

A NON-FLAT RIEMANNIAN MANIFOLD ADMITTING CERTAIN VECTORS FIELDS

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ABSTRACT. It is well-known that Einstein manifolds play an important role in geometry as well as in general relativity. Einstein manifolds form a natural subclass of the class of quasi-Einstein manifolds. The object of the present paper is to study some geometric properties of mixed-generalized quasi-Einstein Manifolds $(MG(QE)_n)$ which admitting certain vector fields. We show the existence of $MG(QE)_n$, by constructing several non-trivial examples. Finally we study warped product on $MG(QE)_n$ and show that $M = I \times M^*$ ($\dim I = 1$ and $\dim M^* = n - 1$) is a $MG(QE)_n$ if M^* is a generalized quasi-Einstein Manifold $(G(QE)_n)$.

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1. Introduction

The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [3]. A non-flat Riemannian manifold (M^n, g) , $(n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$(1) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y)$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$(2) \quad g(X, U) = A(X), \forall X \in \chi(M),$$

U being a unit vector field.

Here a and b are called the *associated scalars*, A is called the associated 1-form and U is called the *generator* of the manifold. Such an n -dimensional manifold will be denoted by $(QE)_n$.

As a generalization of quasi-Einstein manifold in [4], U. C. De and G. C. Ghosh defined the *generalized quasi-Einstein manifold*. A non-flat Riemannian manifold is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$(3) \quad S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b and c are non-zero scalars and A, B are two 1-forms such that

$$(4) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X),$$

U and V being unit vectors which are orthogonal, i.e.,

$$(5) \quad g(U, V) = 0.$$

The vector fields U and V are called the generators of the manifold. This type of manifold will be denoted by $G(QE)_n$.

In [1, 6], A. Bhattacharyya and T. De introduced the notion of *mixed generalized quasi-Einstein manifold*. A non-flat Riemannian manifold is called *mixed generalized*

quasi-Einstein manifold if its Ricci tensor is non-zero and satisfies the condition

$$(6) \quad \begin{aligned} S(X, Y) &= ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) \\ &+ d[A(X)B(Y) + A(Y)B(X)] \end{aligned}$$

where a, b, c, d are non-zero scalars,

$$(7) \quad g(X, U) = A(X) \text{ and } g(X, V) = B(X), \forall X \in \chi(M),$$

and also

$$(8) \quad g(U, V) = 0.$$

A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold becomes to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

On the other hand, Gray([7]) introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class A consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi type tensor, i.e.,

$$(9) \quad (\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z).$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(10) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0.$$

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) ($n > 2$) is called *generalized Ricci recurrent* [5] if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$(11) \quad (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(X)g(Y, Z),$$

where $\alpha(X)$ and $\beta(X)$ are two nowhere vanishing 1-forms such that $\alpha(X) = g(X, \rho)$ and $\beta(X) = g(X, \mu)$; ρ and μ being *associated vector fields* of the 1-forms α and β , respectively. If $\beta = 0$, then the manifold reduce to Ricci recurrent manifold [10].

Motivated by the above studies the present paper deals with the study of mixed-generalized quasi-Einstein manifolds admitting certain vector fields. The paper is organized as follows: In section 2, we have studied that if the generators U and V

are Killing vector fields, then the mixed generalized quasi-Einstein manifold satisfies cyclic parallel Ricci tensor. Section 3 is concerned with $MG(QE)_n$ satisfying Codazzi type of Ricci tensor. In the next two sections we study $MG(QE)_n$ with generators U and V both as concurrent and recurrent vector fields. In section 6, we show the existence of $MG(QE)_n$, by constructing some non-trivial examples. Finally we study warped product on $MG(QE)_n$ and display that $M = I \times M^*$ ($\dim I = 1$ and $\dim M^* = n - 1$) is a $MG(QE)_n$ if M^* is a $G(QE)_n$.

