

THE RESTRICTED FIVE-BODY PROBLEM WITH CYCLIC KITE CONFIGURATION

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ABSTRACT. This manuscript investigates the five-body problem with kite configuration where four bodies are placed at the vertices of a kite. All the vertices of kite are taken at the circumference of a circle. These four bodies are considered as primaries which are moving on the circle with same centre which is taken as origin. The fifth infinitesimal body is moving in the space under the influences of these four primaries but not influencing them. After evaluating the equations of motion of the infinitesimal body, we have determined the Jacobi-integral. In the next section, we have done the computational works, where we have plotted the locations of equilibrium points, zero-velocity curves, regions of motion, Poincaré surfaces of section and basins of attraction in different planes (in-plane and out-of-planes). Moreover, we have examined the linear stability of all the equilibrium points and found that all the equilibrium points are unstable.

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Keywords: Five-body Problem; Cyclic kite configuration; Equilibrium points; Zero-velocity curves; Regions of motion; Poincaré surfaces of section; Basins of attraction; Stability

1. INTRODUCTION

Celestial mechanics is full of the problems with many bodies and many perturbations. We tried to unveil one of the problems where we have considered five bodies in which four bodies are taken as primaries, placed at the vertices of a cyclic kite configuration and fifth body as infinitesimal body, moving in space under the Newtonian gravitational forces of the four primaries but not influencing them.

Many researchers have studied the restricted three-body, four-body and five-body problems with many perturbations. Some of them are as follow:

[24] has explained about the motion of few-body problem and their stabilities in his book: Theory of Orbits. [14] has generalized the classical restricted three-body problem to a particular restricted five-body problem and investigated the locations and stability of equilibrium points through computer algebra. [16] et al. have numerically studied the regions of quasi-periodic motion around non-symmetric periodic orbits in the vicinity of the triangular equilibrium points and observed that regions surround in general the linearly stable segments of the corresponding families and become smaller as the mass ratio increases. [9] has investigated the 3-dimensional periodic orbits which are nearly circular and have stable characteristic exponent in the circular restricted three-body problem. [1] et al. have investigated the stability of equilibrium points under the influence of small perturbations in the Coriolis and the centrifugal forces, radiation and oblateness in the restricted three-body problem. It is observed that Coriolis force has a stabilizing tendency while the centrifugal force, radiation and oblateness of the primaries have destabilizing effects. [12] have discussed the non-linear stability of triangular equilibrium points in the restricted three-body problem with Poynting-Robertson drag where the smaller primary is oblate and the bigger primary is taken as radiating. They have performed first and second order normalization of the Hamiltonian of the problem. To examined the stability, they have applied KAM theorem. [17] has illustrated cyclic central configurations in the four-body problem. [2, 3] et al. have studied the existence of equilibrium points, their stability and the periodic orbits around these points in the restricted three-body problem under the effects of oblateness of the three participating bodies as well as the perturbations. [15] et al. have studied the motion of the infinitesimal body in the generalized frame of the Sitnikov problem. They have considered four bodies in which three are equal oblate primaries,

placed at the vertices of an equilateral triangle and moving in the circular orbits around their common center of mass and fourth one is infinitesimal body moving in the gravitational fields of three primaries in the z -direction but not influencing them. [13] et al. have studied the spatial restricted rhomboidal five-body problem and derived the Hamiltonian function for it. They also found that the behaviour of the infinitesimal mass is quite predictable, periodic solutions of arbitrary long periods are shown to exist. [11] et al. have illustrated the motion of infinitesimal body in the restricted four-body problem with solar wind drag and examined the stability with the help of Poincaré surfaces of section and Lyapunov characteristic exponents. [21] et al. have studied the existence of transversal homoclinic orbits in a planar circular restricted four-body problem based on the perturbation theory of integrable Hamiltonian system. [10] et al. have studied the location of equilibrium points in the CR3BP when both the primaries are finite straight segments and found five equilibrium points. They observed that the collinear points are unstable for all mass ratios but the triangular points are conditional stable for a particular range of mass ratio and for rest it is unstable. They have also presented an application of this problem. [18, 19] have investigated Sitnikov problem with cyclic kite configuration and also performed the connection between three-body configuration and four-body configuration of the problem when one of the masses approaches to zero. [22, 23] have studied the in-plane and out-of-plane equilibrium points and their stability in the restricted four-body problem under the influence of perturbations. [20] et al. have explored numerically and analytically the regions of possible motion on the planar central configurations of rhomboidal and triangular four-and five-body problems. [6, 7, 8] et al. have studied the effect of perturbations on the motion of the infinitesimal body in the CR3BP and CR4BP with the variation of all the masses. By using Meshcherskii transformation, they have evaluated the equations of motion and then studied it numerically for stability and found that all the equilibrium points are unstable. [4, 5] has illustrated the locations and stability of equilibrium points of the effect of Yarkovsky and Poynting-Robertson drag on the circular restricted three-body problem. [25, 26] have performed numerically the Newton-Raphson basins of convergence in the planar circular restricted five-body problem and explained that how the mass parameter affect the geometry as well as

the degree of fractality of the basins of attraction.

