamma[\frac{2}{3}]}}\right)\log(1 + Z),$$

where Γ is the gamma function.

From Eq. (37), with the help of above value of r_1

$$s = R_0 r_1 = \left[2\epsilon_1^{\frac{2}{3}}3^{(-\frac{1}{2})}(3 + \sqrt{3}C_3)\Gamma[\frac{1}{3}] + (2)^{\frac{1}{3}}(\sqrt{3} - C_3)\Gamma[\frac{2}{3}](\epsilon_1 t_0 + \epsilon_0)\right]$$

$$\times \left(\frac{2^{\frac{2}{3}}\sqrt{\sqrt{3} + C_3}\Gamma[\frac{1}{3}]}{(\frac{\epsilon_1}{3})^{\frac{1}{3}}(\sqrt{3} - C_3)C_4\sqrt{\Gamma[\frac{2}{3}]}}\right)\log(1 + Z). \tag{38}$$

From Eq. (37) we see that there is no particle horizon since when $Z \to \infty$ the proper distance s tends to infinity as well.

Using the same method we can easily show that radiation and stiff matter models ($\alpha = -\frac{5}{4}$ and $\alpha = -2$), respectively are free from particle horizons.

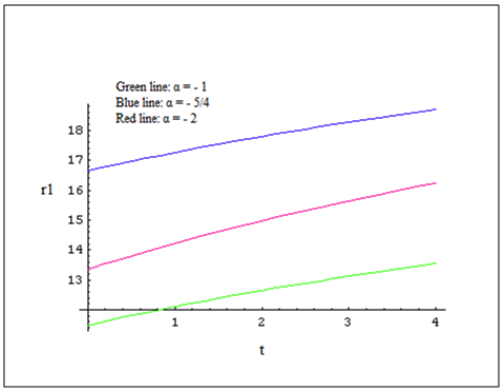


FIGURE 6. Relation between the particle horizons r_1 and cosmic time t of different cases for $\epsilon_0 = 1, \epsilon_1 = 1, C_3 = 1, C_4 = 1$

Table 3. Comparison between the particle horizons r_1 for different values of α

For α	Particle horizons
$\alpha = -1$	$[2\epsilon_1^{\frac{2}{3}} 3^{(-\frac{1}{2})}(3+\sqrt{3}C_3)\Gamma[\frac{1}{3}] + 2^{\frac{1}{3}}(\sqrt{3}-C_3)\Gamma[\frac{2}{3}](\epsilon_1 t_0+\epsilon_0)]$ $\times \left(\frac{2^{\frac{2}{3}}\sqrt{\sqrt{3}+C_3}\Gamma[\frac{1}{3}]}{(\frac{\epsilon_1}{3})^{\frac{1}{3}}(\sqrt{3}-C_3)C_4\sqrt{\Gamma[\frac{2}{3}]}}\right)\log(1+Z)$
$\alpha = -\frac{5}{4}$	$\log[5\epsilon_1^{\frac{2}{3}} 3^{(-\frac{1}{2})}(3+\sqrt{3}C_3)\Gamma[\frac{1}{3}] + 20^{\frac{1}{3}}(\sqrt{3}-C_3)\Gamma[\frac{2}{3}](\epsilon_1 t_0+\epsilon_0)]$ $\times \left(\frac{5^{\frac{2}{3}}(\sqrt{3}+C_3)^{\frac{3}{5}}\Gamma[\frac{1}{3}]}{(4\epsilon_1)^{\frac{1}{3}}3^{(-\frac{4}{15})}(\sqrt{3}-C_3)C_4(\Gamma[\frac{2}{3}])^{\frac{3}{5}}}\right)\log(1+Z)$
$\alpha = -2$	$[4(\frac{\epsilon_1}{3})^{\frac{2}{3}}(3+\sqrt{3}C_3)\Gamma[\frac{1}{3}] + (\frac{2}{3})^{\frac{2}{3}}(\sqrt{3}-C_3)\Gamma[\frac{2}{3}](\epsilon_1 t_0+\epsilon_0)]$ $\times \left(\frac{2^{\frac{4}{3}}(\sqrt{3}+C_3)^{\frac{3}{4}}\Gamma[\frac{1}{3}]}{(\epsilon_1)^{\frac{1}{3}}3^{(-\frac{1}{2})}(\sqrt{3}-C_3)C_4(\Gamma[\frac{2}{3}])^{\frac{3}{4}}}\right)\log(1+Z)$

5. CONCLUSION

In this paper, we considered the cosmological application in the framework of Kaluza-Klein theory, by taking the Newtonian and cosmological constant G and Λ to be a function of t. We used Taylor's expansion of cosmological function, $\Lambda(t)$ up to the first order of the time t and solved the Einstein's field equations by considering the modified equation of state of the form $p = -\rho - f(\rho)$. We have studied the flat universe i.e., $k = 0$ and calculated the cosmological parameters. We observed from the Eqs. (12), (20), (25) and (30) that the value of $\Lambda(t)$ and the scale factor $R(t)$ are finite when $t \to 0$. Similarly, the value of $\Lambda(t)$ and the scale factor $R(t)$ are unbounded when $t \to \infty$.