2. The generators U and V as Killing vector fields

In this section we consider the generators U and V of the manifold as Killing vector fields. Then we have

$$(12) \quad (\mathcal{L}_U g)(X, Y) = 0,$$

and

$$(13) \quad (\mathcal{L}_V g)(X, Y) = 0.$$

where $\mathcal{L}$ denotes the Lie derivative.

From (12) and (13), we get

$$(14) \quad g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0,$$

and

$$(15) \quad g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0.$$

Since $g(\nabla_X U, Y) = (\nabla_X A)(Y)$ and $g(\nabla_X V, Y) = (\nabla_X B)(Y)$, we obtain from (14) and (15) that

$$(16) \quad (\nabla_X A)(Y) + (\nabla_Y A)(X) = 0$$

and

$$(17) \quad (\nabla_X B)(Y) + (\nabla_Y B)(X) = 0$$

for all X, Y .

Similarly, we have

$$(18) \quad (\nabla_X A)(Z) + (\nabla_Z A)(X) = 0,$$

$$(19) \quad (\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0,$$

$$(20) \quad (\nabla_X B)(Z) + (\nabla_Z B)(X) = 0,$$

$$(21) \quad (\nabla_Z B)(Y) + (\nabla_Y A)(Z) = 0$$

for all X, Y, Z . Now from (5), we get

$$\begin{aligned} (\nabla_Z S)(X, Y) &= b[(\nabla_Z A)(X)A(Y) + (\nabla_Z A)(Y)A(X)] + c[(\nabla_Z B)(X)B(Y) \\ &+ (\nabla_Z B)(Y)B(X)] + d[(\nabla_Z A)(X)B(Y) \\ &+ (\nabla_Z B)(Y)A(X) + (\nabla_Z A)(Y)B(X) \\ (22) \quad &+ (\nabla_Z B)(X)A(Y)]. \end{aligned}$$

Using (22), we obtain

$$\begin{aligned} (\nabla_X S)(Y, Z) &+ (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = b[A(Y)\{(\nabla_Z A)(X) \\ &+ (\nabla_X A)(Z)\} + A(X)\{(\nabla_Z A)(Y) + (\nabla_Y A)(Z)\} \\ &+ A(Z)\{(\nabla_Y A)(X) + (\nabla_X A)(Y)\}] + c[B(Y)\{(\nabla_Z B)(X) \\ &+ (\nabla_X B)(Z)\} + B(X)\{(\nabla_Z B)(Y) + (\nabla_Y B)(Z)\} \\ &+ B(Z)\{(\nabla_Y B)(X) + (\nabla_X B)(Y)\}] + d[A(X)\{(\nabla_Z B)(Y) \\ &+ (\nabla_Y B)(Z)\} + A(Y)\{(\nabla_Z B)(X) + (\nabla_X B)(Z)\} \\ &+ A(Z)\{(\nabla_Y B)(X) + (\nabla_X B)(Y)\} + B(X)\{(\nabla_Z A)(Y) \\ &+ (\nabla_Y A)(Z)\} + B(Y)\{(\nabla_Z A)(X) + (\nabla_X A)(Z)\} \\ (23) \quad &+ B(Z)\{(\nabla_Y A)(X) + (\nabla_X A)(Y)\}]. \end{aligned}$$

Now using the equation (16) – (21) and simplifying we obtain from (23) that

$$(24) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

Thus we can state the following theorem:

Theorem 2.1. *If the generators of a $MG(QE)_n$ are Killing vector fields, then the manifold satisfies cyclic parallel Ricci tensor.*

3. $MG(QE)_n$ admits Codazzi type of Ricci tensor

A Riemannian or semi-Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor satisfies the following condition