We have arranged this paper as follow: In the formation of equations of motion section, we have formulated the equations of motion of the infinitesimal body under the effect of cyclic kite configuration of the primaries and hence the Jacobi-Integral constant has been determined. In the computational work section, we have numerically illustrated the equilibrium points, zero-velocity curves corresponding to each Jacobi-constant, regions of motion, Poincaré surfaces of section and basins of attraction in in-plane and out-of-planes by using well known software Mathematica. In the stability of equilibrium points section, we have examined the stability of the equilibrium points of in-plane and out-of-planes. And finally, we have concluded the problem.

This model is quite realistic in this space age and applicable to send an artificial satellite or placed the telescope in space.

2. FORMATION OF EQUATIONS OF MOTION

This problem consists five bodies having masses m_1, m_2, m_3, m_4 and m (with $m_1 = m_3$). Where the primaries m_1, m_2, m_3 and m_4 are placed at the vertices of a cyclic kite with radius R and moving around the circle's centre which is taken as origin O . This kite has two restrictions: Firstly, $P_1P_2P_3$ is an equilateral triangle, i.e. $P_1P_2 = P_2P_3 = P_3P_1 = \ell$ and secondly, $P_1P_3P_4$ is an isosceles triangle, i.e. $P_1P_4 = P_3P_4 = R$ and $P_1P_3 = \ell$. And hence the relation between ℓ and R will be $\ell = \sqrt{3}R$. The fifth infinitesimal body of mass m is moving in the space under the gravitational forces ($\vec{F}_1, \vec{F}_2, \vec{F}_3$ and $\vec{F}_4$, respectively) of these primaries but not influencing them. Let $\vec{r}_1, \vec{r}_2, \vec{r}_3$ and $\vec{r}_4$ be the distances of the primaries from the infinitesimal body respectively. In this study, primary m_1 is placed on x -axis and the line perpendicular to the x -axis and passing through the origin is considered as the y -axis. The line passing through the origin and perpendicular to the plane of motion of the primaries is taken as z -axis. Let us consider the synodic coordinate system ($OXYZ$) coincides initially with the inertial coordinate system ($Oxyz$), and revolving with angular velocity ω about z -axis. Let the coordinates of the primaries P_1, P_2, P_3, P_4 and infinitesimal body P be $(x_1, 0, 0) = (R, 0, 0)$,

$(x_2, y_2, 0) = (-\frac{R}{2}, -\frac{\sqrt{3}R}{2}, 0), (x_3, y_3, 0) = (-\frac{R}{2}, \frac{\sqrt{3}R}{2}, 0), (x_4, y_4, 0) = (\frac{R}{2}, \frac{\sqrt{3}R}{2}, 0)$ and (x, y, z) respectively (Fig. 1).

We can write the equations of motion of the infinitesimal body as

$$(1a) \quad m\left\{\frac{\partial^2 \vec{r}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{r}}{\partial t} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\right\} = \vec{F}.$$

where,

$$(1b) \quad \vec{F} = \sum_{n=1}^4 \frac{Gm_i m}{r_i^3} \vec{r}_i,$$

$$(1c) \quad \vec{r} = x \hat{i} + y \hat{j} + z \hat{k},$$

$$(1d) \quad r_i^2 = (x - x_i)^2 + (y - y_i)^2 + z^2.$$

Using equations (1b, 1c, 1d) in equation (1a), and comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get,

$$(2a) \quad \ddot{x} - 2n\dot{y} - n^2x = - \sum_{n=1}^4 \frac{Gm_i(x - x_i)}{r_i^3},$$

$$(2b) \quad \ddot{y} + 2n\dot{x} - n^2y = - \sum_{n=1}^4 \frac{Gm_i(y - y_i)}{r_i^3},$$

$$(2c) \quad \ddot{z} = - \sum_{n=1}^4 \frac{Gm_i z}{r_i^3}.$$

We choose the unit of distance as the radius of circle, unit of mass as the sum of the masses of the primaries and the unit of time is so chosen that the gravitational constant and mean motion will be unity. i.e. $R = 1, m_1 + m_2 + m_3 + m_4 = 1, G = 1$ and $n = 1$. Again let, $m_2 = \mu, m_4 = \alpha\mu, \Rightarrow m_1 = m_3 = \frac{1-\mu-\alpha\mu}{2}, \alpha \ll 1$.

Hence the equations of motion (2a), (2b) and (2c) become

$$(3a) \quad \ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x},$$

$$(3b) \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y},$$

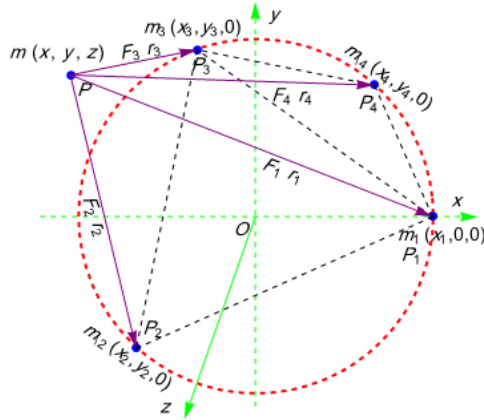


FIGURE 1. Kite configuration of the five-body problem.

$$(3c) \quad \ddot{z} = \frac{\partial \Omega}{\partial z}.$$

$$\text{where, } \Omega = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu-\alpha\mu)}{2r_1} + \frac{\mu}{r_2} + \frac{(1-\mu-\alpha\mu)}{2r_3} + \frac{\alpha\mu}{r_4}.$$

The Jacobi-Integral can be written as

$$(4) \quad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega - C.$$

where, C is the Jacobi-Integral constant.