The universe acceleration gave some motivation to study non standard cosmological singularities such as big rip, sudden future singularity, Type III and Type IV and generalized sudden singularity. However, most of these singularities are weak singularities which do not exhibit incompleteness and allow the evolution of both point like objects and strings through them. The analytical properties of the scale factor $R(t)$, energy density $\rho(t)$ and Hubble parameter $H(t)$ are investigated near to the future singularity $t = t_s(= -\frac{\epsilon_0}{\epsilon_1})$ by assuming different form of equation of state of the form $p = -\rho - f(\rho)$. It is observed that from the solution of the dust, radiation and stiff matter the generalized sudden singularity occurs at finite time $t_s(= -\frac{\epsilon_0}{\epsilon_1})$ i.e. $R \to R_s, \rho \to \rho_s, H \to H_s, |p| \to p_s, \dot{H} \to H_s$ when $t \to t_s$ $(t_s < \infty)$. The obtained results are given in the Table 1 to 3 for dust, radiation and stiff matter and discussed graphically in the figure 1 to 6.

From Fig. 1, it is shown that the scale factor in the three models are increasing with time, which ensures that our universe is expanding. The Hubble parameter Vs time for three different cases are given in Fig. 2. From the Fig. 3, it is observed that the density in the three models are decreasing with time, which means that our universe is expanding. The three models are free from particle horizons, which is consistent with the previous result by Sharif and Farida Khanum [23]. In Fig. 4 and 5, we respectively discussed the three cases for $\dot{H}(t)$ and $q(t)$ Vs time. The relation between particle horizon and cosmic time t are given in Fig. 6.

References

[1] Al-Rawaf A. S., A Cosmological Model with a Generalized Cosmological Constant, *Mod. Phys. Lett. A* **13**, (1998) (1998).

[2] Appelquist T., Chodos A. and Freund P. G. O., Modern Kaluza-Klein Theories, *Addison-Wesley* (1987).

[3] Azri H. and Boinames A., Cosmological Consequences of a Variable Cosmological Constant Model, *Int. J. Mod. Phys. D.* **26** , 1750060 (2017).

[4] Berman M. S., Cosmological Models with a Variable Cosmological Term, *Phys. Rev. D* **43**, 1075 (1991).

[5] Berman M. S., Cosmological Models with a Variable Gravitational and Cosmological constants, *Gen. Rel. Grav.* **23**, 465-469 (1991).

[6] Canuto V., Hsieh S. H., Adams P. J., Mach's principle, the cosmological constant, and the scale-covariant theory of gravity. Phys. Rev. Lett. D. **18**(10), 3577 (1978).

[7] Chen W., and Wu Y. S., Implications of a Cosmological Constant Varying as R 2, *Phys. Rev. D* **41**, 695 (1990).

[8] Dolgov A. D., Higher Spin Fields and the Problem of the Cosmological Constant, *Phys. Rev. D* **55**, 5881 (1997).

[9] Henke C., Quantum Vacuum energy in General Relativity, *Eur. Phys. J. C.* **78** , 126-129 (2018).

[10] Kaluza T., Zum unittsproblem der physik. Sitzungsberichte der Kniglich Preuischen Akademie der Wissenschaften (Berlin), *Sitz. Preuss. Acad. Wiss.* **F1**, 966-972 (1921).

[11] Kazanas D., Dynamics of the Universe and Spontaneous Symmetry Bbreaking, *Astrophys. J. Lett.* **241**, L59-L63 (1980).

[12] Klein O., Quantum Theory and Five-Dimensional Theory of Relativity, *Surveys High Energ. Phys.* **37**, 895-906 (1926).

[13] Klein O., The Atomicity of Electricity as a Quantum Theory Law, *Nature (Lond.)* **118**, 516 (1926).

[14] Lee H. C., An Introduction to Kaluza-Klein Theories, *World Scientific Pub. Co. Inc.* (1984).

[15] Lima J. A. S. and Maia J. M. F., Some Cosmological Consequences of a Λ-term Varying as βH2+ αR-n, *Mod. Phys. Lett. A* **48**, 591-599 (1993).

[16] Overduin J. M. and Wesson P. S., Kaluza-Klein Gravity, *Phys. Rep.* **283**, 303-378 (1997).

[17] Overduin J. M. and Cooperstock F. I., Evolution of the Scale Factor with a Variable Cosmological Term, *Phys. Rev. D* **58**, 43506 (1998).

[18] Padmanabhan T., Cosmological Constantthe Weight of the Vacuum, *Phys. Rep.* **380**, 235-320 (2003).

[19] Peebles P. J. E. and Ratra B., The Cosmological Constant and Dark Energy, *Rev. Mod. Phys.* **75**, 559 (2003).

[20] Pollack M. D., On the connection between the cosmological constant and the gravitational constant in a theory of induced gravity, *Phys. Lett. B* **148**, 287-290 (1984).

[21] Polyakov A. M., Phase Transitions and the Universe, *Phys. Usp.* **25**, 187-187 (1982).

[22] Sahni V. and Starobinsky A., The Case for a Positive Cosmological Λ-term, *Int. J. Mod. Phys. D.* **9**(4), 373-443 (2000).

[23] Sharif M. and Farida K., Kaluza-Klein Cosmology with Varying G and Λ, *Astrophys. Space. Sci.* **334**, 209-214 (2011).

[24] Steinhardt P. J. and Turok N., Why the Cosmological Constant is Small and Positive, *Sci* **312**, 1180-1183 (2006).

[25] Wanas M. T., Nashed G. L. and Nowaya A. A., Cosmological Applications in Kaluza-Klein Theory, *Chin. Phys. B* **21**, (2012).