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z).$$

Using (8) and (22), we obtain

$$\begin{aligned}
 b[(\nabla_X A)(Y)A(Z) &+ (\nabla_X A)(Z)A(Y)] + c[(\nabla_X B)(Y)B(Z) + (\nabla_X B)(Z)B(Y)] \\
 &+ d[(\nabla_X A)(Y)B(Z) + (\nabla_X B)(Z)A(Y) \\
 &+ (\nabla_X A)(Z)B(Y) + (\nabla_X B)(Y)A(Z)] \\
 &- b[(\nabla_Y A)(Z)A(X) + (\nabla_Y A)(X)A(Z)] \\
 &- c[(\nabla_Y B)(Z)B(X) + (\nabla_Y B)(X)B(Z)] \\
 &- d[(\nabla_Y A)(Z)B(X) + (\nabla_Y B)(X)A(Z) \\
 (25) \quad &+ (\nabla_Y B)(Z)A(X) + (\nabla_Y A)(X)B(Z)].
 \end{aligned}$$

Putting $Z = U$ in (25) and using (7) and $(\nabla_X A)(U) = 0$, we get

$$\begin{aligned}
 b[(\nabla_Y A)(X) &- \nabla_X A)(Y)] + d[(\nabla_Y B)(X) \\
 (26) \quad &- \nabla_X B)(Y)] = 0.
 \end{aligned}$$

i.e.,

$$(27) \quad b\mathbf{d}A(X, Y) = -d\mathbf{d}B(X, Y).$$

Similarly, putting $Z = V$ in (25) and using (7) and $(\nabla_X B)(V) = 0$, we get

$$(28) \quad c\mathbf{d}B(X, Y) = -d\mathbf{d}A(X, Y).$$

From the equation (27) and (28), we get

either

$$d = \pm\sqrt{bc}$$

or

$$\mathbf{d}A(X, Y) = 0$$

and

$$\mathbf{d}B(X, Y) = 0.$$

Thus we can state the following:

Theorem 3.1. *If a $MG(QE)_n$ satisfies the Codazzi type of Ricci tensor, then either $d = \pm\sqrt{bc}$ or the associated 1-forms A and B are closed.*

Again putting $X = Z = U$ in (25), we have

$$(29) \quad b(\nabla_U A)(Y) + d(\nabla_U B)(Y) = 0,$$

which means that

$$(30) \quad bg(Y, \nabla_U U) + dg(Y, \nabla_U V) = 0.$$

Similarly, putting $X = Z = V$ in (25) we have

$$(31) \quad b(\nabla_V B)(Y) + d(\nabla_V A)(Y) = 0,$$

and so

$$(32) \quad cg(Y, \nabla_V V) + dg(Y, \nabla_V U) = 0.$$

If U and V are parallel vector fields, then $\nabla_U V = 0 = \nabla_V U$, and thus from the equation (30) and (32), we get

$$g(Y, \nabla_U U) = 0, \text{ i.e., } \nabla_U U = 0$$

and

$$g(Y, \nabla_V V) = 0, \text{ i.e., } \nabla_V V = 0$$

for all Y .

This leads to the following theorem:

Theorem 3.2. *If a $MG(QE)_n$ admits Codazzi type of Ricci tensor, then the integral curves of the parallel vector fields U and V are geodesic.*

4. The generators U and V as concurrent vector fields

A vector field ξ is said to be concurrent if [11]

$$(33) \quad \nabla_X \xi = \rho X,$$

where ρ is a non-zero constant. If $\rho = 0$, the vector field reduces to a parallel vector field. In this section we consider the vector fields U and V corresponding to the associated 1-forms A and B respectively are concurrent. Then

$$(34) \quad (\nabla_X A)(Y) = \alpha g(X, Y)$$

and

$$(35) \quad (\nabla_X B)(Y) = \beta g(X, Y),$$

where α and β are non-zero constants.

Using (34) and (35) in (22), we get

$$\begin{aligned} (\nabla_Z S)(X, Y) &= b[\alpha g(X, Z)A(Y) + \alpha g(Y, Z)A(X)] + c[\beta g(X, Z)B(Y) \\ &\quad + \beta g(Y, Z)B(X)] + d[\alpha g(X, Z)B(Y) \\ &\quad + \beta g(Y, Z)A(X) + \alpha g(Y, Z)B(X) \\ (36) \quad &\quad + \beta g(X, Z)A(Y)]. \end{aligned}$$

Contracting (36) over X and Y we obtain

$$(37) \quad dr(Z) = 2[(b\alpha + d\beta)A(Z) + (c\beta + d\alpha)B(Z)],$$

where r is the scalar curvature of the manifold.