3. COMPUTATIONAL WORK

This section is devoted to illustrate the computational work in which we have plotted equilibrium points in different planes (in-plane and out-of-planes), zero-velocity curves, regions of motion, Poincaré surfaces of section and basins of attraction. All these graphs are unveiling the behaviour of infinitesimal body. In all the calculations, we have taken $\mu = 0.019$ and $\alpha = 0.01$.

3.1. Equilibrium points. We have plotted the equilibrium points by solving $\frac{\partial \Omega}{\partial x} = 0$, $\frac{\partial \Omega}{\partial y} = 0$ and $\frac{\partial \Omega}{\partial z} = 0$, in two different planes:

- In-plane motion $(x, y, z = 0)$,
- Out-of planes motion $((x, y = 0, z)$ and $(x = 0, y, z))$.

3.1.1. In-plane motion. Here, we have drawn the position of equilibrium points in $x-y$ -plane (Fig.2a) and found four equilibrium points in which one (L_1) is on the x -axis and other three (L_2, L_3, L_4) are non-collinear. All the equilibrium points are near by the primaries as L_1 is near m_1 , L_2 and L_3 are near m_2 and L_4 is near m_3 . There is no equilibrium points near the primary m_4 .

3.1.2. Out-of-plane motion. We have considered two out-of-planes of motion as $(x, y = 0, z)$ and $(x = 0, y, z)$. During the out-of-plane $(x, y = 0, z)$, we found two collinear equilibrium points which are situated either side of primary m_1 on the x -axis (Fig.2b). During the out-of-plane $(x = 0, y, z)$, we found only one equilibrium point which is situated on the y -axis (Fig.2c).

3.2. Zero-velocity curves. Here, we have illustrated the zero-velocity curves by using equation (4) of Jacobi-Integral in three-different planes $(x, y, z = 0)$ (Fig.3a), $(x, y = 0, z)$ (Fig.3b) and $(x = 0, y, z)$ (Fig.3c). To draw these, we have evaluated the values of Jacobian constant C corresponding to each equilibrium points for all three planes i.e., C_i , where $1 \leq i \leq 4$.

In $x-y$ -plane, we have found three zero-velocity curves corresponding to each Jacobian constant (Fig. 3a). At the value of $C_{L_{1,4}}$, we have the same zero-velocity curve with two islands near the primaries m_1 and m_3 where L_1 and L_4 are representing the gateway of these two islands. Corresponding to the value of C_{L_2} , we have the shape like cardioid with one island near the primary m_2 , where L_2 is representing the gateway of island. Corresponding to the value of C_{L_3} , we have the shape like half-oval containing equilibrium point L_3 . It is observed that all the curves are bounded.

In $x-z$ - plane, we have two zero-velocity curves corresponding to each Jacobian constant (Fig. 3b). At the value of C_{L_1} , we have unbounded zero-velocity curve with one island near the primary m_1 where L_1 is representing the gateway of island. Corresponding to the value of C_{L_2} , we have the shape like DNA cell with intersection point at L_2 .

In $y-z$ - plane, we have one equilibrium point therefore corresponding to this we have just one DNA cell shape zero-velocity curve with intersection point at L_1 .

3.3. Regions of motion. In this subsection, we have illustrated the regions of motion for the dynamical behaviour of infinitesimal body under the effect of cyclic