Again from (5) we have

$$(38) \quad r = na + b + c.$$

Since $a, b, c, d \in R$, it follows that $dr(X) = 0$, for all X . Thus equation (37) gives

$$(39) \quad (b\alpha + d\beta)A(Z) + (c\beta + d\alpha)B(Z) = 0.$$

Since α and β are not zero, using (39) in (5), we finally get

$$(40) \quad S(X, Y) = ag(X, Y) + [b + c \frac{(b\alpha + d\beta)^2}{(d\alpha + c\beta)^2} - 2d \frac{(b\alpha + d\beta)}{(d\alpha + c\beta)}]A(X)A(Y).$$

Thus the manifold reduces to a quasi-Einstein manifold. Hence we can state the following theorem:

Theorem 4.1. *If the associated vector fields of a $MG(QE)_n$ are concurrent vector fields, then the manifold reduces to a quasi-Einstein manifold.*

5. The generators U and V as recurrent vector fields

A vector field ξ corresponding to the associated 1-form η is said to be recurrent if [11]

$$(41) \quad (\nabla_X \eta)(Y) = \psi(X)\eta(Y),$$

where ψ is a non-zero 1-form.

Here, we suppose that the generators U and V corresponding to the associated 1-forms A and B are recurrent. Then we have

$$(42) \quad (\nabla_X A)(Y) = \lambda(X)A(Y)$$

and

$$(43) \quad (\nabla_X B)(Y) = \mu(X)B(Y),$$

where λ and μ are non-zero 1-forms.

Now, using (42) and (43) in (22) we get

$$(44) \quad \begin{aligned} (\nabla_Z S)(X, Y) &= 2b\lambda(Z)A(X)A(Y) + 2c\mu(Z)B(X)B(Y) + d[\lambda(Z) \\ &+ \mu(Z)][A(X)B(Y) + A(Y)B(X)]. \end{aligned}$$

We assume that the 1-forms λ and μ are equal, i.e.,

$$(45) \quad \lambda(Z) = \mu(Z)$$

for all Z . From the equation (44) and (45), we get

$$(46) \quad \begin{aligned} (\nabla_Z S)(X, Y) &= 2\lambda(Z)[bA(X)A(Y) + cB(X)B(Y) \\ &+ d\{A(X)B(Y) + A(Y)B(X)\}]. \end{aligned}$$

Using (5) and (46), we have

$$(47) \quad (\nabla_Z S)(X, Y) = \alpha_1(Z)S(X, Y) + \alpha_2(Z)g(X, Y),$$

where $\alpha_1(Z) = 2\lambda(Z)$ and $\alpha_2(Z) = -2a\lambda(Z)$.

Thus we have the following theorem:

Theorem 5.1. *If the generators of a $MG(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence, then the manifold is a generalized Ricci recurrent manifold.*

6. Examples

If we take any example of non-Einstein manifold, we can transfer into $MG(QE)_n$. Here we have given some nontrivial concrete examples of non-flat 4-dimensional Riemannian manifolds which can be converted into 4-dimensional $MG(QE)_n$.

Example 6.1. Let us consider a Riemannian metric g on the 4-dimensional real number space M^4 by

$$ds^2 = g_{ij}dx^i dx^j = (e^{x^2})(dx^1)^2 + (x^1 x^3)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 .

Example 6.2. Let us consider a Riemannian metric g on the 4-dimensional real number space M^4 by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1 \sin x^2)^2(dx^3)^2 + e^{x^1}(dx^4)^2$$

where $i, j = 1, 2, 3, 4$ and $0 < x^1 < 1$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 .