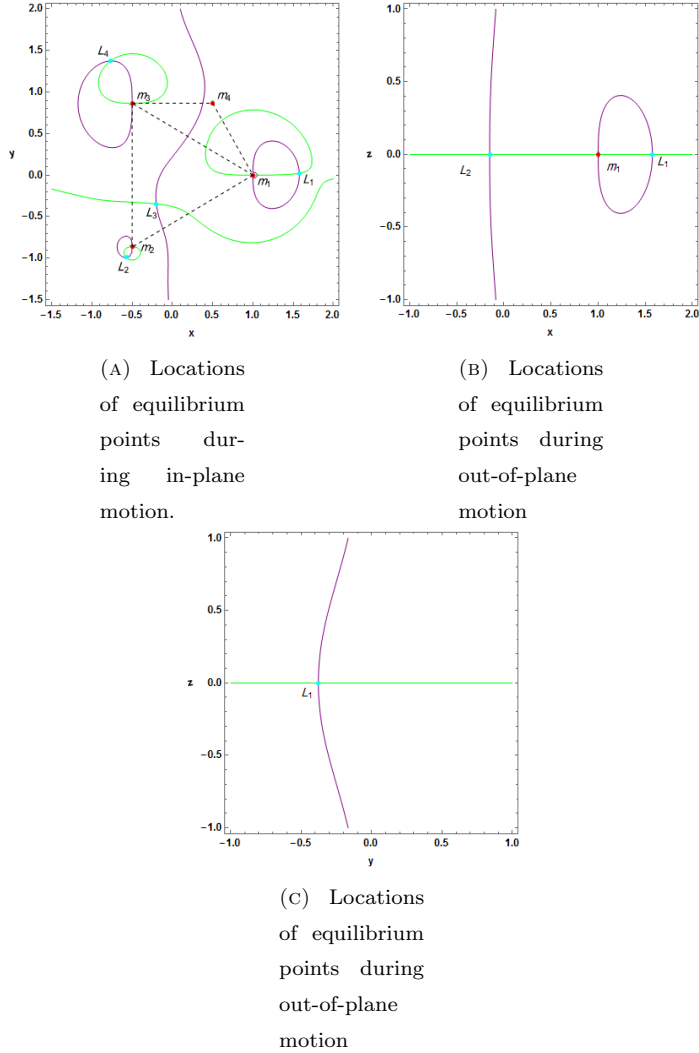


FIGURE 2. Locations of equilibrium points in different planes. Where cyan colors denote the position of the equilibrium points and red colors represent the positions of the primaries.

kite configuration of the primaries in three planes ($x-y$, (Fig. 4a), $x-z$, (Fig. 4b) and $y-z$ (Fig. 4c)). In the figure 4a, we got three color regions blue, green and black corresponding to the jacobian constants $C_{L_{1,4}}$, C_{L_2} and C_{L_3} respectively. In

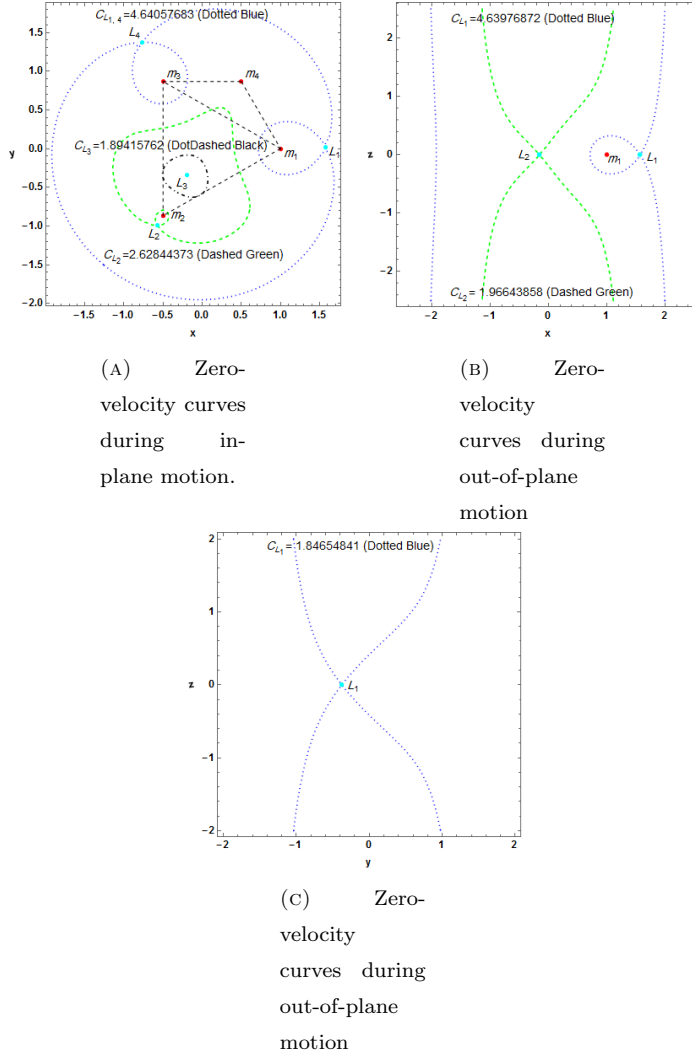


FIGURE 3. Zero-velocity curves in different three planes

the figure 4b, we got two color regions blue and green corresponding to the jacobian constants C_{L_1} and C_{L_2} respectively. In the figure 4c, we got blue color region corresponding to the jacobian constant C_{L_1} . These color regions are the forbidden regions which decrease as the decrease in the value of corresponding Jacobian constants and the infinitesimal body can move freely in the rest regions.

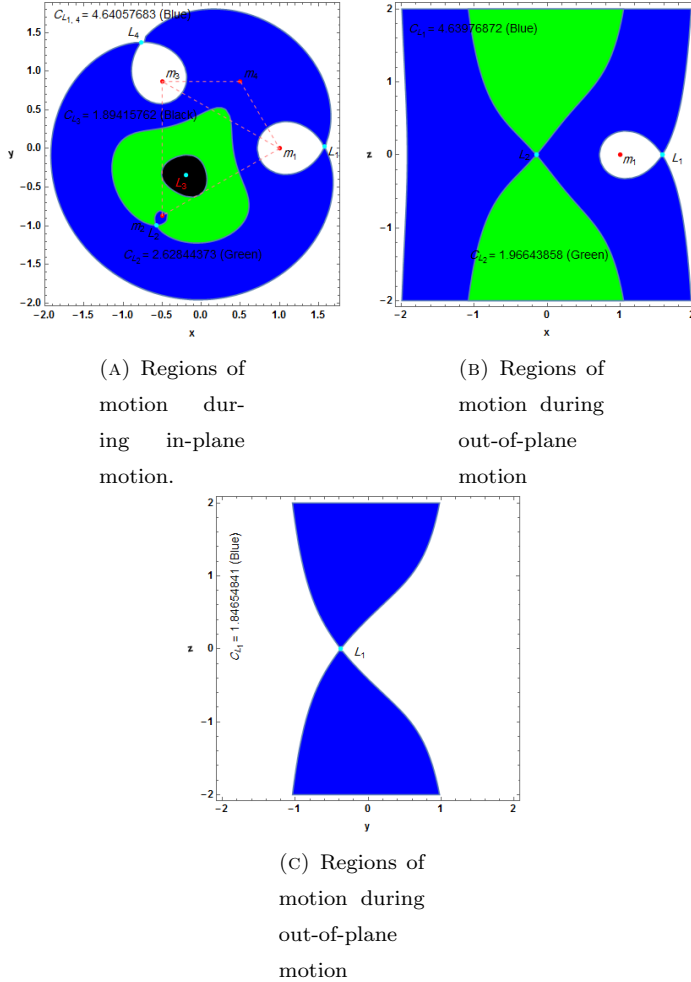


FIGURE 4. Regions of motion in different three planes

3.4. Poincaré surfaces of section. We have drawn the Poincaré surfaces of section of the motion of infinitesimal body under the effect of cyclic kite configuration of the primaries. From the figure 5, it is observed that the shape of the surface is as the wings of butterfly and there are no chaotic regions.

3.5. Basins of Attraction. The basins of attraction are revealed in the frame of the circular restricted five-body problem with cyclic kite configuration of the primaries by using the simple, fast and accurate Newton-Raphson method to solve the

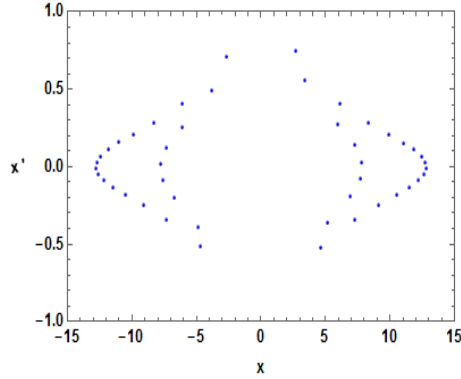


FIGURE 5. Poincaré surfaces of section.

multivariate functions. The basin of convergence or converging region is composed by all the initial conditions that lead to a specific equilibrium points. It is an issue of great importance to get the basins of convergence which reflect some most important qualitative properties of the dynamical systems. Using this iterative method, we have drawn the basins of convergence in three planes ($x - y$ plane (Fig. 6a), $x - z$ plane (Fig. 7a) and $y - z$ plane (Fig. 7b)). The algorithm of our problem in $x - y$ plane when $z = 0$, is given by

$$(5a) \quad x_{n+1} = x_n - \left(\frac{\Omega_x \Omega_{yy} - \Omega_y \Omega_{xy}}{\Omega_{xx} \Omega_{yy} - \Omega_{xy} \Omega_{yx}} \right)_{(x_n, y_n)},$$

$$(5b) \quad y_{n+1} = y_n - \left(\frac{\Omega_y \Omega_{xx} - \Omega_x \Omega_{yx}}{\Omega_{xx} \Omega_{yy} - \Omega_{xy} \Omega_{yx}} \right)_{(x_n, y_n)}.$$

Where x_n, y_n are the values of x and y coordinates of the n^{th} step of the Newton-Raphson iterative process. If the initial point converges rapidly to one of the equilibrium points then this point (x, y) will be a member of the basin of convergence of the root. This process stops when the successive approximation converges to an attractor. We used color code for the classification of the equilibrium points on the planes.

In $x - y$ -plane (Fig. 6a), we observed that the equilibrium points L_1 represents blue color region, L_2 represents red color region, L_3 represents light green color region and L_4 represents yellow color region. The basins of attraction corresponding to three equilibrium points (L_1, L_3, L_4) extend to infinity while corresponding to L_2

covers a finite region. In $x-z$ and $y-z$ -planes (Fig. 7 a, b), the equilibrium points represent cyan color regions and the basins of attraction corresponding to these equilibrium points extend to infinity. In all the figures black points and red points represent the locations of the equilibrium points and the primaries respectively.