Example 6.3. We define a Riemannian metric g on the 4-dimensional real number space M^4 by the formula

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2$$

where $i, j = 1, 2, 3, 4$, $p = \frac{e^{x^1}}{\rho^2}$ and ρ is a constant and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 .

Example 6.4. We define a another Riemannian metric g on the 4-dimensional

real number space M^4 by the formula

$$ds^2 = g_{ij}dx^i dx^j = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

where $i, j = 1, 2, 3, 4$, $0 < x^1 < \frac{\pi}{2}$ but $x^1 \neq \frac{\pi}{4}$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 .

Example 6.5. Let us consider a Riemannian metric g on the 4-dimensional real number space M^4 by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + e^{x^2}[e^{x^2}(dx^2)^2 + e^{x^3}(dx^3)^2 + e^{x^4}(dx^4)^2]$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 .

As all the above metrics are non-Einstein, so, we can easily construct $MG(QE)_n$ using those metrics.

Next we show the existence of $MG(QE)_n$ by constructing some non-trivial concrete examples.

Example 6.6. Let us consider a semi-Riemannian metric g on M^4 by

$$(48) \quad ds^2 = g_{ij}dx^i dx^j = x^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2,$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 . Then the only non vanishing components of the Christoffel symbols, the curvature tensors and the derivatives of the components of curvature tensors are

$$(49) \quad \Gamma_{11}^2 = \Gamma_{33}^2 = -\frac{1}{2x^2}, \quad \Gamma_{22}^2 = \Gamma_{12}^1 = \Gamma_{23}^3 = \frac{1}{2x^2},$$

$$(50) \quad R_{1221} = R_{2332} = -\frac{1}{2x^2}, \quad R_{1331} = \frac{1}{4x^2}, \quad R_{1232} = 0,$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors R_{ij} are

$$(51) \quad R_{11} = R_{33} = -\frac{1}{4x^2}, \quad R_{22} = -\frac{1}{x^2}.$$

It can be easily shown that the scalar curvature of the resulting manifold (M^4, g) is $-\frac{3}{2(x^2)^3} \neq 0$. We shall show that (M^4, g) is a $MG(QE)_n$.

Let us consider the associated scalars as follows:

$$(52) \quad a = \frac{1}{(x^2)^3}, \quad b = -\frac{1}{(x^2)^4}, \quad c = -\frac{2}{(x^2)^4}.$$

Again let now consider the associated 1-forms as follows:

$$(53) \quad A_i(x) = \begin{cases} \frac{\sqrt{5}(x^2)}{2}, & \text{for } i = 1, 3 \\ 0 & \text{for } otherwise, \end{cases}$$

and

$$(54) \quad B_i(x) = \begin{cases} (x^2), & \text{for } i = 2 \\ 0 & \text{for } otherwise \end{cases}$$

at any point $x \in M^4$. To verify the relation (5), it is sufficient to check the following equations:

$$(55) \quad R_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + 2dA_1B_1,$$

$$(56) \quad R_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + 2dA_2B_2,$$

$$(57) \quad R_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + 2dA_3B_3.$$

Since for the other cases (5) holds trivially. By virtue of the equation (52), (53), (54), (55), we get R.H.S of (55) = $ag_{11} + bA_1A_1 + cB_1B_1 + 2dA_1B_1$ = L.H.S of (55). By similar argument it can be shown that (56) and (57) are also true. We shall now show that the associated vectors A_i and B_i are unit.

Here

$$g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0.$$

Therefore the vectors A_i and B_i are unit and also they are orthogonal.

So, (M^4, g) is a $MG(QE)_n$.