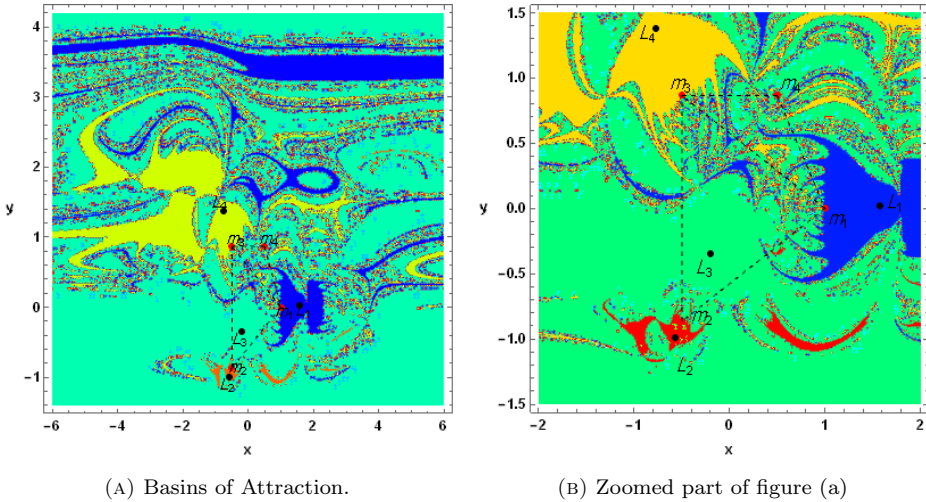


FIGURE 6. Basins of Attraction of five-body problem with kite configuration of the primaries in $x-y$ plane

4. STABILITY OF EQUILIBRIUM POINTS

This section is devoted to the mathematical investigation of the stability of equilibrium points in the frame of five-body problem with kite configuration. For which we considered the small displacement from the equilibrium points as well as the small velocity to the infinitesimal body. If the motion of the infinitesimal body is depart from the proximity of the point and it never come back, then this equilibrium point is known as unstable. But if the body oscillates about the point, it is said to be stable point. In the celestial mechanics, this method has applied frequently.

Let (x_0, y_0, z_0) be the coordinates of the equilibrium points in the inertial frame. Let (α, β, γ) denote small displacements of the infinitesimal body from the equilibrium points. These are functions of time and hence there will be velocity and acceleration components $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ and $(\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma})$ respectively. Therefore, we can

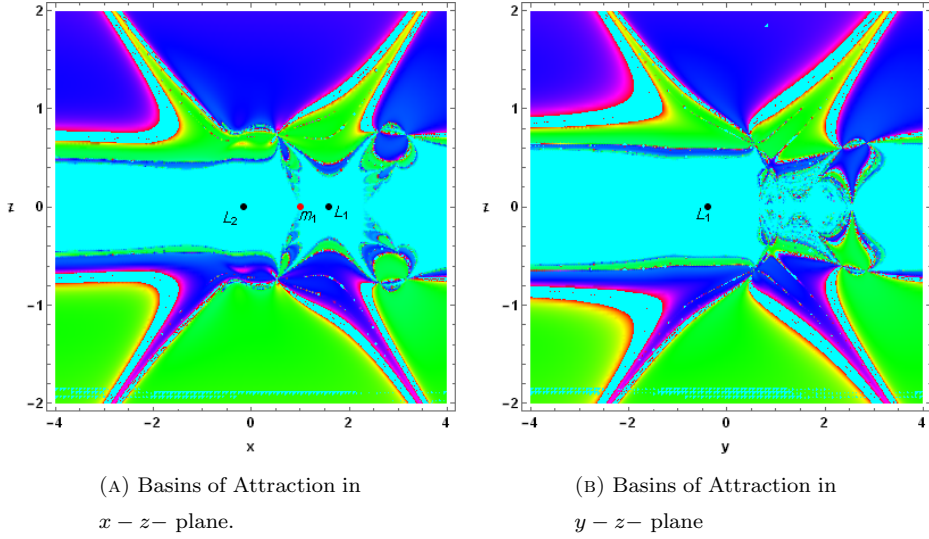


FIGURE 7. Basins of Attraction of five-body problem with kite configuration of the primaries

write the variational equations from the equations (3a), (3b) and (3c) as

$$(6a) \quad \ddot{\alpha} - 2\dot{\beta} = \Omega_{xx}^0 \alpha + \Omega_{xy}^0 \beta + \Omega_{xz}^0 \gamma,$$

$$(6b) \quad \ddot{\alpha} + 2\dot{\beta} = \Omega_{yx}^0 \alpha + \Omega_{yy}^0 \beta + \Omega_{yz}^0 \gamma,$$

$$(6c) \quad \dot{\gamma} = \Omega_{zx}^0 \alpha + \Omega_{zy}^0 \beta + \Omega_{zz}^0 \gamma.$$

Where the superscript zero (0) denote the values at the equilibrium point (x_0, y_0, z_0) . Let the solutions of these equations be $\alpha = A e^{\lambda t}$, $\beta = B e^{\lambda t}$ and $\gamma = C e^{\lambda t}$. Where A, B, C and λ are parameters to be found. Therefore, it will satisfy the variational equations (6a), (6b) and (6c), and become:

$$(7a) \quad A[\lambda^2 - \Omega_{xx}^0] + B[-2\lambda - \Omega_{xy}^0] + C[-\Omega_{xz}^0] = 0,$$

$$(7b) \quad A[2\lambda - \Omega_{yx}^0] + B[\lambda^2 - \Omega_{yy}^0] + C[-\Omega_{yz}^0] = 0,$$

$$(7c) \quad A[-\Omega_{zx}^0] + B[-\Omega_{zy}^0] + C[\lambda^2 - \Omega_{zz}^0] = 0.$$