Example 6.7. Let us consider a semi-Riemannian metric g on $\mathbb{R}^4$ by

$$(58) \quad ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2,$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4 . Then the only non vanishing components of the Christoffel symbols, the curvature tensors and the Ricci tensors are

$$(59) \quad \Gamma_{14}^1 = \Gamma_{24}^2 = \Gamma_{34}^3 = \frac{2}{3x^4}, \quad \Gamma_{11}^4 = \Gamma_{22}^4 = \Gamma_{33}^4 = -\frac{2}{3}(x^4)^{\frac{1}{3}},$$

$$(60) \quad R_{11} = R_{22} = R_{33} = -\frac{2}{9(x^4)^{\frac{2}{3}}}, \quad R_{44} = -\frac{2}{3(x^4)^2}.$$

It can be shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is $r = -\frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant.

We shall show that M^4 is a $MG(QE)_n$. Let us consider the associated scalars and the associated 1-forms as follows:

$$(61) \quad a = -\frac{2}{3(x^4)^2}, \quad b = \frac{2}{9(x^4)^{\frac{1}{3}}}, \quad c = \frac{4}{9(x^4)^{\frac{1}{6}}}, \quad d = -\frac{2}{9(x^4)^{\frac{1}{4}}},$$

and

$$(62) \quad A_i(x) = \begin{cases} \frac{1}{\sqrt{(x^4)^{\frac{1}{3}}}}, & \text{for } i = 1, 2, 3 \\ 0 & \text{for } otherwise, \end{cases}$$

and

$$(63) \quad B_i(x) = \begin{cases} \frac{1}{\sqrt{(x^4)^{\frac{1}{3}}}}, & \text{for } i = 1, 2, 3 \\ 0 & \text{for } otherwise \end{cases}$$

at any point $x \in R^4$. To verify the relation (5), it is sufficient to check the following equations:

$$(64) \quad R_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + 2dA_1B_1,$$

$$(65) \quad R_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + 2dA_2B_2,$$

$$(66) \quad R_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + 2dA_3B_3.$$

Since for the other cases (5) holds trivially. By using (61), (62), (63), it can be easily prove that the equation (64), (65), (66) are true. So, (M^4, g) is a $MG(QE)_n$.

7. Warped product on $MG(QE)_n$

The study of warped product manifold was initiated by Kručkovič [9] in 1957. Again in 1969 Bishop and O'Neill [2] also obtained the same notion of the warped product manifolds while they were constructing a large class of manifolds of negative curvature. Warped product are generalizations of the Cartesian product of Riemannian manifolds. Let $(\bar{M}, \bar{g})$ and (M^*, g^*) be two Riemannian or semi-Riemannian manifolds. Let $\bar{M}$ and M^* be covered with coordinate charts $(U; x^1, x^2, \dots, x^p)$ and $(V; y^{p+1}, y^{p+2}, \dots, y^n)$ respectively. Then the warped product $M = \bar{M} \times_f M^*$ is the product manifold of dimension n furnished with the metric

$$(67) \quad g = \pi^*(\bar{g}) + (f \circ \pi)\sigma(g^*),$$

where $\pi : M \rightarrow M^*$ and $\sigma : M \rightarrow M^*$ are natural projections such that the warped product manifold $\bar{M} \times_f M^*$ is covered with the co-ordinate chart

$$(U \times V; x^1, x^2, \dots, x^p, x^{p+1} = y^{p+1}, x^{p+2} = y^{p+2}, \dots, x^n = y^n).$$

Then the local components of the metric g with respect to this co-ordinate chart are given by

$$(68) \quad g_{ij}(x) = \begin{cases} \bar{g}_{ij}, & \text{for } i = a \text{ and } j = b, \\ f g_{ij}^*, & \text{for } i = \alpha \text{ and } j = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $a, b, c, \dots \in 1, 2, \dots, p$ and $\alpha, \beta, \gamma, \dots \in p+1, p+2, \dots, p=n$ and $a = i, j, k, \dots \in 1, 2, \dots, n$. Here $\bar{M}$ is called the base, M^* is called the fibre and f is called the warping function the warped product $\bar{M} \times_f M^*$. We denote by Γ_{jk}^i , R_{ijkl} , R_{ij} , and r as the components of Levi-Civita connection ∇ , the Riemann-Christoffel curvature tensor R , Ricci tensor S and the scalar curvature of (M, g) respectively. Moreover we consider that, when Ω is a quantity formed with respect to 1, we denote by $\bar{\Omega}$ and Ω^* , the similar quantities formed with respect to $\bar{g}$ and g^* respectively. Then the non-zero local components of Levi-Civita connection ∇ of (M, g) are given by

$$(69) \quad \Gamma_{bc}^a = \bar{\Gamma}_{bc}^a, \Gamma_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^{*\alpha}, \Gamma_{\beta\gamma}^a = \frac{1}{2} \bar{g}^{ab} f_b g_{\beta\gamma}^*, \Gamma_{a\beta}^\alpha = \frac{1}{2f} f_a \delta_\beta^\alpha,$$

where $f_a = \partial_a f = \frac{\partial f}{\partial x^a}$. The local components

$$R_{hijk} = g_{hl} R_{ijk}^l = g_{hl} (\partial_k \Gamma_{ij}^l - \partial_j \Gamma_{ik}^l + \Gamma_{ij}^m \Gamma_{mk}^l - \Gamma_{ik}^m \Gamma_{mj}^l),$$

$\partial_k = \frac{\partial}{\partial x^k}$, of the Riemann-Christoffel curvature tensor R of (M, g) which may not vanish identically are the following:

$$(70) \quad R_{abcd} = \bar{R}_{abcd}, R_{a\alpha\beta} = -f T_{ab} g_{\alpha\beta}^*, R_{\alpha\beta\gamma\delta} = f R_{\alpha\beta\gamma\delta}^* - f^2 G_{\alpha\beta\gamma\delta}^*,$$

where $G_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl}$ and

$$T_{ab} = -\frac{1}{2f} (\nabla_b f_a - \frac{1}{2f} f_a f_b), \text{tr}(T) = g^{ab} T_{ab},$$

$$P = \frac{1}{4f^2} g^{ab} f_a f_b,$$

$$Q = f(n - p - 1)P - \text{tr}(T).$$

Again the non-zero local components of the Ricci tensor $R_{jk} = g^{il} R_{ijkl}$ of (M, g) are given by

$$(71) \quad R_{ab} = \bar{R}_{ab} + (n - p)T_{ab}, \quad R_{\alpha\beta} = R_{\alpha\beta}^* - Qg_{\alpha\beta}^*.$$

The scalar curvature r of (M, g) is given by

$$(72) \quad r = \bar{r} + \frac{r^*}{f} - (n - p)[(n - p - 1)P - 2\text{tr}(T)].$$

Here, we consider warped product manifold $M = I \times_f M^*$, $\dim I = 1$, $\dim M^* = n - 1$ ($n \geq 3$), $f = \exp\{\frac{g}{2}\}$. We take the metric on I as $(dt)^2$ and M^* is generalized quasi-Einstein manifold.

Using the above consideration and (71), we get

$$R_{tt} = \bar{R}_{tt} + (n - 1)T_{tt}$$

which implies

$$(73) \quad R_{tt} = -\frac{(n - 1)}{16} [(q')^2 + 4q''],$$

since $\bar{R}_{tt}$ of I is zero.

Also

$$R_{\alpha\beta} = R_{\alpha\beta}^* - Qg_{\alpha\beta}^*,$$

which implies

$$(74) \quad R_{\alpha\beta} = R_{\alpha\beta}^* - \frac{e^{\frac{q}{2}}}{16} [(2n-3)(q')^2 + 4(n-1)q''] g_{\alpha\beta}^*,$$

where ‘ $\prime$ ’ and ‘ $''$ ’ denote the 1st order and 2nd order partial derivatives respectively with respect to t . Since M^* is $G(QE)_n$, we obtain