There will have a non-trivial solution for A, B and C, if

$$(8) \quad \begin{vmatrix} \lambda^2 - \Omega_{xx}^0 & -2\lambda - \Omega_{xy}^0 & -\Omega_{xz}^0 \\ 2\lambda - \Omega_{yx}^0 & \lambda^2 - \Omega_{yy}^0 & -\Omega_{yz}^0 \\ -\Omega_{zx}^0 & -\Omega_{zy}^0 & \lambda^2 - \Omega_{zz}^0 \end{vmatrix} = 0$$

and this simplifies to

$$(9) \quad \lambda^6 + \lambda^4(4 - \Omega_{xx}^0 - \Omega_{yy}^0 - \Omega_{zz}^0) + \lambda^2(-(\Omega_{xy}^0)^2 - (\Omega_{xz}^0)^2 + \Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{yz}^0)^2 - 4\Omega_{zz}^0 \\ + \Omega_{xx}^0\Omega_{zz}^0 + \Omega_{yy}^0\Omega_{zz}^0) + ((\Omega_{xz}^0)^2\Omega_{yy}^0 - 2\Omega_{xy}^0\Omega_{xz}^0\Omega_{yz}^0 + \Omega_{xx}^0(\Omega_{yz}^0)^2 + (\Omega_{xy}^0)^2\Omega_{zz}^0 - \\ \Omega_{xx}^0\Omega_{yy}^0\Omega_{zz}^0) = 0.$$

We solved the characteristic equation (9), for each equilibrium points and found six complex characteristic roots (Table-1) in which at least one of the roots is positive real part (bold font). Therefore all the equilibrium points either in-plane or out-of-planes motion are unstable.

5. CONCLUSION

This manuscript unveiled the effect of the kite configuration of the primaries on the motion of the infinitesimal body. We have evaluated the equations of motion and hence the Jacobi-integral constant C. In the next, we have studied numerically the equilibrium points, zero-velocity curves corresponding to each Jacobi-constant, regions of motion, Poincaré surfaces of section and basins of attraction. We have drawn equilibrium points in in-plane and out-of-planes motion. During in-plane motion, we have four equilibrium points while in out-of-planes motion, we have at-most two equilibrium points. We have drawn the zero-velocity curves in in-plane and out-of-planes corresponding to each Jacobi-constant and found different shapes. We also illustrated the regions of motion corresponding to each Jacobi-constant by colored regions which shows that the motion is forbidden in the colored regions and infinitesimal body can move freely in the rest regions. In the Poincaré surfaces of section, we have the shape like the wings of butterfly and there are no chaos. We also have drawn the basins of convergence in in-plane and out-of-planes and

TABLE 1. Characteristic roots corresponding to each equilibrium points in in-plane and out-of-planes motion

<i>S.No.</i>	Equilibrium points	Corresponding Characteristic roots
	In-plane $(x, y, 0)$	
1	(1.5748279202728284, 0.02247029658753783, 0)	-1.8652945800777807, - 1.619067179609094 I, 1.619067179609094 I, - 1.690545869942661 I, 1.690545869942661 I, 1.8652945800777807
2	(-0.5728627219855881, -0.9922273402412433, 0)	-3.2840905292468325, - 2.508430441008324 I, 2.508430441008324 I, - 2.548141936178444 I, 2.548141936178444 I, 3.284090529246833
3	(-0.20012467131512354, -0.34662609856581555, 0)	-0.5390772804420898 - 0.9974721899550215 I, -0.5390772804420898 + 0.9974721899550215 I, - 0.7689649466070257 I, 0.7689649466070257 I, 0.5390772804420898 - 0.9974721899550215 I, 0.5390772804420898 + 0.9974721899550215 I
4	(-0.7679541124610354, 1.3750761338390525, 0)	-1.865294580077801, - 1.6190671796091034 I, 1.6190671796091034 I, - 1.6905458699426679 I, 1.6905458699426679 I, 1.8652945800778007
	Out-of-plane $(x, 0, z)$	
5	(1.5752136984892648, 0, 0)	-1.866386818989472, - 1.619133440202689 I, 1.619133440202689 I, - 1.6916875186969476 I, 1.6916875186969476 I, 1.866386818989472
6	(-0.15152416795341708, 0, 0)	-0.6343120485701083 - 0.9636357096491857 I, -0.6343120485701083 + 0.9636357096491857 I, - 0.9734043291973976 I, 0.9734043291973976 I, 0.6343120485701084 - 0.9636357096491857 I, 0.6343120485701084 + 0.9636357096491857 I
	Out-of-plane $(0, y, z)$	
7	(0, -0.37778423045739645, , 0)	-0.5531518106174178 - 0.9877183280275212 I, -0.5531518106174178 + 0.9877183280275212 I, - 0.8128830544030864 I, 0.8128830544030864 I, 0.5531518106174178 - 0.9877183280275212 I, 0.5531518106174178 + 0.9877183280275212 I

found different attracting regions in different planes. Finally, we have examined the stability of all the equilibrium points and observed that there are at-least one positive real parts in the complex characteristic roots. Hence we concluded that all the equilibrium points are unstable.