$$(75) \quad R_{\alpha\beta}^* = \lambda g_{\alpha\beta}^* + \mu A_\alpha^* A_\beta^* + \gamma [A_\alpha^* B_\beta^* + A_\beta^* B_\alpha^*],$$

where λ , μ and γ are certain non-zero scalars and A_α^* and B_α^* are unit covariant vector such that $g^{*\alpha\beta} A_\alpha^* A_\beta^* = 1$, $g^{*\alpha\beta} B_\alpha^* B_\beta^* = 1$ and $g^{*\alpha\beta} A_\alpha^* B_\beta^* = 0$ and

$$(76) \quad A_\alpha(x) = \begin{cases} \bar{A}_\alpha, & \text{for } \alpha = 1, \\ A_\alpha^* & \text{otherwise.} \end{cases}$$

Using (75) in (74) we get

$$(77) \quad \begin{aligned} R_{\alpha\beta} &= \lambda g_{\alpha\beta}^* + \mu A_\alpha^* A_\beta^* - \frac{e^{\frac{q}{2}}}{16} [(2n-3)(q')^2 + 4(n-1)q''] g_{\alpha\beta}^* \\ &+ \gamma [A_\alpha^* B_\beta^* + A_\beta^* B_\alpha^*]. \end{aligned}$$

Again, using (68) and (76) in (77), we obtain

$$(78) \quad \begin{aligned} R_{\alpha\beta} &= -\frac{1}{16} [(2n-3)(q')^2 + 4(n-1)q''] g_{\alpha\beta} + \frac{\lambda}{e^{\frac{q}{2}}} g_{\alpha\beta} + \mu A_\alpha A_\beta \\ &+ \gamma [A_\alpha B_\beta + A_\beta B_\alpha]. \end{aligned}$$

Now if we choose $g_{\alpha\beta} = e^{\frac{q}{2}} B_\alpha B_\beta$, where

$$(79) \quad B_\alpha(x) = \begin{cases} \bar{B}_\alpha, & \text{for } \alpha = 1, \\ B_\alpha^* & \text{otherwise,} \end{cases}$$

then

$$(80) \quad \begin{aligned} R_{\alpha\beta} &= -\frac{1}{16} [(2n-3)(q')^2 + 4(n-1)q''] g_{\alpha\beta} + \lambda B_\alpha B_\beta + \mu A_\alpha A_\beta \\ &+ \gamma [A_\alpha B_\beta + A_\beta B_\alpha]. \end{aligned}$$

Again from (73) we obtain

$$(81) \quad \begin{aligned} R_{tt} &= \frac{1}{16} [(2n-3)(q')^2 + 4(n-1)q''] g_{tt} - \frac{1}{16} [(2n-3)(q')^2 + 4(n-1)q''] \\ &- \frac{(n-1)}{16} [(q')^2 + 4q''], \end{aligned}$$

since $\bar{g}_{tt} = 1$ and $g_{tt} = \bar{g}_{tt}$ in I. Thus (81) can be written as

$$(82) \quad R_{tt} = \frac{1}{16} [(2n-3)(q')^2 + 4(n-1)q''] g_{tt} - \frac{3n-4}{16} (q')^2 + \frac{2(n-1)}{4} q''.$$

Since $\dim I = 1$, we can take

$$(83) \quad \bar{A}_t = \sqrt{\{q'' + (q')^2\}} - q'$$

and

$$(84) \quad \bar{B}_t = q',$$

where q' and q'' are the functions of M .

Thus using (76), (79), (83) and (84), in (82) we get

$$(85) \quad \begin{aligned} R_{tt} &= \frac{1}{16}[(2n-3)(q')^2 + 4(n-1)q'']g_{tt} + \frac{(n-1)}{2}A_tA_t - \frac{3n-4}{16}B_tB_t \\ &+ \frac{(n-1)}{2}[A_tB_t + B_tA_t]. \end{aligned}$$

Thus from (80) and (85) we can conclude that $M = I \times_f M^*$ is a $MG(QE)_n$ if M^* is a $G(QE)_n$.

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