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REFERENCES

- [1] AbdulRaheem A, Singh J., Combined effects of perturbations, radiation and oblateness on the stability of equilibrium points in the restricted three-body problem, *The Astronomical Journal*, 131, 1880-1885 (2006).
- [2] Abouelmagd EI, Sharaf MA., The motion around the libration points in the restricted three-body problem with the effect of radiation and oblateness, *Astrophys Space Sci.*, 344, 321-332 (2013).
- [3] Abouelmagd EI, Asiri HM, Sharaf MA., The effect of oblateness in the perturbed restricted three-body problem, *Meccanica*, 48, 2479-2490 (2013). DOI 10.1007/s11012-013-9762-3.
- [4] Alhussain ZA., The Yarkovsky effect on the circular restricted three-body problem with perturbations, *Differential equations and control theory*, 19(3), 153-168 (2018).
- [5] Alhussain ZA., Effects of Poynting-Robertson drag on the circular restricted three-body problem with variable masses, *J. of Taibah university for science*, 12(4), 455-463 (2018).
- [6] Ansari AA, Alhussain ZA, Kellil R., The effect of perturbations on the circular restricted four-body problem with variable masses, *J. Math. Computer Sci.*, 17, 365377 (2017).
- [7] Ansari AA, Alhussain ZA, Kellil R., Locations and stability of the libration points in the CR3BP with perturbations. *Journal of Mathematical Analysis* ISSN: 2217-3412, URL: www.ilirias.com/jma, 8(5), 131-144 (2017).
- [8] Ansari AA., Effect of Albedo on the motion of the infinitesimal body in circular restricted three-body problem with variable masses, *Italian J. of Pure and Applied Mathematics*, 38, 581-600 (2017).
- [9] Broucke RA, Stability orbits of planets of binary star system in three-dimensional restricted problem, *Celest Mech Dyn Astro.*, 81(4), 321-341 (2001).
- [10] Jain R, Sinha D., Stability and regions of motion in the restricted three-body problem when both the primaries are finite straight segments, *Astrophys Space Sci.*, 351, 87-100 (2014), DOI 10.1007/s10509-013-1698-5.

- [11] Kumari R, Kushvah BS., Equilibrium points and zero velocity surfaces in the restricted four-body problem with solar wind drag, *Astrophys Space Sci.*, 344(2), 347-359 (2013).
- [12] Kushvah BS, Sharma JP, Ishwar B., Non-linear stability in the generalised photogravitational restricted three-body problem with P-R drag, *Astrophys Space Sci.*, 312(3-4), 279-293 (2007), DOI 10.1007/s10509-007-9688-0.
- [13] Marchesin M, Vidal C, Special restricted rhomboidal five-body problem and horizontal stability of its periodic solutions, *Celest Mech Dyn Astro.*, 115, 261-279 (2013), DOI 10.1007/s10569-012-9462-7.
- [14] Ollongren A, On a particular restricted five-body problem: An analysis with computer algebra, *J. Symbolic computation*, 6, 117-126 (1988).
- [15] Pandey LP, Ahmad I, Periodic orbits and bifurcation in the Sitnikov four-body problem when all the primaries are oblate, *Astron Astrophys*, 345, 73-83 (2013).
- [16] Ragos O, Papadakis KE, Zagouras CG, Stability regions and quasi-periodic motion in the vicinity of triangular equilibrium points, *Celest Mech Dyn Astro.*, 67(4), 251-274 (1997).
- [17] Robert GE, Cyclic central configurations in the four-body problem, In: AMS-HC, 1-30, 2011.
- [18] Shahbaz Ullah M, Hassan MR, Connection between three-body configuration and four-body configuration of the Sitnikov problem when one of the masses approaches to zero: circular case, *Astron. Astrophys*, 353, 53-64 (2014).
- [19] Shahbaz Ullah M, Bhatnagar KB, Hassan MR, Sitnikov problem in the cyclic kite configuration, *Astrophys Space Sci.*, 354, 301-309 (2014).
- [20] Shoaib M, Kashif AR, Csillik IS, On the planar central configurations of rhomboidal and triangular four-and five-body problems, *Astrophys Space Sci.*, 362, 182 (2017).
- [21] She Z, Cheng X, Li C, The existence of transversal homoclinic orbits in a planar circular restricted four-body problem, *Celest Mech Dyn Astro.*, 115(3), 299-309 (2013).
- [22] Singh J, Vincent AE, Effect of perturbations in the coriolis and centrifugal forces on the stability of equilibrium points in the restricted four-body problem, *Few-Body Syst*, 56, 713-723 (2015), DOI 10.1007/s00601-015-1019-3.
- [23] Singh J, Vincent AE, Out-of-plane equilibrium points in the photogravitational restricted four-body problem with oblateness, *British Journal of Mathematics and Computer Sci*, 19(5), 1-15 (2016), DOI: 10.9734/BJMCS/2016/29698.
- [24] Szebehely V, *Theory of Orbits*, New York(NY), Academic Press, 1997.
- [25] Zotos EE, Revealing the basins of convergence in the planar equilateral restricted four-body problem, *Astrophys Space Sci*, 362(2) (2017).
- [26] Zotos EE, Suraj MS, Basins of attraction of equilibrium points in the planar circular restricted five-body problem, *Astrophys Space Sci*, 363, 20 (2018), <http://doi.org/10.1007/s10509-017-3240-7>.